

The points in a plane perpendicular to the x -axis all have the same x -coordinate, this being the number at which that plane cuts the x -axis. The y - and z -coordinates can be any numbers. Similarly, the points in a plane perpendicular to the y -axis have a common y -coordinate and the points in a plane perpendicular to the z -axis have a common z -coordinate. To write equations for these planes, we name the common coordinate's value. The plane $x = 2$ is the plane perpendicular to the x -axis at $x = 2$. The plane $y = 3$ is the plane perpendicular to the y -axis at $y = 3$. The plane $z = 5$ is the plane perpendicular to the z -axis at $z = 5$. Figure 12.3 shows the planes $x = 2$, $y = 3$, and $z = 5$, together with their intersection point $(2, 3, 5)$.

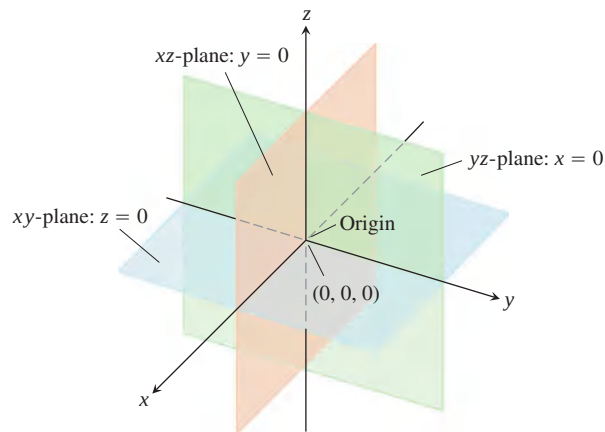


FIGURE 12.2 The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

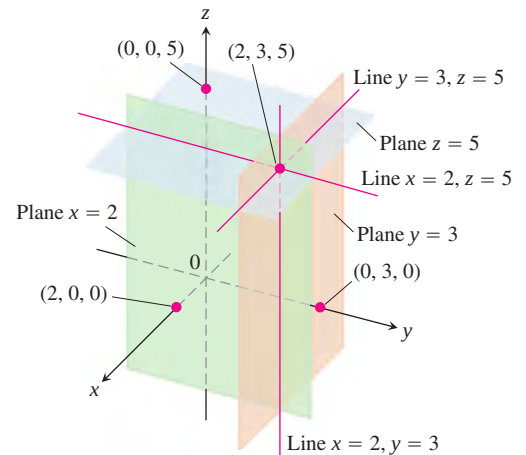


FIGURE 12.3 The planes $x = 2$, $y = 3$, and $z = 5$ determine three lines through the point $(2, 3, 5)$.

The planes $x = 2$ and $y = 3$ in Figure 12.3 intersect in a line parallel to the z -axis. This line is described by the *pair* of equations $x = 2$, $y = 3$. A point (x, y, z) lies on the line if and only if $x = 2$ and $y = 3$. Similarly, the line of intersection of the planes $y = 3$ and $z = 5$ is described by the equation pair $y = 3$, $z = 5$. This line runs parallel to the x -axis. The line of intersection of the planes $x = 2$ and $z = 5$, parallel to the y -axis, is described by the equation pair $x = 2$, $z = 5$.

In the following examples, we match coordinate equations and inequalities with the sets of points they define in space.



EXAMPLE 1 Interpreting Equations and Inequalities Geometrically

(a) $z \geq 0$

The half-space consisting of the points on and above the xy -plane.

(b) $x = -3$

The plane perpendicular to the x -axis at $x = -3$. This plane lies parallel to the yz -plane and 3 units behind it.

(c) $z = 0, x \leq 0, y \geq 0$

The second quadrant of the xy -plane.

(d) $x \geq 0, y \geq 0, z \geq 0$

The first octant.

(e) $-1 \leq y \leq 1$

The slab between the planes $y = -1$ and $y = 1$ (planes included).

(f) $y = -2, z = 2$

The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x -axis. ■

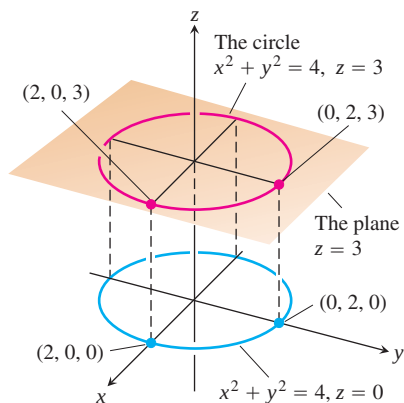


FIGURE 12.4 The circle $x^2 + y^2 = 4$ in the plane $z = 3$ (Example 2).

EXAMPLE 2 Graphing Equations

What points $P(x, y, z)$ satisfy the equations

$$x^2 + y^2 = 4 \quad \text{and} \quad z = 3?$$

Solution The points lie in the horizontal plane $z = 3$ and, in this plane, make up the circle $x^2 + y^2 = 4$. We call this set of points “the circle $x^2 + y^2 = 4$ in the plane $z = 3$ ” or, more simply, “the circle $x^2 + y^2 = 4, z = 3$ ” (Figure 12.4). ■

Distance and Spheres in Space

The formula for the distance between two points in the xy -plane extends to points in space.

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof We construct a rectangular box with faces parallel to the coordinate planes and the points P_1 and P_2 at opposite corners of the box (Figure 12.5). If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then the three box edges P_1A , AB , and BP_2 have lengths

$$|P_1A| = |x_2 - x_1|, \quad |AB| = |y_2 - y_1|, \quad |BP_2| = |z_2 - z_1|.$$

Because triangles P_1BP_2 and P_1AB are both right-angled, two applications of the Pythagorean theorem give

$$|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2 \quad \text{and} \quad |P_1B|^2 = |P_1A|^2 + |AB|^2$$

(see Figure 12.5).

So

$$\begin{aligned} |P_1P_2|^2 &= |P_1B|^2 + |BP_2|^2 \\ &= |P_1A|^2 + |AB|^2 + |BP_2|^2 \quad \text{Substitute } |P_1B|^2 = |P_1A|^2 + |AB|^2. \\ &= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

Therefore

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad \blacksquare$$

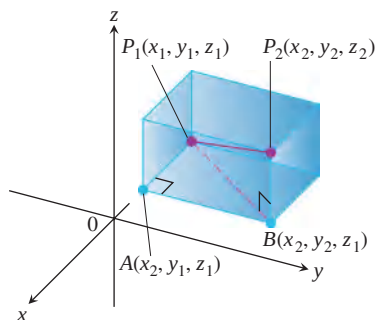


FIGURE 12.5 We find the distance between P_1 and P_2 by applying the Pythagorean theorem to the right triangles P_1AB and P_1BP_2 .



EXAMPLE 3 Finding the Distance Between Two Points

The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$\begin{aligned} |P_1P_2| &= \sqrt{(-2 - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\ &= \sqrt{16 + 4 + 25} \\ &= \sqrt{45} \approx 6.708. \end{aligned} \quad \blacksquare$$



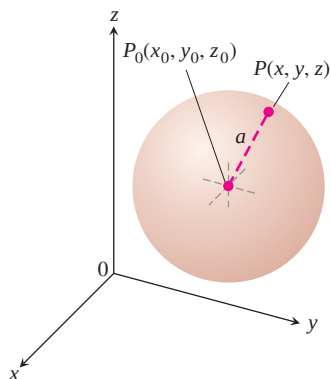


FIGURE 12.6 The standard equation of the sphere of radius a centered at the point (x_0, y_0, z_0) is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

We can use the distance formula to write equations for spheres in space (Figure 12.6). A point $P(x, y, z)$ lies on the sphere of radius a centered at $P_0(x_0, y_0, z_0)$ precisely when $|P_0P| = a$ or

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2.$$

The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

EXAMPLE 4 Finding the Center and Radius of a Sphere

Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Solution We find the center and radius of a sphere the way we find the center and radius of a circle: Complete the squares on the x -, y -, and z -terms as necessary and write each quadratic as a squared linear expression. Then, from the equation in standard form, read off the center and radius. For the sphere here, we have

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = -1 + \frac{9}{4} + 4 = \frac{21}{4}.$$

From this standard form, we read that $x_0 = -3/2$, $y_0 = 0$, $z_0 = 2$, and $a = \sqrt{21}/2$. The center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$. ■

EXAMPLE 5 Interpreting Equations and Inequalities

(a) $x^2 + y^2 + z^2 < 4$

The interior of the sphere $x^2 + y^2 + z^2 = 4$.

(b) $x^2 + y^2 + z^2 \leq 4$

The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.

(c) $x^2 + y^2 + z^2 > 4$

The exterior of the sphere $x^2 + y^2 + z^2 = 4$.

(d) $x^2 + y^2 + z^2 = 4, z \leq 0$

The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy -plane (the plane $z = 0$). ■

Just as polar coordinates give another way to locate points in the xy -plane (Section 10.5), alternative coordinate systems, different from the Cartesian coordinate system developed here, exist for three-dimensional space. We examine two of these coordinate systems in Section 15.6.



EXERCISES 12.1

Sets, Equations, and Inequalities

In Exercises 1–12, give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations.

1. $x = 2, y = 3$
2. $x = -1, z = 0$
3. $y = 0, z = 0$
4. $x = 1, y = 0$
5. $x^2 + y^2 = 4, z = 0$
6. $x^2 + y^2 = 4, z = -2$
7. $x^2 + z^2 = 4, y = 0$
8. $y^2 + z^2 = 1, x = 0$
9. $x^2 + y^2 + z^2 = 1, x = 0$
10. $x^2 + y^2 + z^2 = 25, y = -4$
11. $x^2 + y^2 + (z + 3)^2 = 25, z = 0$
12. $x^2 + (y - 1)^2 + z^2 = 4, y = 0$

In Exercises 13–18, describe the sets of points in space whose coordinates satisfy the given inequalities or combinations of equations and inequalities.

13. a. $x \geq 0, y \geq 0, z = 0$ b. $x \geq 0, y \leq 0, z = 0$
14. a. $0 \leq x \leq 1$ b. $0 \leq x \leq 1, 0 \leq y \leq 1$
c. $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$
15. a. $x^2 + y^2 + z^2 \leq 1$ b. $x^2 + y^2 + z^2 > 1$
16. a. $x^2 + y^2 \leq 1, z = 0$ b. $x^2 + y^2 \leq 1, z = 3$
c. $x^2 + y^2 \leq 1$, no restriction on z
17. a. $x^2 + y^2 + z^2 = 1, z \geq 0$
b. $x^2 + y^2 + z^2 \leq 1, z \geq 0$
18. a. $x = y, z = 0$ b. $x = y$, no restriction on z

In Exercises 19–28, describe the given set with a single equation or with a pair of equations.

19. The plane perpendicular to the
a. x -axis at $(3, 0, 0)$ b. y -axis at $(0, -1, 0)$
c. z -axis at $(0, 0, -2)$
20. The plane through the point $(3, -1, 2)$ perpendicular to the
a. x -axis b. y -axis c. z -axis
21. The plane through the point $(3, -1, 1)$ parallel to the
a. xy -plane b. yz -plane c. xz -plane
22. The circle of radius 2 centered at $(0, 0, 0)$ and lying in the
a. xy -plane b. yz -plane c. xz -plane
23. The circle of radius 2 centered at $(0, 2, 0)$ and lying in the
a. xy -plane b. yz -plane c. plane $y = 2$
24. The circle of radius 1 centered at $(-3, 4, 1)$ and lying in a plane parallel to the
a. xy -plane b. yz -plane c. xz -plane

25. The line through the point $(1, 3, -1)$ parallel to the
a. x -axis b. y -axis c. z -axis
26. The set of points in space equidistant from the origin and the point $(0, 2, 0)$
27. The circle in which the plane through the point $(1, 1, 3)$ perpendicular to the z -axis meets the sphere of radius 5 centered at the origin
28. The set of points in space that lie 2 units from the point $(0, 0, 1)$ and, at the same time, 2 units from the point $(0, 0, -1)$

Write inequalities to describe the sets in Exercises 29–34.

29. The slab bounded by the planes $z = 0$ and $z = 1$ (planes included)
30. The solid cube in the first octant bounded by the coordinate planes and the planes $x = 2, y = 2$, and $z = 2$
31. The half-space consisting of the points on and below the xy -plane
32. The upper hemisphere of the sphere of radius 1 centered at the origin
33. The (a) interior and (b) exterior of the sphere of radius 1 centered at the point $(1, 1, 1)$
34. The closed region bounded by the spheres of radius 1 and radius 2 centered at the origin. (*Closed* means the spheres are to be included. Had we wanted the spheres left out, we would have asked for the *open* region bounded by the spheres. This is analogous to the way we use *closed* and *open* to describe intervals: *closed* means endpoints included, *open* means endpoints left out. Closed sets include boundaries; open sets leave them out.)

Distance

In Exercises 35–40, find the distance between points P_1 and P_2 .

35. $P_1(1, 1, 1), P_2(3, 3, 0)$
36. $P_1(-1, 1, 5), P_2(2, 5, 0)$
37. $P_1(1, 4, 5), P_2(4, -2, 7)$
38. $P_1(3, 4, 5), P_2(2, 3, 4)$
39. $P_1(0, 0, 0), P_2(2, -2, -2)$
40. $P_1(5, 3, -2), P_2(0, 0, 0)$

Spheres

Find the centers and radii of the spheres in Exercises 41–44.

41. $(x + 2)^2 + y^2 + (z - 2)^2 = 8$
42. $\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 + \left(z + \frac{1}{2}\right)^2 = \frac{21}{4}$
43. $(x - \sqrt{2})^2 + (y - \sqrt{2})^2 + (z + \sqrt{2})^2 = 2$
44. $x^2 + \left(y + \frac{1}{3}\right)^2 + \left(z - \frac{1}{3}\right)^2 = \frac{29}{9}$

Find equations for the spheres whose centers and radii are given in Exercises 45–48.



	Center	Radius
45.	$(1, 2, 3)$	$\sqrt{14}$
46.	$(0, -1, 5)$	2
47.	$(-2, 0, 0)$	$\sqrt{3}$
48.	$(0, -7, 0)$	7



Find the centers and radii of the spheres in Exercises 49–52.

49. $x^2 + y^2 + z^2 + 4x - 4z = 0$
 50. $x^2 + y^2 + z^2 - 6y + 8z = 0$

51. $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$

52. $3x^2 + 3y^2 + 3z^2 + 2y - 2z = 9$



Theory and Examples

53. Find a formula for the distance from the point $P(x, y, z)$ to the
 a. x -axis b. y -axis c. z -axis
54. Find a formula for the distance from the point $P(x, y, z)$ to the
 a. xy -plane b. yz -plane c. xz -plane
55. Find the perimeter of the triangle with vertices $A(-1, 2, 1)$, $B(1, -1, 3)$, and $C(3, 4, 5)$.
56. Show that the point $P(3, 1, 2)$ is equidistant from the points $A(2, -1, 3)$ and $B(4, 3, 1)$.

12.2

Vectors



Some of the things we measure are determined simply by their magnitudes. To record mass, length, or time, for example, we need only write down a number and name an appropriate unit of measure. We need more information to describe a force, displacement, or velocity. To describe a force, we need to record the direction in which it acts as well as how large it is. To describe a body's displacement, we have to say in what direction it moved as well as how far. To describe a body's velocity, we have to know where the body is headed as well as how fast it is going.

Component Form

A quantity such as force, displacement, or velocity is called a *vector* and is represented by a **directed line segment** (Figure 12.7). The arrow points in the direction of the action and its length gives the magnitude of the action in terms of a suitably chosen unit. For example, a force vector points in the direction in which the force acts; its length is a measure of the force's strength; a velocity vector points in the direction of motion and its length is the speed of the moving object. Figure 12.8 displays the velocity vector $\mathbf{v}$ at a specific location for a particle moving along a path in the plane or in space. (This application of vectors is studied in Chapter 13.)

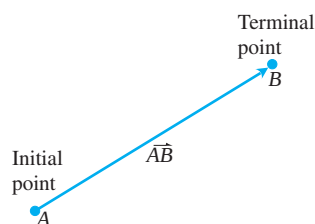


FIGURE 12.7 The directed line segment $\overrightarrow{AB}$.

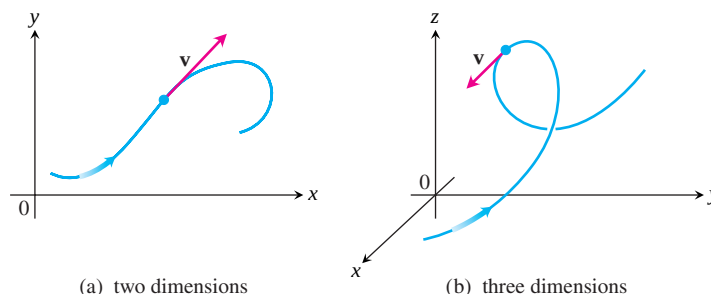


FIGURE 12.8 The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

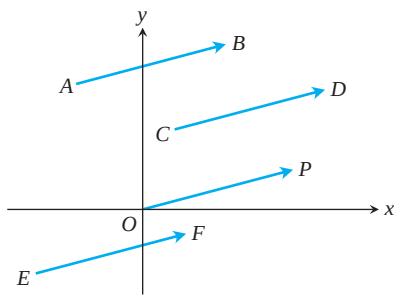


FIGURE 12.9 The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\vec{AB} = \vec{CD} = \vec{OP} = \vec{EF}$.

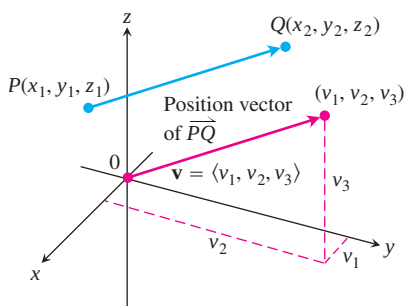


FIGURE 12.10 A vector $\vec{PQ}$ in standard position has its initial point at the origin. The directed line segments $\vec{PQ}$ and $\mathbf{v}$ are parallel and have the same length.

DEFINITIONS Vector, Initial and Terminal Point, Length

A **vector** in the plane is a directed line segment. The directed line segment $\vec{AB}$ has **initial point** A and **terminal point** B ; its **length** is denoted by $|\vec{AB}|$. Two vectors are **equal** if they have the same length and direction.

The arrows we use when we draw vectors are understood to represent the same vector if they have the same length, are parallel, and point in the same direction (Figure 12.9) regardless of the initial point.

In textbooks, vectors are usually written in lowercase, boldface letters, for example $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Sometimes we use uppercase boldface letters, such as $\mathbf{F}$, to denote a force vector. In handwritten form, it is customary to draw small arrows above the letters, for example $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{F}$.

We need a way to represent vectors algebraically so that we can be more precise about the direction of a vector.

Let $\mathbf{v} = \vec{PQ}$. There is one directed line segment equal to $\vec{PQ}$ whose initial point is the origin (Figure 12.10). It is the representative of $\mathbf{v}$ in **standard position** and is the vector we normally use to represent $\mathbf{v}$. We can specify $\mathbf{v}$ by writing the coordinates of its terminal point (v_1, v_2, v_3) when $\mathbf{v}$ is in standard position. If $\mathbf{v}$ is a vector in the plane its terminal point (v_1, v_2) has two coordinates.

DEFINITION Component Form

If $\mathbf{v}$ is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) , then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If $\mathbf{v}$ is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

So a two-dimensional vector is an ordered pair $\mathbf{v} = \langle v_1, v_2 \rangle$ of real numbers, and a three-dimensional vector is an ordered triple $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ of real numbers. The numbers v_1 , v_2 , and v_3 are called the **components** of $\mathbf{v}$.

Observe that if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is represented by the directed line segment $\vec{PQ}$, where the initial point is $P(x_1, y_1, z_1)$ and the terminal point is $Q(x_2, y_2, z_2)$, then $x_1 + v_1 = x_2$, $y_1 + v_2 = y_2$, and $z_1 + v_3 = z_2$ (see Figure 12.10). Thus, $v_1 = x_2 - x_1$, $v_2 = y_2 - y_1$, and $v_3 = z_2 - z_1$ are the components of $\vec{PQ}$.

In summary, given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ equal to $\vec{PQ}$ is

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$



If $\mathbf{v}$ is two-dimensional with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as points in the plane, then $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$. There is no third component for planar vectors. With this understanding, we will develop the algebra of three-dimensional vectors and simply drop the third component when the vector is two-dimensional (a planar vector).

Two vectors are equal if and only if their standard position vectors are identical. Thus $\langle u_1, u_2, u_3 \rangle$ and $\langle v_1, v_2, v_3 \rangle$ are equal if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.

The **magnitude** or **length** of the vector $\vec{PQ}$ is the length of any of its equivalent directed line segment representations. In particular, if $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ is the standard position vector for $\vec{PQ}$, then the distance formula gives the magnitude or length of $\mathbf{v}$, denoted by the symbol $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

The **magnitude** or **length** of the vector $\mathbf{v} = \vec{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(See Figure 12.10.)

The only vector with length 0 is the **zero vector** $\mathbf{0} = \langle 0, 0 \rangle$ or $\mathbf{0} = \langle 0, 0, 0 \rangle$. This vector is also the only vector with no specific direction.



EXAMPLE 1 Component Form and Length of a Vector

Find the (a) component form and (b) length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution

(a) The standard position vector $\mathbf{v}$ representing $\vec{PQ}$ has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \quad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

and

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of $\vec{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

(b) The length or magnitude of $\mathbf{v} = \vec{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3. \quad \blacksquare$$

EXAMPLE 2 Force Moving a Cart

A small cart is being pulled along a smooth horizontal floor with a 20-lb force $\mathbf{F}$ making a 45° angle to the floor (Figure 12.11). What is the *effective* force moving the cart forward?

Solution The effective force is the horizontal component of $\mathbf{F} = \langle a, b \rangle$, given by

$$a = |\mathbf{F}| \cos 45^\circ = (20) \left(\frac{\sqrt{2}}{2} \right) \approx 14.14 \text{ lb.}$$

Notice that $\mathbf{F}$ is a two-dimensional vector. \(\blacksquare\)

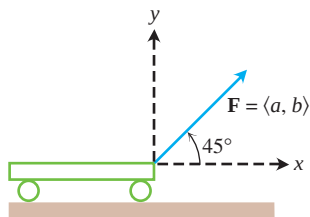


FIGURE 12.11 The force pulling the cart forward is represented by the vector $\mathbf{F}$ of magnitude 20 (pounds) making an angle of 45° with the horizontal ground (positive x -axis) (Example 2).

Vector Algebra Operations

Two principal operations involving vectors are *vector addition* and *scalar multiplication*. A **scalar** is simply a real number, and is called such when we want to draw attention to its differences from vectors. Scalars can be positive, negative, or zero.

DEFINITIONS Vector Addition and Multiplication of a Vector by a Scalar

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

We add vectors by adding the corresponding components of the vectors. We multiply a vector by a scalar by multiplying each component by the scalar. The definitions apply to planar vectors except there are only two components, $\langle u_1, u_2 \rangle$ and $\langle v_1, v_2 \rangle$.

The definition of vector addition is illustrated geometrically for planar vectors in Figure 12.12a, where the initial point of one vector is placed at the terminal point of the other. Another interpretation is shown in Figure 12.12b (called the **parallelogram law** of addition), where the sum, called the **resultant vector**, is the diagonal of the parallelogram. In physics, forces add vectorially as do velocities, accelerations, and so on. So the force acting on a particle subject to electric and gravitational forces is obtained by adding the two force vectors.

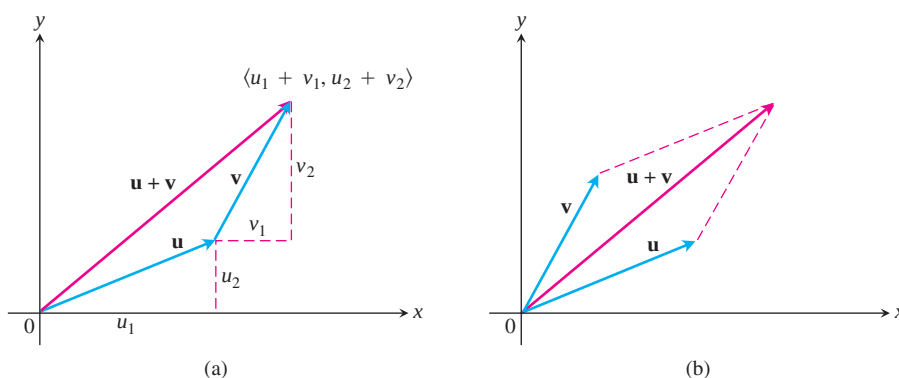


FIGURE 12.12 (a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition.

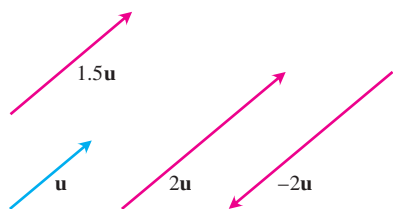


FIGURE 12.13 Scalar multiples of $\mathbf{u}$.

Figure 12.13 displays a geometric interpretation of the product $k\mathbf{u}$ of the scalar k and vector $\mathbf{u}$. If $k > 0$, then $k\mathbf{u}$ has the same direction as $\mathbf{u}$; if $k < 0$, then the direction of $k\mathbf{u}$ is opposite to that of $\mathbf{u}$. Comparing the lengths of $\mathbf{u}$ and $k\mathbf{u}$, we see that

$$\begin{aligned} |k\mathbf{u}| &= \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2} = \sqrt{k^2(u_1^2 + u_2^2 + u_3^2)} \\ &= \sqrt{k^2} \sqrt{u_1^2 + u_2^2 + u_3^2} = |k| |\mathbf{u}|. \end{aligned}$$

The length of $k\mathbf{u}$ is the absolute value of the scalar k times the length of $\mathbf{u}$. The vector $(-1)\mathbf{u} = -\mathbf{u}$ has the same length as $\mathbf{u}$ but points in the opposite direction.

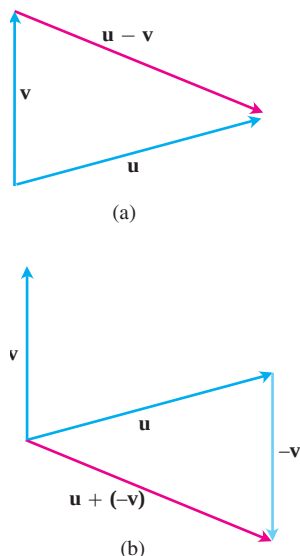


FIGURE 12.14 (a) The vector $\mathbf{u} - \mathbf{v}$, when added to $\mathbf{v}$, gives $\mathbf{u}$.
(b) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors, we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$

Note that $(\mathbf{u} - \mathbf{v}) + \mathbf{v} = \mathbf{u}$, so adding the vector $(\mathbf{u} - \mathbf{v})$ to $\mathbf{v}$ gives $\mathbf{u}$ (Figure 12.14a).

Figure 12.14b shows the difference $\mathbf{u} - \mathbf{v}$ as the sum $\mathbf{u} + (-\mathbf{v})$.

EXAMPLE 3 Performing Operations on Vectors

Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find

(a) $2\mathbf{u} + 3\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\left| \frac{1}{2}\mathbf{u} \right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left| \frac{1}{2}\mathbf{u} \right| = \left| \left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle \right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$ ■

Vector operations have many of the properties of ordinary arithmetic. These properties are readily verified using the definitions of vector addition and multiplication by a scalar.

Properties of Vector Operations

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$	6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$	8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$	

An important application of vectors occurs in navigation.

EXAMPLE 4 Finding Ground Speed and Direction

A Boeing® 767® airplane, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?



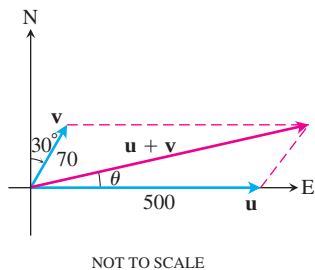


FIGURE 12.15 Vectors representing the velocities of the airplane $\mathbf{u}$ and tailwind $\mathbf{v}$ in Example 4.

Solution If $\mathbf{u}$ = the velocity of the airplane alone and $\mathbf{v}$ = the velocity of the tailwind, then $|\mathbf{u}| = 500$ and $|\mathbf{v}| = 70$ (Figure 12.15). The velocity of the airplane with respect to the ground is given by the magnitude and direction of the *resultant vector* $\mathbf{u} + \mathbf{v}$. If we let the positive x -axis represent east and the positive y -axis represent north, then the component forms of $\mathbf{u}$ and $\mathbf{v}$ are

$$\mathbf{u} = \langle 500, 0 \rangle \quad \text{and} \quad \mathbf{v} = \langle 70 \cos 60^\circ, 70 \sin 60^\circ \rangle = \langle 35, 35\sqrt{3} \rangle.$$

Therefore,

$$\mathbf{u} + \mathbf{v} = \langle 535, 35\sqrt{3} \rangle$$

$$|\mathbf{u} + \mathbf{v}| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4$$

and

$$\theta = \tan^{-1} \frac{35\sqrt{3}}{535} \approx 6.5^\circ. \quad \text{Figure 12.15}$$

The new ground speed of the airplane is about 538.4 mph, and its new direction is about 6.5° north of east. ■

Unit Vectors

A vector $\mathbf{v}$ of length 1 is called a **unit vector**. The **standard unit vectors** are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as follows:

$$\begin{aligned} \mathbf{v} &= \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle \\ &= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle \\ &= v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}. \end{aligned}$$

We call the scalar (or number) v_1 the **i-component** of the vector $\mathbf{v}$, v_2 the **j-component**, and v_3 the **k-component**. In component form, the vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

(Figure 12.16).

Whenever $\mathbf{v} \neq \mathbf{0}$, its length $|\mathbf{v}|$ is not zero and

$$\left| \frac{1}{|\mathbf{v}|} \mathbf{v} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1.$$

That is, $\mathbf{v}/|\mathbf{v}|$ is a unit vector in the direction of $\mathbf{v}$, called **the direction** of the nonzero vector $\mathbf{v}$.

EXAMPLE 5 Finding a Vector's Direction

Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

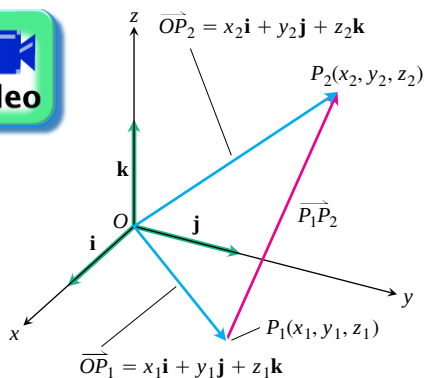


FIGURE 12.16 The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

Solution We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

The unit vector $\mathbf{u}$ is the direction of $\overrightarrow{P_1P_2}$. ■



EXAMPLE 6 Expressing Velocity as Speed Times Direction

If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times a unit vector in the direction of motion.

Solution Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ has the same direction as $\mathbf{v}$:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = 5 \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} \right).$$

Length
(speed) Direction of motion

In summary, we can express any nonzero vector $\mathbf{v}$ in terms of its two important features, length and direction, by writing $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$.

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of $\mathbf{v}$;
2. the equation $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ in terms of its length and direction.

HISTORICAL BIOGRAPHY

Hermann Grassmann
(1809–1877)



EXAMPLE 7 A Force Vector

A force of 6 newtons is applied in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. Express the force $\mathbf{F}$ as a product of its magnitude and direction.

Solution The force vector has magnitude 6 and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$, so

$$\begin{aligned} \mathbf{F} &= 6 \frac{\mathbf{v}}{|\mathbf{v}|} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = 6 \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} \\ &= 6 \left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k} \right). \end{aligned}$$

■

Midpoint of a Line Segment

Vectors are often useful in geometry. For example, the coordinates of the midpoint of a line segment are found by averaging.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

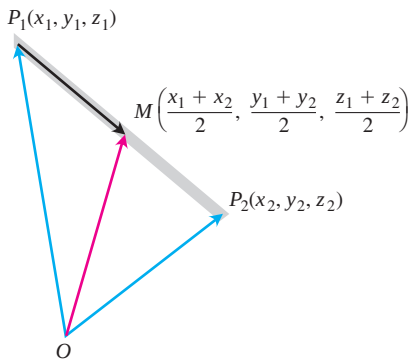


FIGURE 12.17 The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2 .

To see why, observe (Figure 12.17) that

$$\begin{aligned} \vec{OM} &= \vec{OP}_1 + \frac{1}{2}(\vec{P}_1\vec{P}_2) = \vec{OP}_1 + \frac{1}{2}(\vec{OP}_2 - \vec{OP}_1) \\ &= \frac{1}{2}(\vec{OP}_1 + \vec{OP}_2) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}. \end{aligned}$$

EXAMPLE 8 Finding Midpoints

The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2} \right) = (5, 1, 2).$$



EXERCISES 12.2

Vectors in the Plane

In Exercises 1–8, let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the (a) component form and (b) magnitude (length) of the vector.

1. $3\mathbf{u}$	2. $-2\mathbf{v}$
3. $\mathbf{u} + \mathbf{v}$	4. $\mathbf{u} - \mathbf{v}$
5. $2\mathbf{u} - 3\mathbf{v}$	6. $-2\mathbf{u} + 5\mathbf{v}$
7. $\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v}$	8. $-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v}$

In Exercises 9–16, find the component form of the vector.

- The vector $\overrightarrow{PQ}$, where $P = (1, 3)$ and $Q = (2, -1)$
- The vector $\overrightarrow{OP}$ where O is the origin and P is the midpoint of segment RS , where $R = (2, -1)$ and $S = (-4, 3)$
- The vector from the point $A = (2, 3)$ to the origin
- The sum of $\overrightarrow{AB}$ and $\overrightarrow{CD}$, where $A = (1, -1)$, $B = (2, 0)$, $C = (-1, 3)$, and $D = (-2, 2)$

- The unit vector that makes an angle $\theta = 2\pi/3$ with the positive x -axis
- The unit vector that makes an angle $\theta = -3\pi/4$ with the positive x -axis
- The unit vector obtained by rotating the vector $\langle 0, 1 \rangle$ 120° counterclockwise about the origin
- The unit vector obtained by rotating the vector $\langle 1, 0 \rangle$ 135° counterclockwise about the origin

Vectors in Space

In Exercises 17–22, express each vector in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

- $\overrightarrow{P_1P_2}$ if P_1 is the point $(5, 7, -1)$ and P_2 is the point $(2, 9, -2)$
- $\overrightarrow{P_1P_2}$ if P_1 is the point $(1, 2, 0)$ and P_2 is the point $(-3, 0, 5)$
- $\overrightarrow{AB}$ if A is the point $(-7, -8, 1)$ and B is the point $(-10, 8, 1)$
- $\overrightarrow{AB}$ if A is the point $(1, 0, 3)$ and B is the point $(-1, 4, 5)$

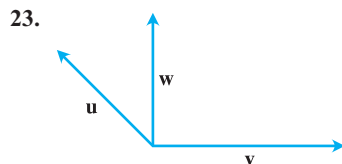




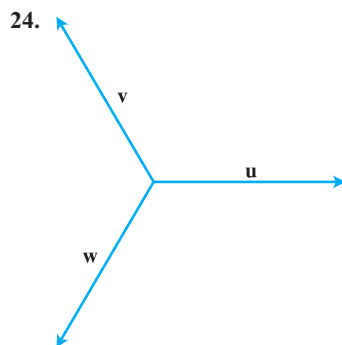
21. $5\mathbf{u} - \mathbf{v}$ if $\mathbf{u} = \langle 1, 1, -1 \rangle$ and $\mathbf{v} = \langle 2, 0, 3 \rangle$
 22. $-2\mathbf{u} + 3\mathbf{v}$ if $\mathbf{u} = \langle -1, 0, 2 \rangle$ and $\mathbf{v} = \langle 1, 1, 1 \rangle$

Geometry and Calculation

In Exercises 23 and 24, copy vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ head to tail as needed to sketch the indicated vector.



- a. $\mathbf{u} + \mathbf{v}$ b. $\mathbf{u} + \mathbf{v} + \mathbf{w}$
 c. $\mathbf{u} - \mathbf{v}$ d. $\mathbf{u} - \mathbf{w}$



- a. $\mathbf{u} - \mathbf{v}$ b. $\mathbf{u} - \mathbf{v} + \mathbf{w}$
 c. $2\mathbf{u} - \mathbf{v}$ d. $\mathbf{u} + \mathbf{v} + \mathbf{w}$

Length and Direction

In Exercises 25–30, express each vector as a product of its length and direction.



25. $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ 26. $9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
 27. $5\mathbf{k}$ 28. $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$
 29. $\frac{1}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$ 30. $\frac{\mathbf{i}}{\sqrt{3}} + \frac{\mathbf{j}}{\sqrt{3}} + \frac{\mathbf{k}}{\sqrt{3}}$

31. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

Length	Direction
a. 2	$\mathbf{i}$
b. $\sqrt{3}$	$-\mathbf{k}$
c. $\frac{1}{2}$	$\frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$
d. 7	$\frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$

32. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

Length	Direction
a. 7	$-\mathbf{j}$
b. $\sqrt{2}$	$-\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{k}$
c. $\frac{13}{12}$	$\frac{3}{13}\mathbf{i} - \frac{4}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$
d. $a > 0$	$\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$

33. Find a vector of magnitude 7 in the direction of $\mathbf{v} = 12\mathbf{i} - 5\mathbf{k}$.
 34. Find a vector of magnitude 3 in the direction opposite to the direction of $\mathbf{v} = (1/2)\mathbf{i} - (1/2)\mathbf{j} - (1/2)\mathbf{k}$.



Vectors Determined by Points; Midpoints

In Exercises 35–38, find

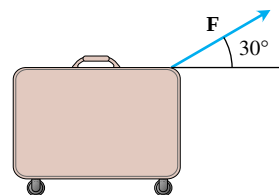
- a. the direction of $\overrightarrow{P_1P_2}$ and
 b. the midpoint of line segment P_1P_2 .

35. $P_1(-1, 1, 5)$ $P_2(2, 5, 0)$
 36. $P_1(1, 4, 5)$ $P_2(4, -2, 7)$
 37. $P_1(3, 4, 5)$ $P_2(2, 3, 4)$
 38. $P_1(0, 0, 0)$ $P_2(2, -2, -2)$
 39. If $\overrightarrow{AB} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and B is the point $(5, 1, 3)$, find A .
 40. If $\overrightarrow{AB} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$ and A is the point $(-2, -3, 6)$, find B .

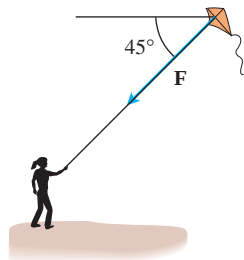


Theory and Applications

41. **Linear combination** Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$, and $\mathbf{w} = \mathbf{i} - \mathbf{j}$. Find scalars a and b such that $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$.
 42. **Linear combination** Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$, and $\mathbf{w} = \mathbf{i} + \mathbf{j}$. Write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1$ is parallel to $\mathbf{v}$ and $\mathbf{u}_2$ is parallel to $\mathbf{w}$. (See Exercise 41.)
 43. **Force vector** You are pulling on a suitcase with a force $\mathbf{F}$ (pictured here) whose magnitude is $|\mathbf{F}| = 10$ lb. Find the $\mathbf{i}$ - and $\mathbf{j}$ -components of $\mathbf{F}$.

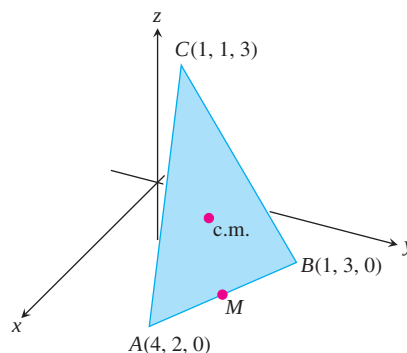


44. **Force vector** A kite string exerts a 12-lb pull ($|\mathbf{F}| = 12$) on a kite and makes a 45° angle with the horizontal. Find the horizontal and vertical components of $\mathbf{F}$.



- 45. Velocity** An airplane is flying in the direction 25° west of north at 800 km/h. Find the component form of the velocity of the airplane, assuming that the positive x -axis represents due east and the positive y -axis represents due north.
- 46. Velocity** An airplane is flying in the direction 10° east of south at 600 km/h. Find the component form of the velocity of the airplane, assuming that the positive x -axis represents due east and the positive y -axis represents due north.
- 47. Location** A bird flies from its nest 5 km in the direction 60° north of east, where it stops to rest on a tree. It then flies 10 km in the direction due southeast and lands atop a telephone pole. Place an xy -coordinate system so that the origin is the bird's nest, the x -axis points east, and the y -axis points north.
- At what point is the tree located?
 - At what point is the telephone pole?
- 48.** Use similar triangles to find the coordinates of the point Q that divides the segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ into two lengths whose ratio is $p/q = r$.
- 49. Medians of a triangle** Suppose that A , B , and C are the corner points of the thin triangular plate of constant density shown here.
- Find the vector from C to the midpoint M of side AB .
 - Find the vector from C to the point that lies two-thirds of the way from C to M on the median CM .

- c. Find the coordinates of the point in which the medians of $\triangle ABC$ intersect. According to Exercise 29, Section 6.4, this point is the plate's center of mass.



- 50.** Find the vector from the origin to the point of intersection of the medians of the triangle whose vertices are $A(1, -1, 2)$, $B(2, 1, 3)$, and $C(-1, 2, -1)$.
- 51.** Let $ABCD$ be a general, not necessarily planar, quadrilateral in space. Show that the two segments joining the midpoints of opposite sides of $ABCD$ bisect each other. (*Hint*: Show that the segments have the same midpoint.)
- 52.** Vectors are drawn from the center of a regular n -sided polygon in the plane to the vertices of the polygon. Show that the sum of the vectors is zero. (*Hint*: What happens to the sum if you rotate the polygon about its center?)
- 53.** Suppose that A , B , and C are vertices of a triangle and that a , b , and c are, respectively, the midpoints of the opposite sides. Show that $\vec{Aa} + \vec{Bb} + \vec{Cc} = \vec{0}$.
- 54. Unit vectors in the plane** Show that a unit vector in the plane can be expressed as $\mathbf{u} = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}$, obtained by rotating $\mathbf{i}$ through an angle θ in the counterclockwise direction. Explain why this form gives *every* unit vector in the plane.

12.3

The Dot Product

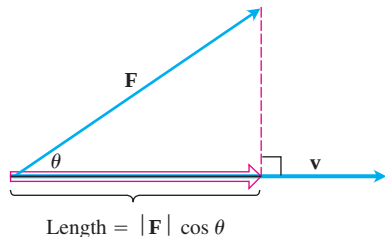


FIGURE 12.18 The magnitude of the force $\mathbf{F}$ in the direction of vector $\mathbf{v}$ is the length $|\mathbf{F}| \cos \theta$ of the projection of $\mathbf{F}$ onto $\mathbf{v}$.

If a force $\mathbf{F}$ is applied to a particle moving along a path, we often need to know the magnitude of the force in the direction of motion. If $\mathbf{v}$ is parallel to the tangent line to the path at the point where $\mathbf{F}$ is applied, then we want the magnitude of $\mathbf{F}$ in the direction of $\mathbf{v}$. Figure 12.18 shows that the scalar quantity we seek is the length $|\mathbf{F}| \cos \theta$, where θ is the angle between the two vectors $\mathbf{F}$ and $\mathbf{v}$.

In this section, we show how to calculate easily the angle between two vectors directly from their components. A key part of the calculation is an expression called the *dot product*. Dot products are also called *inner* or *scalar* products because the product results in a scalar, not a vector. After investigating the dot product, we apply it to finding the projection of one vector onto another (as displayed in Figure 12.18) and to finding the work done by a constant force acting through a displacement.



Project



Project

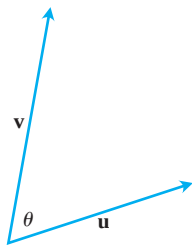


FIGURE 12.19 The angle between $\mathbf{u}$ and $\mathbf{v}$.

Angle Between Vectors

When two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are placed so their initial points coincide, they form an angle θ of measure $0 \leq \theta \leq \pi$ (Figure 12.19). If the vectors do not lie along the same line, the angle θ is measured in the plane containing both of them. If they do lie along the same line, the angle between them is 0 if they point in the same direction, and π if they point in opposite directions. The angle θ is the **angle between $\mathbf{u}$ and $\mathbf{v}$** . Theorem 1 gives a formula to determine this angle.

THEOREM 1 Angle Between Two Vectors

The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1} \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}| |\mathbf{v}|} \right).$$

Before proving Theorem 1 (which is a consequence of the law of cosines), let's focus attention on the expression $u_1 v_1 + u_2 v_2 + u_3 v_3$ in the calculation for θ .

DEFINITION Dot Product

The **dot product** $\mathbf{u} \cdot \mathbf{v}$ (“ $\mathbf{u}$ dot $\mathbf{v}$ ”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

EXAMPLE 1 Finding Dot Products

$$\begin{aligned} \text{(a)} \quad \langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle &= (1)(-6) + (-2)(2) + (-1)(-3) \\ &= -6 - 4 + 3 = -7 \end{aligned}$$

$$\text{(b)} \quad \left(\frac{1}{2} \mathbf{i} + 3 \mathbf{j} + \mathbf{k} \right) \cdot (4 \mathbf{i} - \mathbf{j} + 2 \mathbf{k}) = \left(\frac{1}{2} \right)(4) + (3)(-1) + (1)(2) = 1$$

The dot product of a pair of two-dimensional vectors is defined in a similar fashion:

$$\langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2.$$

Proof of Theorem 1 Applying the law of cosines (Equation (6), Section 1.6) to the triangle in Figure 12.20, we find that

$$|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos \theta \quad \text{Law of cosines}$$

$$2|\mathbf{u}||\mathbf{v}| \cos \theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2.$$

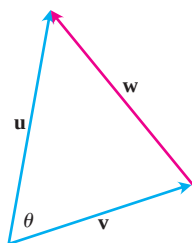


FIGURE 12.20 The parallelogram law of addition of vectors gives $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

Because $\mathbf{w} = \mathbf{u} - \mathbf{v}$, the component form of $\mathbf{w}$ is $\langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$. So

$$|\mathbf{u}|^2 = (\sqrt{u_1^2 + u_2^2 + u_3^2})^2 = u_1^2 + u_2^2 + u_3^2$$

$$|\mathbf{v}|^2 = (\sqrt{v_1^2 + v_2^2 + v_3^2})^2 = v_1^2 + v_2^2 + v_3^2$$

$$\begin{aligned} |\mathbf{w}|^2 &= (\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2})^2 \\ &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\ &= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2 \end{aligned}$$

and

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Therefore,

$$2|\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3)$$

$$|\mathbf{u}||\mathbf{v}|\cos\theta = u_1v_1 + u_2v_2 + u_3v_3$$

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}$$

So

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right) \quad \blacksquare$$

With the notation of the dot product, the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ can be written as

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right).$$



EXAMPLE 2 Finding the Angle Between Two Vectors in Space

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution We use the formula above:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

$$= \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians.} \quad \blacksquare$$

The angle formula applies to two-dimensional vectors as well.

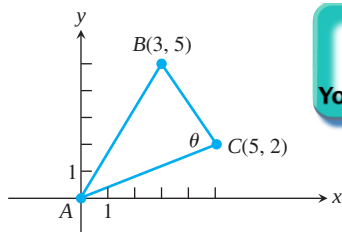


FIGURE 12.21 The triangle in Example 3.



EXAMPLE 3 Finding an Angle of a Triangle

Find the angle θ in the triangle ABC determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$ (Figure 12.21).

Solution The angle θ is the angle between the vectors $\vec{CA}$ and $\vec{CB}$. The component forms of these two vectors are

$$\vec{CA} = \langle -5, -2 \rangle \quad \text{and} \quad \vec{CB} = \langle -2, 3 \rangle.$$

First we calculate the dot product and magnitudes of these two vectors.

$$\vec{CA} \cdot \vec{CB} = (-5)(-2) + (-2)(3) = 4$$

$$|\vec{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

$$|\vec{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

Then applying the angle formula, we have

$$\begin{aligned} \theta &= \cos^{-1} \left(\frac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}| |\vec{CB}|} \right) \\ &= \cos^{-1} \left(\frac{4}{(\sqrt{29})(\sqrt{13})} \right) \\ &\approx 78.1^\circ \quad \text{or} \quad 1.36 \text{ radians.} \end{aligned}$$

Perpendicular (Orthogonal) Vectors

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular or **orthogonal** if the angle between them is $\pi/2$. For such vectors, we have $\mathbf{u} \cdot \mathbf{v} = 0$ because $\cos(\pi/2) = 0$. The converse is also true. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors with $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta = 0$, then $\cos \theta = 0$ and $\theta = \cos^{-1} 0 = \pi/2$.

DEFINITION Orthogonal Vectors

Vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

EXAMPLE 4 Applying the Definition of Orthogonality

(a) $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

(b) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0$.

(c) $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ since

$$\begin{aligned} \mathbf{0} \cdot \mathbf{u} &= \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) \\ &= 0. \end{aligned}$$

Dot Product Properties and Vector Projections

The dot product obeys many of the laws that hold for ordinary products of real numbers (scalars).

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0$.

HISTORICAL BIOGRAPHY

Carl Friedrich Gauss
(1777–1855)

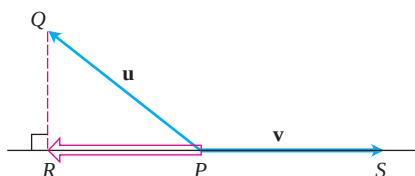
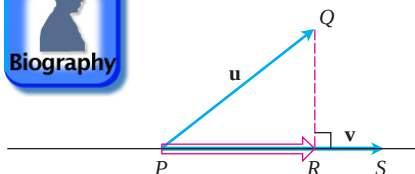


FIGURE 12.22 The vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

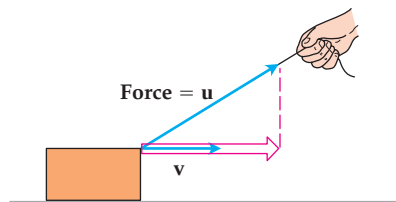


FIGURE 12.23 If we pull on the box with force $\mathbf{u}$, the effective force moving the box forward in the direction $\mathbf{v}$ is the projection of $\mathbf{u}$ onto $\mathbf{v}$.

Proofs of Properties 1 and 3 The properties are easy to prove using the definition. For instance, here are the proofs of Properties 1 and 3.

1. $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \mathbf{v} \cdot \mathbf{u}$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$
 $= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3)$
 $= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 + u_3 v_3 + u_3 w_3$
 $= (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3)$
 $= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ ■

We now return to the problem of projecting one vector onto another, posed in the opening to this section. The **vector projection** of $\mathbf{u} = \overrightarrow{PQ}$ onto a nonzero vector $\mathbf{v} = \overrightarrow{PS}$ (Figure 12.22) is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS . The notation for this vector is

$\text{proj}_{\mathbf{v}} \mathbf{u}$ (“the vector projection of $\mathbf{u}$ onto $\mathbf{v}$ ”).

If $\mathbf{u}$ represents a force, then $\text{proj}_{\mathbf{v}} \mathbf{u}$ represents the effective force in the direction of $\mathbf{v}$ (Figure 12.23).

If the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is acute, $\text{proj}_{\mathbf{v}} \mathbf{u}$ has length $|\mathbf{u}| \cos \theta$ and direction $\mathbf{v}/|\mathbf{v}|$ (Figure 12.24). If θ is obtuse, $\cos \theta < 0$ and $\text{proj}_{\mathbf{v}} \mathbf{u}$ has length $-|\mathbf{u}| \cos \theta$ and direction $-\mathbf{v}/|\mathbf{v}|$. In both cases,

$$\begin{aligned} \text{proj}_{\mathbf{v}} \mathbf{u} &= (|\mathbf{u}| \cos \theta) \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} & |\mathbf{u}| \cos \theta &= \frac{|\mathbf{u}| |\mathbf{v}| \cos \theta}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}. \end{aligned}$$

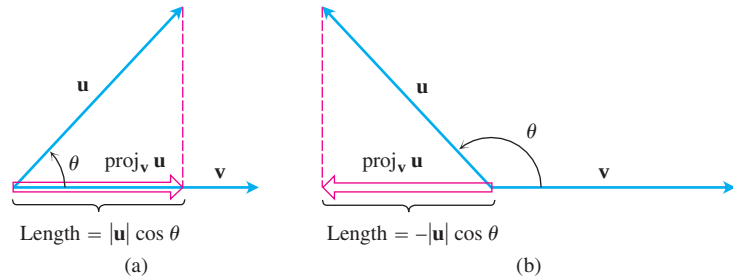


FIGURE 12.24 The length of $\text{proj}_{\mathbf{v}} \mathbf{u}$ is (a) $|\mathbf{u}| \cos \theta$ if $\cos \theta \geq 0$ and (b) $-|\mathbf{u}| \cos \theta$ if $\cos \theta < 0$.

The number $|\mathbf{u}| \cos \theta$ is called the **scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$** . To summarize,

Vector projection of $\mathbf{u}$ onto $\mathbf{v}$:

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \quad (1)$$

Scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$|\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \quad (2)$$

Note that both the vector projection of $\mathbf{u}$ onto $\mathbf{v}$ and the scalar component of $\mathbf{u}$ onto $\mathbf{v}$ depend only on the direction of the vector $\mathbf{v}$ and not its length (because we dot $\mathbf{u}$ with $\mathbf{v}/|\mathbf{v}|$, which is the direction of $\mathbf{v}$).

EXAMPLE 5 Finding the Vector Projection

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$.

Solution We find $\text{proj}_{\mathbf{v}} \mathbf{u}$ from Equation (1):

$$\begin{aligned} \text{proj}_{\mathbf{v}} \mathbf{u} &= \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9} (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9} \mathbf{i} + \frac{8}{9} \mathbf{j} + \frac{8}{9} \mathbf{k}. \end{aligned}$$

We find the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ from Equation (2):

$$\begin{aligned} |\mathbf{u}| \cos \theta &= \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{1}{3} \mathbf{i} - \frac{2}{3} \mathbf{j} - \frac{2}{3} \mathbf{k} \right) \\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}. \end{aligned}$$

Equations (1) and (2) also apply to two-dimensional vectors. ■

EXAMPLE 6 Finding Vector Projections and Scalar Components

Find the vector projection of a force $\mathbf{F} = 5\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{v} = \mathbf{i} - 3\mathbf{j}$ and the scalar component of $\mathbf{F}$ in the direction of $\mathbf{v}$.

Solution The vector projection is

$$\begin{aligned}\text{proj}_{\mathbf{v}} \mathbf{F} &= \left(\frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \\ &= \frac{5 - 6}{1 + 9} (\mathbf{i} - 3\mathbf{j}) = -\frac{1}{10} (\mathbf{i} - 3\mathbf{j}) \\ &= -\frac{1}{10} \mathbf{i} + \frac{3}{10} \mathbf{j}.\end{aligned}$$

The scalar component of $\mathbf{F}$ in the direction of $\mathbf{v}$ is

$$|\mathbf{F}| \cos \theta = \frac{\mathbf{F} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{5 - 6}{\sqrt{1 + 9}} = -\frac{1}{\sqrt{10}}.$$

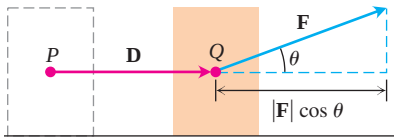
Work

FIGURE 12.25 The work done by a constant force $\mathbf{F}$ during a displacement $\mathbf{D}$ is $(|\mathbf{F}| \cos \theta)|\mathbf{D}|$.

In Chapter 6, we calculated the work done by a constant force of magnitude F in moving an object through a distance d as $W = Fd$. That formula holds only if the force is directed along the line of motion. If a force $\mathbf{F}$ moving an object through a displacement $\mathbf{D} = \overrightarrow{PQ}$ has some other direction, the work is performed by the component of $\mathbf{F}$ in the direction of $\mathbf{D}$. If θ is the angle between $\mathbf{F}$ and $\mathbf{D}$ (Figure 12.25), then

$$\begin{aligned}\text{Work} &= (\text{scalar component of } \mathbf{F} \text{ in the direction of } \mathbf{D})(\text{length of } \mathbf{D}) \\ &= (|\mathbf{F}| \cos \theta)|\mathbf{D}| \\ &= \mathbf{F} \cdot \mathbf{D}.\end{aligned}$$

DEFINITION Work by Constant Force

The **work** done by a constant force $\mathbf{F}$ acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}| |\mathbf{D}| \cos \theta,$$

where θ is the angle between $\mathbf{F}$ and $\mathbf{D}$.

EXAMPLE 7 Applying the Definition of Work

If $|\mathbf{F}| = 40$ N (newtons), $|\mathbf{D}| = 3$ m, and $\theta = 60^\circ$, the work done by $\mathbf{F}$ in acting from P to Q is

$$\begin{aligned}\text{Work} &= |\mathbf{F}| |\mathbf{D}| \cos \theta && \text{Definition} \\ &= (40)(3) \cos 60^\circ && \text{Given values} \\ &= (120)(1/2) \\ &= 60 \text{ J (joules)}.\end{aligned}$$

We encounter more challenging work problems in Chapter 16 when we learn to find the work done by a variable force along a *path* in space.

Writing a Vector as a Sum of Orthogonal Vectors

We know one way to write a vector $\mathbf{u} = \langle u_1, u_2 \rangle$ or $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ as a sum of two orthogonal vectors:

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} \quad \text{or} \quad \mathbf{u} = u_1\mathbf{i} + (u_2\mathbf{j} + u_3\mathbf{k})$$

(since $\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$).

Sometimes, however, it is more informative to express $\mathbf{u}$ as a different sum. In mechanics, for instance, we often need to write a vector $\mathbf{u}$ as a sum of a vector parallel to a given vector $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$. As an example, in studying the motion of a particle moving along a path in the plane (or space), it is desirable to know the components of the acceleration vector in the direction of the tangent to the path (at a point) and of the normal to the path. (These *tangential* and *normal components* of acceleration are investigated in Section 13.4.) The acceleration vector can then be expressed as the sum of its (vector) tangential and normal components (which reflect important geometric properties about the nature of the path itself, such as *curvature*). Velocity and acceleration vectors are studied in the next chapter.

Generally, for vectors $\mathbf{u}$ and $\mathbf{v}$, it is easy to see from Figure 12.26 that the vector

$$\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$$

is orthogonal to the projection vector $\text{proj}_{\mathbf{v}} \mathbf{u}$ (which has the same direction as $\mathbf{v}$). The following calculation verifies this observation:

$$\begin{aligned} (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \cdot \text{proj}_{\mathbf{v}} \mathbf{u} &= \left(\mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \right) \cdot \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} && \text{Equation (1)} \\ &= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) (\mathbf{u} \cdot \mathbf{v}) - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right)^2 (\mathbf{v} \cdot \mathbf{v}) && \text{Dot product properties 2 and 3} \\ &= \frac{(\mathbf{u} \cdot \mathbf{v})^2}{|\mathbf{v}|^2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{|\mathbf{v}|^2} && \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \text{ cancels} \\ &= 0. \end{aligned}$$

So the equation

$$\mathbf{u} = \text{proj}_{\mathbf{v}} \mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u})$$

expresses $\mathbf{u}$ as a sum of orthogonal vectors.

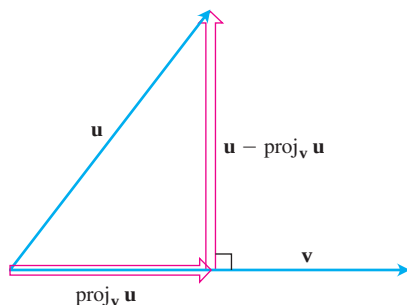


FIGURE 12.26 Writing $\mathbf{u}$ as the sum of vectors parallel and orthogonal to $\mathbf{v}$.



How to Write $\mathbf{u}$ as a Vector Parallel to $\mathbf{v}$ Plus a Vector Orthogonal to $\mathbf{v}$

$$\begin{aligned} \mathbf{u} &= \text{proj}_{\mathbf{v}} \mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) \\ &= \underbrace{\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}}_{\text{Parallel to } \mathbf{v}} + \underbrace{\left(\mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} \right)}_{\text{Orthogonal to } \mathbf{v}} \end{aligned}$$

EXAMPLE 8 Force on a Spacecraft

A force $\mathbf{F} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is applied to a spacecraft with velocity vector $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$. Express $\mathbf{F}$ as a sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

Solution

$$\begin{aligned}
 \mathbf{F} &= \text{proj}_{\mathbf{v}} \mathbf{F} + (\mathbf{F} - \text{proj}_{\mathbf{v}} \mathbf{F}) \\
 &= \frac{\mathbf{F} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} + \left(\mathbf{F} - \frac{\mathbf{F} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \right) \\
 &= \left(\frac{6 - 1}{9 + 1} \right) \mathbf{v} + \left(\mathbf{F} - \left(\frac{6 - 1}{9 + 1} \right) \mathbf{v} \right) \\
 &= \frac{5}{10} (3\mathbf{i} - \mathbf{j}) + \left(2\mathbf{i} + \mathbf{j} - 3\mathbf{k} - \frac{5}{10} (3\mathbf{i} - \mathbf{j}) \right) \\
 &= \left(\frac{3}{2} \mathbf{i} - \frac{1}{2} \mathbf{j} \right) + \left(\frac{1}{2} \mathbf{i} + \frac{3}{2} \mathbf{j} - 3\mathbf{k} \right).
 \end{aligned}$$

The force $(3/2)\mathbf{i} - (1/2)\mathbf{j}$ is the effective force parallel to the velocity $\mathbf{v}$. The force $(1/2)\mathbf{i} + (3/2)\mathbf{j} - 3\mathbf{k}$ is orthogonal to $\mathbf{v}$. To check that this vector is orthogonal to $\mathbf{v}$, we find the dot product:

$$\left(\frac{1}{2} \mathbf{i} + \frac{3}{2} \mathbf{j} - 3\mathbf{k} \right) \cdot (3\mathbf{i} - \mathbf{j}) = \frac{3}{2} - \frac{3}{2} = 0.$$



EXERCISES 12.3

Dot Product and Projections

In Exercises 1–8, find

- $\mathbf{v} \cdot \mathbf{u}$, $|\mathbf{v}|$, $|\mathbf{u}|$
- the cosine of the angle between $\mathbf{v}$ and $\mathbf{u}$
- the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$
- the vector $\text{proj}_{\mathbf{u}} \mathbf{v}$.

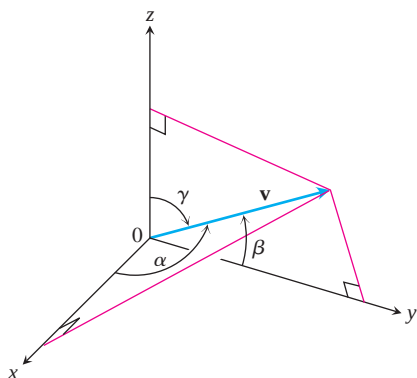
- $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \sqrt{5}\mathbf{k}$, $\mathbf{u} = -2\mathbf{i} + 4\mathbf{j} - \sqrt{5}\mathbf{k}$
- $\mathbf{v} = (3/5)\mathbf{i} + (4/5)\mathbf{j}$, $\mathbf{u} = 5\mathbf{i} + 12\mathbf{j}$
- $\mathbf{v} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$, $\mathbf{u} = 3\mathbf{j} + 4\mathbf{k}$
- $\mathbf{v} = 2\mathbf{i} + 10\mathbf{j} - 11\mathbf{k}$, $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
- $\mathbf{v} = 5\mathbf{j} - 3\mathbf{k}$, $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
- $\mathbf{v} = -\mathbf{i} + \mathbf{j}$, $\mathbf{u} = \sqrt{2}\mathbf{i} + \sqrt{3}\mathbf{j} + 2\mathbf{k}$
- $\mathbf{v} = 5\mathbf{i} + \mathbf{j}$, $\mathbf{u} = 2\mathbf{i} + \sqrt{17}\mathbf{j}$
- $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$, $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right\rangle$

T Angles Between Vectors

Find the angles between the vectors in Exercises 9–12 to the nearest hundredth of a radian.

- $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$
- $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{k}$
- $\mathbf{u} = \sqrt{3}\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = \sqrt{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
- $\mathbf{u} = \mathbf{i} + \sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$
- Triangle** Find the measures of the angles of the triangle whose vertices are $A = (-1, 0)$, $B = (2, 1)$, and $C = (1, -2)$.
- Rectangle** Find the measures of the angles between the diagonals of the rectangle whose vertices are $A = (1, 0)$, $B = (0, 3)$, $C = (3, 4)$, and $D = (4, 1)$.
- Direction angles and direction cosines** The *direction angles* α , β , and γ of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ are defined as follows:
 α is the angle between $\mathbf{v}$ and the positive x -axis ($0 \leq \alpha \leq \pi$)
 β is the angle between $\mathbf{v}$ and the positive y -axis ($0 \leq \beta \leq \pi$)
 γ is the angle between $\mathbf{v}$ and the positive z -axis ($0 \leq \gamma \leq \pi$).





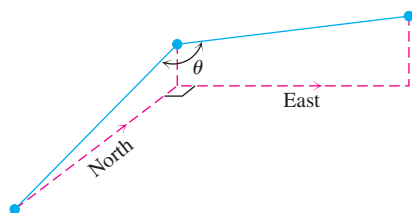
a. Show that

$$\cos \alpha = \frac{a}{|\mathbf{v}|}, \quad \cos \beta = \frac{b}{|\mathbf{v}|}, \quad \cos \gamma = \frac{c}{|\mathbf{v}|},$$

and $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. These cosines are called the *direction cosines* of $\mathbf{v}$.

b. **Unit vectors are built from direction cosines** Show that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a unit vector, then a , b , and c are the direction cosines of $\mathbf{v}$.

16. **Water main construction** A water main is to be constructed with a 20% grade in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east.



Decomposing Vectors

In Exercises 17–19, write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

17. $\mathbf{u} = 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$

18. $\mathbf{u} = \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$

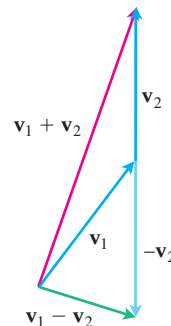
19. $\mathbf{u} = 8\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

20. **Sum of vectors** $\mathbf{u} = \mathbf{i} + (\mathbf{j} + \mathbf{k})$ is already the sum of a vector parallel to $\mathbf{i}$ and a vector orthogonal to $\mathbf{i}$. If you use $\mathbf{v} = \mathbf{i}$, in the decomposition $\mathbf{u} = \text{proj}_{\mathbf{v}} \mathbf{u} + (\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u})$, do you get $\text{proj}_{\mathbf{v}} \mathbf{u} = \mathbf{i}$ and $(\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}) = \mathbf{j} + \mathbf{k}$? Try it and find out.

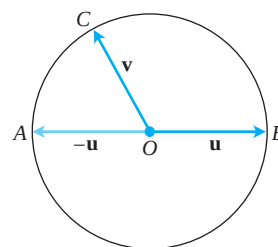
Geometry and Examples

21. **Sums and differences** In the accompanying figure, it looks as if $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ are orthogonal. Is this mere coincidence, or are there circumstances under which we may expect the sum of

two vectors to be orthogonal to their difference? Give reasons for your answer.



22. **Orthogonality on a circle** Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B . Show that $\overline{CA}$ and $\overline{CB}$ are orthogonal.

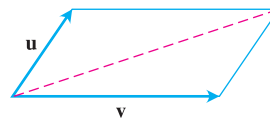


23. **Diagonals of a rhombus** Show that the diagonals of a rhombus (parallelogram with sides of equal length) are perpendicular.

24. **Perpendicular diagonals** Show that squares are the only rectangles with perpendicular diagonals.

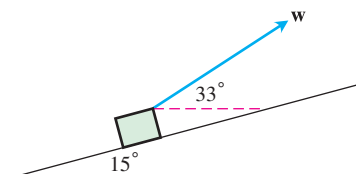
25. **When parallelograms are rectangles** Prove that a parallelogram is a rectangle if and only if its diagonals are equal in length. (This fact is often exploited by carpenters.)

26. **Diagonal of parallelogram** Show that the indicated diagonal of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$ if $|\mathbf{u}| = |\mathbf{v}|$.



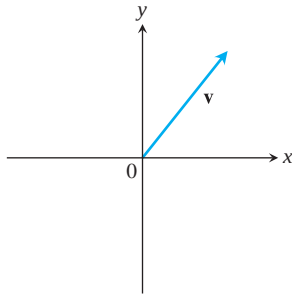
27. **Projectile motion** A gun with muzzle velocity of 1200 ft/sec is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

28. **Inclined plane** Suppose that a box is being towed up an inclined plane as shown in the figure. Find the force $\mathbf{w}$ needed to make the component of the force parallel to the inclined plane equal to 2.5 lb.



Theory and Examples

29. **a. Cauchy-Schwarz inequality** Use the fact that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ to show that the inequality $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$ holds for any vectors $\mathbf{u}$ and $\mathbf{v}$.
- b.** Under what circumstances, if any, does $|\mathbf{u} \cdot \mathbf{v}|$ equal $|\mathbf{u}||\mathbf{v}|$? Give reasons for your answer.
30. Copy the axes and vector shown here. Then shade in the points (x, y) for which $(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{v} \leq 0$. Justify your answer.



31. **Orthogonal unit vectors** If $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal unit vectors and $\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2$, find $\mathbf{v} \cdot \mathbf{u}_1$.
32. **Cancellation in dot products** In real-number multiplication, if $uv_1 = uv_2$ and $u \neq 0$, we can cancel the u and conclude that $v_1 = v_2$. Does the same rule hold for the dot product: If $\mathbf{u} \cdot \mathbf{v}_1 = \mathbf{u} \cdot \mathbf{v}_2$ and $\mathbf{u} \neq \mathbf{0}$, can you conclude that $\mathbf{v}_1 = \mathbf{v}_2$? Give reasons for your answer.

Equations for Lines in the Plane

33. **Line perpendicular to a vector** Show that the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is perpendicular to the line $ax + by = c$ by establishing that the slope of $\mathbf{v}$ is the negative reciprocal of the slope of the given line.
34. **Line parallel to a vector** Show that the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is parallel to the line $bx - ay = c$ by establishing that the slope of the line segment representing $\mathbf{v}$ is the same as the slope of the given line.

In Exercises 35–38, use the result of Exercise 33 to find an equation for the line through P perpendicular to $\mathbf{v}$. Then sketch the line. Include $\mathbf{v}$ in your sketch as a vector starting at the origin.

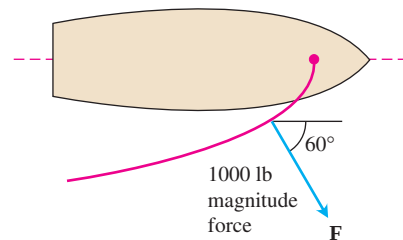
35. $P(2, 1)$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$
 36. $P(-1, 2)$, $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$
 37. $P(-2, -7)$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$
 38. $P(11, 10)$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

In Exercises 39–42, use the result of Exercise 34 to find an equation for the line through P parallel to $\mathbf{v}$. Then sketch the line. Include $\mathbf{v}$ in your sketch as a vector starting at the origin.

39. $P(-2, 1)$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$ 40. $P(0, -2)$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$
 41. $P(1, 2)$, $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ 42. $P(1, 3)$, $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

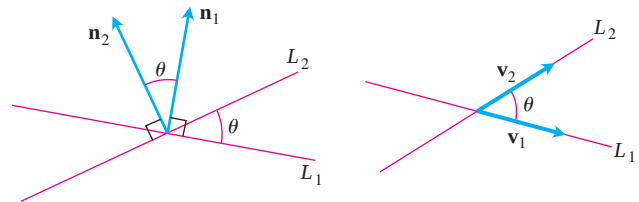
Work

43. **Work along a line** Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point $(1, 1)$ (distance in meters).
44. **Locomotive** The union Pacific's *Big Boy* locomotive could pull 6000-ton trains with a tractive effort (pull) of 602,148 N (135,375 lb). At this level of effort, about how much work did *Big Boy* do on the (approximately straight) 605-km journey from San Francisco to Los Angeles?
45. **Inclined plane** How much work does it take to slide a crate 20 m along a loading dock by pulling on it with a 200 N force at an angle of 30° from the horizontal?
46. **Sailboat** The wind passing over a boat's sail exerted a 1000-lb magnitude force $\mathbf{F}$ as shown here. How much work did the wind perform in moving the boat forward 1 mi? Answer in foot-pounds.



Angles Between Lines in the Plane

The acute angle between intersecting lines that do not cross at right angles is the same as the angle determined by vectors normal to the lines or by the vectors parallel to the lines.



Use this fact and the results of Exercise 33 or 34 to find the acute angles between the lines in Exercises 47–52.

47. $3x + y = 5$, $2x - y = 4$
 48. $y = \sqrt{3}x - 1$, $y = -\sqrt{3}x + 2$
 49. $\sqrt{3}x - y = -2$, $x - \sqrt{3}y = 1$
 50. $x + \sqrt{3}y = 1$, $(1 - \sqrt{3})x + (1 + \sqrt{3})y = 8$
 51. $3x - 4y = 3$, $x - y = 7$
 52. $12x + 5y = 1$, $2x - 2y = 3$

Angles Between Differentiable Curves

The angles between two differentiable curves at a point of intersection are the angles between the curves' tangent lines at these points. Find

the angles between the curves in Exercises 53–56. Note that if $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is a vector in the plane, then the vector has slope b/a provided $a \neq 0$.

53. $y = (3/2) - x^2$, $y = x^2$ (two points of intersection)

54. $x = (3/4) - y^2$, $x = y^2 - (3/4)$ (two points of intersection)

55. $y = x^3$, $x = y^2$ (two points of intersection)

56. $y = -x^2$, $y = \sqrt{x}$ (two points of intersection)

12.4

The Cross Product

In studying lines in the plane, when we needed to describe how a line was tilting, we used the notions of slope and angle of inclination. In space, we want a way to describe how a *plane* is tilting. We accomplish this by multiplying two vectors in the plane together to get a third vector perpendicular to the plane. The direction of this third vector tells us the “inclination” of the plane. The product we use to multiply the vectors together is the *vector* or *cross product*, the second of the two vector multiplication methods we study in calculus.

Cross products are widely used to describe the effects of forces in studies of electricity, magnetism, fluid flows, and orbital mechanics. This section presents the mathematical properties that account for the use of cross products in these fields.

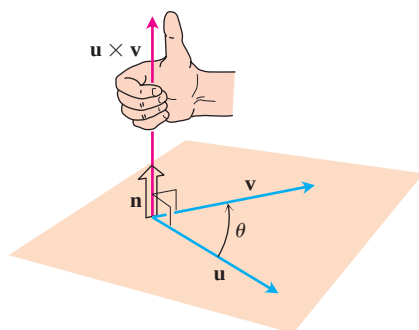


FIGURE 12.27 The construction of $\mathbf{u} \times \mathbf{v}$.

The Cross Product of Two Vectors in Space

We start with two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in space. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, they determine a plane. We select a unit vector $\mathbf{n}$ perpendicular to the plane by the **right-hand rule**. This means that we choose $\mathbf{n}$ to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from $\mathbf{u}$ to $\mathbf{v}$ (Figure 12.27). Then the **cross product** $\mathbf{u} \times \mathbf{v}$ (“ $\mathbf{u}$ cross $\mathbf{v}$ ”) is the *vector* defined as follows.

DEFINITION Cross Product

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}| |\mathbf{v}| \sin \theta) \mathbf{n}$$

Unlike the dot product, the cross product is a vector. For this reason it’s also called the **vector product** of $\mathbf{u}$ and $\mathbf{v}$, and applies *only* to vectors in space. The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ because it is a scalar multiple of $\mathbf{n}$.

Since the sines of 0 and π are both zero, it makes sense to define the cross product of two parallel nonzero vectors to be $\mathbf{0}$. If one or both of $\mathbf{u}$ and $\mathbf{v}$ are zero, we also define $\mathbf{u} \times \mathbf{v}$ to be zero. This way, the cross product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is zero if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel or one or both of them are zero.

Parallel Vectors

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

The cross product obeys the following laws.

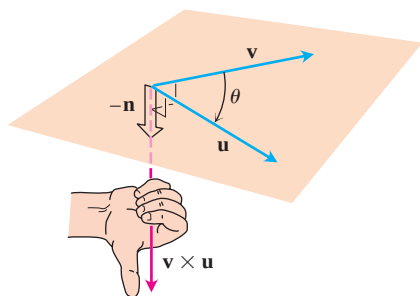


FIGURE 12.28 The construction of $\mathbf{v} \times \mathbf{u}$.

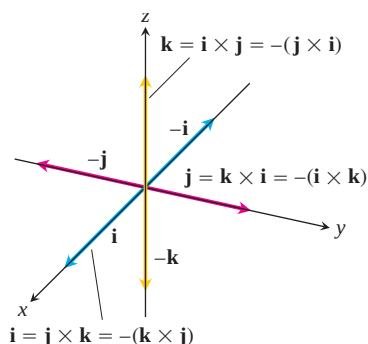


FIGURE 12.29 The pairwise cross products of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r , s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
4. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

To visualize Property 4, for example, notice that when the fingers of a right hand curl through the angle θ from $\mathbf{v}$ to $\mathbf{u}$, the thumb points the opposite way and the unit vector we choose in forming $\mathbf{v} \times \mathbf{u}$ is the negative of the one we choose in forming $\mathbf{u} \times \mathbf{v}$ (Figure 12.28).

Property 1 can be verified by applying the definition of cross product to both sides of the equation and comparing the results. Property 2 is proved in Appendix 6. Property 3 follows by multiplying both sides of the equation in Property 2 by -1 and reversing the order of the products using Property 4. Property 5 is a definition. As a rule, cross product multiplication is *not associative* so $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ does not generally equal $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$. (See Additional Exercise 15.)

When we apply the definition to calculate the pairwise cross products of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, we find (Figure 12.29)

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$

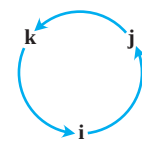


Diagram for recalling these products

and

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

$|\mathbf{u} \times \mathbf{v}|$ Is the Area of a Parallelogram

Because $\mathbf{n}$ is a unit vector, the magnitude of $\mathbf{u} \times \mathbf{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\sin \theta| |\mathbf{n}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.$$

This is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ (Figure 12.30), $|\mathbf{u}|$ being the base of the parallelogram and $|\mathbf{v}| |\sin \theta|$ the height.

Determinant Formula for $\mathbf{u} \times \mathbf{v}$

Our next objective is to calculate $\mathbf{u} \times \mathbf{v}$ from the components of $\mathbf{u}$ and $\mathbf{v}$ relative to a Cartesian coordinate system.

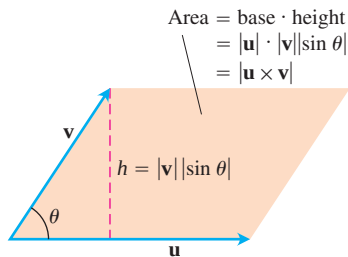


FIGURE 12.30 The parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

Determinants

2×2 and 3×3 determinants are evaluated as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE

$$\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} = (2)(3) - (1)(-4) \\ = 6 + 4 = 10$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$$

$$- a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

EXAMPLE

$$\begin{vmatrix} -5 & 3 & 1 \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = (-5) \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \\ - (3) \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + (1) \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \\ = -5(1 - 3) - 3(2 + 4) \\ + 1(6 + 4) \\ = 10 - 18 + 10 = 2$$

(For more information, see the Web site at www.aw-bc.com/thomas.)

Suppose that

$$\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}, \quad \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}.$$

Then the distributive laws and the rules for multiplying $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ tell us that

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}) \times (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \\ &= u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\ &\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\ &\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} \\ &= (u_2 v_3 - u_3 v_2) \mathbf{i} - (u_1 v_3 - u_3 v_1) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}. \end{aligned}$$

The terms in the last line are the same as the terms in the expansion of the symbolic determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

We therefore have the following rule.

Calculating Cross Products Using Determinants

If $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ and $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

**EXAMPLE 1** Calculating Cross Products with Determinants

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} \mathbf{k} \\ &= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k} \\ \mathbf{v} \times \mathbf{u} &= -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} \end{aligned}$$

EXAMPLE 2 Finding Vectors Perpendicular to a Plane

Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

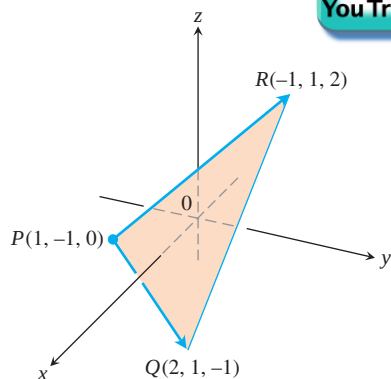


FIGURE 12.31 The area of triangle PQR is half of $|\vec{PQ} \times \vec{PR}|$ (Example 2).

Solution The vector $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\vec{PQ} = (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\vec{PR} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\begin{aligned}\vec{PQ} \times \vec{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 6\mathbf{k}. \end{aligned}$$

EXAMPLE 3 Finding the Area of a Triangle

Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.31).

Solution The area of the parallelogram determined by P , Q , and R is

$$\begin{aligned}|\vec{PQ} \times \vec{PR}| &= |6\mathbf{i} + 6\mathbf{k}| && \text{Values from Example 2.} \\ &= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.\end{aligned}$$

The triangle's area is half of this, or $3\sqrt{2}$.

EXAMPLE 4 Finding a Unit Normal to a Plane

Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution Since $\vec{PQ} \times \vec{PR}$ is perpendicular to the plane, its direction $\mathbf{n}$ is a unit vector perpendicular to the plane. Taking values from Examples 2 and 3, we have

$$\mathbf{n} = \frac{\vec{PQ} \times \vec{PR}}{|\vec{PQ} \times \vec{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

For ease in calculating the cross product using determinants, we usually write vectors in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ rather than as ordered triples $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

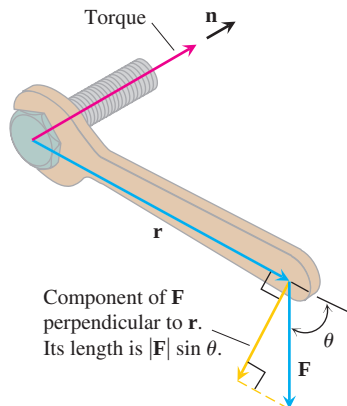


FIGURE 12.32 The torque vector describes the tendency of the force $\mathbf{F}$ to drive the bolt forward.

Torque

When we turn a bolt by applying a force $\mathbf{F}$ to a wrench (Figure 12.32), the torque we produce acts along the axis of the bolt to drive the bolt forward. The magnitude of the torque depends on how far out on the wrench the force is applied and on how much of the force is perpendicular to the wrench at the point of application. The number we use to measure the torque's magnitude is the product of the length of the lever arm $\mathbf{r}$ and the scalar component of $\mathbf{F}$ perpendicular to $\mathbf{r}$. In the notation of Figure 12.32,

$$\text{Magnitude of torque vector} = |\mathbf{r}| |\mathbf{F}| \sin \theta,$$

or $|\mathbf{r} \times \mathbf{F}|$. If we let $\mathbf{n}$ be a unit vector along the axis of the bolt in the direction of the torque, then a complete description of the torque vector is $\mathbf{r} \times \mathbf{F}$, or

$$\text{Torque vector} = (|\mathbf{r}| |\mathbf{F}| \sin \theta) \mathbf{n}.$$

Recall that we defined $\mathbf{u} \times \mathbf{v}$ to be $\mathbf{0}$ when $\mathbf{u}$ and $\mathbf{v}$ are parallel. This is consistent with the torque interpretation as well. If the force $\mathbf{F}$ in Figure 12.32 is parallel to the wrench, meaning that we are trying to turn the bolt by pushing or pulling along the line of the wrench's handle, the torque produced is zero.

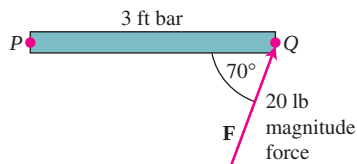


FIGURE 12.33 The magnitude of the torque exerted by $\mathbf{F}$ at P is about 56.4 ft-lb (Example 5).

EXAMPLE 5 Finding the Magnitude of a Torque

The magnitude of the torque generated by force $\mathbf{F}$ at the pivot point P in Figure 12.33 is

$$\begin{aligned} |\vec{PQ} \times \mathbf{F}| &= |\vec{PQ}| |\mathbf{F}| \sin 70^\circ \\ &\approx (3)(20)(0.94) \\ &\approx 56.4 \text{ ft-lb.} \end{aligned}$$

Triple Scalar or Box Product

The product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is called the **triple scalar product** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (in that order). As you can see from the formula

$$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta|,$$

the absolute value of the product is the volume of the parallelepiped (parallelogram-sided box) determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (Figure 12.34). The number $|\mathbf{u} \times \mathbf{v}|$ is the area of the base parallelogram. The number $|\mathbf{w}| |\cos \theta|$ is the parallelepiped's height. Because of this geometry, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is also called the **box product** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

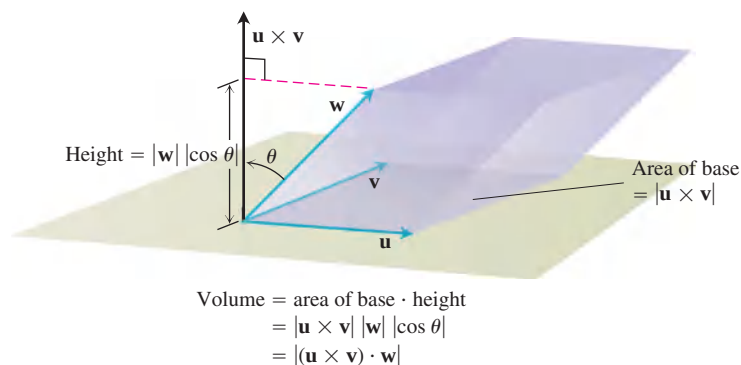


FIGURE 12.34 The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

The dot and cross may be interchanged in a triple scalar product without altering its value.

By treating the planes of $\mathbf{v}$ and $\mathbf{w}$ and of $\mathbf{w}$ and $\mathbf{u}$ as the base planes of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, we see that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}.$$

Since the dot product is commutative, we also have

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}).$$

The triple scalar product can be evaluated as a determinant:

$$\begin{aligned}
 (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left[\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right] \cdot \mathbf{w} \\
 &= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
 &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.
 \end{aligned}$$

Calculating the Triple Scalar Product

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



EXAMPLE 6 Finding the Volume of a Parallelepiped

Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution Using the rule for calculating determinants, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed. ■

EXERCISES 12.4

Cross Product Calculations

In Exercises 1–8, find the length and direction (when defined) of $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

1. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$
2. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j}$
3. $\mathbf{u} = 2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
4. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{0}$
5. $\mathbf{u} = 2\mathbf{i}$, $\mathbf{v} = -3\mathbf{j}$
6. $\mathbf{u} = \mathbf{i} \times \mathbf{j}$, $\mathbf{v} = \mathbf{j} \times \mathbf{k}$
7. $\mathbf{u} = -8\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
8. $\mathbf{u} = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$

In Exercises 9–14, sketch the coordinate axes and then include the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$ as vectors starting at the origin.

9. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$
10. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j}$
11. $\mathbf{u} = \mathbf{i} - \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
12. $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$
13. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$
14. $\mathbf{u} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = \mathbf{i}$

Triangles in Space

In Exercises 15–18,

- a. Find the area of the triangle determined by the points P , Q , and R .
- b. Find a unit vector perpendicular to plane PQR .





15. $P(1, -1, 2)$, $Q(2, 0, -1)$, $R(0, 2, 1)$
 16. $P(1, 1, 1)$, $Q(2, 1, 3)$, $R(3, -1, 1)$
 17. $P(2, -2, 1)$, $Q(3, -1, 2)$, $R(3, -1, 1)$
 18. $P(-2, 2, 0)$, $Q(0, 1, -1)$, $R(-1, 2, -2)$

Triple Scalar Products

In Exercises 19–22, verify that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ and find the volume of the parallelepiped (box) determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

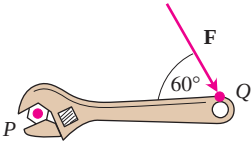
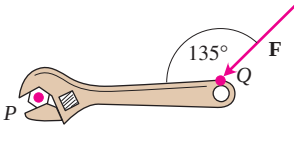


$\mathbf{u}$	$\mathbf{v}$	$\mathbf{w}$
19. $2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
20. $\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
21. $2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$
22. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} - \mathbf{k}$	$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

Theory and Examples

23. **Parallel and perpendicular vectors** Let $\mathbf{u} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} - 5\mathbf{k}$, $\mathbf{w} = -15\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. Which vectors, if any, are (a) perpendicular? (b) Parallel? Give reasons for your answers.
24. **Parallel and perpendicular vectors** Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{k}$, $\mathbf{r} = -(\pi/2)\mathbf{i} - \pi\mathbf{j} + (\pi/2)\mathbf{k}$. Which vectors, if any, are (a) perpendicular? (b) Parallel? Give reasons for your answers.

In Exercises 39 and 40, find the magnitude of the torque exerted by $\mathbf{F}$ on the bolt at P if $|\vec{PQ}| = 8$ in. and $|\mathbf{F}| = 30$ lb. Answer in foot-pounds.

25.  26. 

27. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

- a. $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$ b. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|$
 c. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$ d. $\mathbf{u} \times (-\mathbf{u}) = \mathbf{0}$
 e. $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$
 f. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
 g. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$
 h. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

28. Which of the following are *always true*, and which are *not always true*? Give reasons for your answers.

- a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ b. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
 c. $(-\mathbf{u}) \times \mathbf{v} = -(\mathbf{u} \times \mathbf{v})$

- d. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ (any number c)
 e. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$ (any number c)
 f. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ g. $(\mathbf{u} \times \mathbf{u}) \cdot \mathbf{u} = 0$
 h. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v})$

29. Given nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, use dot product and cross product notation, as appropriate, to describe the following.
- The vector projection of $\mathbf{u}$ onto $\mathbf{v}$
 - A vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$
 - A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and $\mathbf{w}$
 - The volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$
30. Given nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, use dot product and cross product notation to describe the following.
- A vector orthogonal to $\mathbf{u} \times \mathbf{v}$ and $\mathbf{u} \times \mathbf{w}$
 - A vector orthogonal to $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$
 - A vector of length $|\mathbf{u}|$ in the direction of $\mathbf{v}$
 - The area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{w}$
31. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors. Which of the following make sense, and which do not? Give reasons for your answers.
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ b. $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$
 - $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ d. $\mathbf{u} \cdot (\mathbf{v} \cdot \mathbf{w})$
32. **Cross products of three vectors** Show that except in degenerate cases, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the plane of $\mathbf{u}$ and $\mathbf{v}$, whereas $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$. What *are* the degenerate cases?
33. **Cancellation in cross products** If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.
34. **Double cancellation** If $\mathbf{u} \neq \mathbf{0}$ and if $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then does $\mathbf{v} = \mathbf{w}$? Give reasons for your answer.

Area in the Plane

Find the areas of the parallelograms whose vertices are given in Exercises 35–38.

35. $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$, $D(0, -1)$
 36. $A(0, 0)$, $B(7, 3)$, $C(9, 8)$, $D(2, 5)$
 37. $A(-1, 2)$, $B(2, 0)$, $C(7, 1)$, $D(4, 3)$
 38. $A(-6, 0)$, $B(1, -4)$, $C(3, 1)$, $D(-4, 5)$

Find the areas of the triangles whose vertices are given in Exercises 39–42.

39. $A(0, 0)$, $B(-2, 3)$, $C(3, 1)$
 40. $A(-1, -1)$, $B(3, 3)$, $C(2, 1)$
 41. $A(-5, 3)$, $B(1, -2)$, $C(6, -2)$
 42. $A(-6, 0)$, $B(10, -5)$, $C(-2, 4)$

43. **Triangle area** Find a formula for the area of the triangle in the xy -plane with vertices at $(0, 0)$, (a_1, a_2) , and (b_1, b_2) . Explain your work.

44. **Triangle area** Find a concise formula for the area of a triangle with vertices (a_1, a_2) , (b_1, b_2) , and (c_1, c_2) .

12.5 Lines and Planes in Space

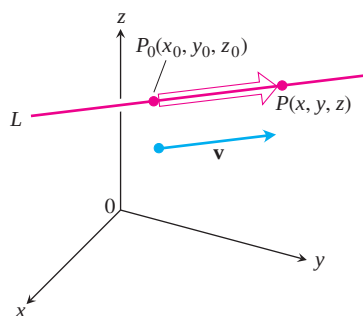


FIGURE 12.35 A point P lies on L through P_0 parallel to $\mathbf{v}$ if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$.

In the calculus of functions of a single variable, we used our knowledge of lines to study curves in the plane. We investigated tangents and found that, when highly magnified, differentiable curves were effectively linear.

To study the calculus of functions of more than one variable in the next chapter, we start with planes and use our knowledge of planes to study the surfaces that are the graphs of functions in space.

This section shows how to use scalar and vector products to write equations for lines, line segments, and planes in space.

Lines and Line Segments in Space

In the plane, a line is determined by a point and a number giving the slope of the line. In space a line is determined by a point and a *vector* giving the direction of the line.

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$ (Figure 12.35). Thus, $\overrightarrow{P_0P} = t\mathbf{v}$ for some scalar parameter t . The value of t depends on the location of the point P along the line, and the domain of t is $(-\infty, \infty)$. The expanded form of the equation $\overrightarrow{P_0P} = t\mathbf{v}$ is

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}),$$

which can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}). \quad (1)$$

If $\mathbf{r}(t)$ is the position vector of a point $P(x, y, z)$ on the line and $\mathbf{r}_0$ is the position vector of the point $P_0(x_0, y_0, z_0)$, then Equation (1) gives the following vector form for the equation of a line in space.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty, \quad (2)$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Equating the corresponding components of the two sides of Equation (1) gives three scalar equations involving the parameter t :

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3.$$

These equations give us the standard parametrization of the line for the parameter interval $-\infty < t < \infty$.

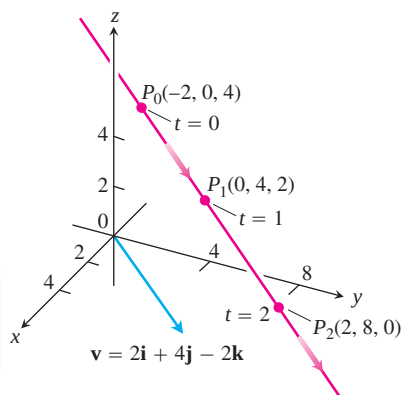


FIGURE 12.36 Selected points and parameter values on the line $x = -2 + 2t, y = 4t, z = 4 - 2t$. The arrows show the direction of increasing t (Example 1).

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \quad (3)$$

EXAMPLE 1 Parametrizing a Line Through a Point Parallel to a Vector

Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (Figure 12.36).

Solution With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$$x = -2 + 2t, \quad y = 4t, \quad z = 4 - 2t.$$

EXAMPLE 2 Parametrizing a Line Through Two Points

Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution The vector

$$\begin{aligned} \overrightarrow{PQ} &= (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} \\ &= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} \end{aligned}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the “base point” and written

$$x = 1 + 4t, \quad y = -1 - 3t, \quad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t .

Notice that parametrizations are not unique. Not only can the “base point” change, but so can the parameter. The equations $x = -3 + 4t^3, y = 2 - 3t^3$, and $z = -3 + 7t^3$ also parametrize the line in Example 2.

To parametrize a line segment joining two points, we first parametrize the line through the points. We then find the t -values for the endpoints and restrict t to lie in the closed interval bounded by these values. The line equations together with this added restriction parametrize the segment.

EXAMPLE 3 Parametrizing a Line Segment

Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Figure 12.37).

Solution We begin with equations for the line through P and Q , taking them, in this case, from Example 2:

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t.$$

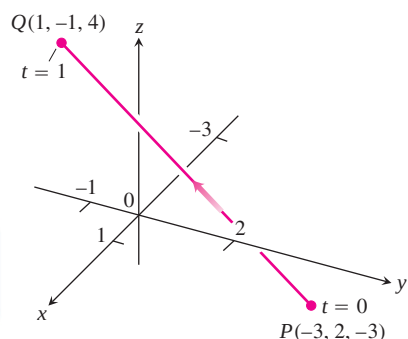


FIGURE 12.37 Example 3 derives a parametrization of line segment PQ . The arrow shows the direction of increasing t .

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \leq t \leq 1$ to parametrize the segment:

$$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \leq t \leq 1. \quad \blacksquare$$

The vector form (Equation (2)) for a line in space is more revealing if we think of a line as the path of a particle starting at position $P_0(x_0, y_0, z_0)$ and moving in the direction of vector $\mathbf{v}$. Rewriting Equation (2), we have

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 + t\mathbf{v} \\ &= \mathbf{r}_0 + t|\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}. \end{aligned} \quad (4)$$

In other words, the position of the particle at time t is its initial position plus its distance moved (speed $\times$ time) in the direction $\mathbf{v}/|\mathbf{v}|$ of its straight-line motion.

EXAMPLE 4 Flight of a Helicopter

A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Solution We place the origin at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}). \end{aligned}$$

When $t = 10$ sec,

$$\begin{aligned} \mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle. \end{aligned}$$

After 10 sec of flight from the origin toward $(1, 1, 1)$, the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of $(60 \text{ ft/sec})(10 \text{ sec}) = 600$ ft, which is the length of the vector $\mathbf{r}(10)$. $\blacksquare$

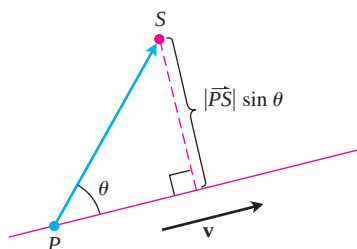


FIGURE 12.38 The distance from S to the line through P parallel to $\mathbf{v}$ is $|\vec{PS}| \sin \theta$, where θ is the angle between $\vec{PS}$ and $\mathbf{v}$.

The Distance from a Point to a Line in Space

To find the distance from a point S to a line that passes through a point P parallel to a vector $\mathbf{v}$, we find the absolute value of the scalar component of $\vec{PS}$ in the direction of a vector normal to the line (Figure 12.38). In the notation of the figure, the absolute value of the scalar component is, $|\vec{PS}| \sin \theta$, which is $\frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.

Distance from a Point S to a Line Through P Parallel to $\mathbf{v}$

$$d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} \quad (5)$$



EXAMPLE 5 Finding Distance from a Point to a Line

Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\vec{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\vec{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

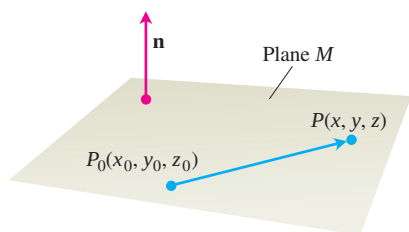


FIGURE 12.39 The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \vec{P_0P} = 0$.

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and its “tilt” or orientation. This “tilt” is defined by specifying a vector that is perpendicular or normal to the plane.

Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\vec{P_0P}$ is orthogonal to $\mathbf{n}$ (Figure 12.39). Thus, the dot product $\mathbf{n} \cdot \vec{P_0P} = 0$. This equation is equivalent to

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot [(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}] = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$, where
 $D = Ax_0 + By_0 + Cz_0$

**EXAMPLE 6** Finding an Equation for a Plane

Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution The component equation is

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

Simplifying, we obtain

$$5x + 15 + 2y - z + 7 = 0$$

$$5x + 2y - z = -22. \quad \blacksquare$$

Notice in Example 6 how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of x , y , and z in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

**EXAMPLE 7** Finding an Equation for a Plane Through Three Points

Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

$$3x + 2y + 6z = 6. \quad \blacksquare$$

Lines of Intersection

Just as lines are parallel if and only if they have the same direction, two planes are **parallel** if and only if their normals are parallel, or $\mathbf{n}_1 = k\mathbf{n}_2$ for some scalar k . Two planes that are not parallel intersect in a line.

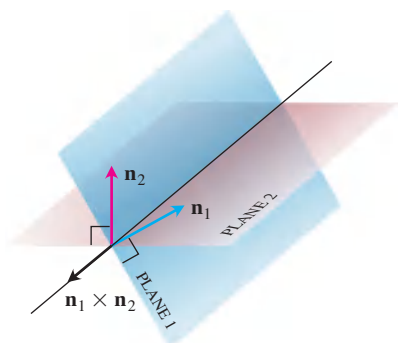


FIGURE 12.40 How the line of intersection of two planes is related to the planes' normal vectors (Example 8).



EXAMPLE 8 Finding a Vector Parallel to the Line of Intersection of Two Planes

Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The line of intersection of two planes is perpendicular to both planes' normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ (Figure 12.40) and therefore parallel to $\mathbf{n}_1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Any nonzero scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$ will do as well. ■

EXAMPLE 9 Parametrizing the Line of Intersection of Two Planes

Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Solution We find a vector parallel to the line and a point on the line and use Equations (3).

Example 8 identifies $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously identifies one of these points as $(3, -1, 0)$. The line is

$$x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t.$$

The choice $z = 0$ is arbitrary and we could have chosen $z = 1$ or $z = -1$ just as well. Or we could have let $x = 0$ and solved for y and z . The different choices would simply give different parametrizations of the same line. ■

Sometimes we want to know where a line and a plane intersect. For example, if we are looking at a flat plate and a line segment passes through it, we may be interested in knowing what portion of the line segment is hidden from our view by the plate. This application is used in computer graphics (Exercise 74).



EXAMPLE 10 Finding the Intersection of a Line and a Plane

Find the point where the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Solution The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t \right)$$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$\begin{aligned} 3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) &= 6 \\ 8 + 6t - 4t + 6 + 6t &= 6 \\ 8t &= -8 \\ t &= -1. \end{aligned}$$

The point of intersection is

$$(x, y, z)|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right). \quad \blacksquare$$

The Distance from a Point to a Plane

If P is a point on a plane with normal $\mathbf{n}$, then the distance from any point S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$. That is, the distance from S to the plane is

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \quad (6)$$

where $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is normal to the plane.



EXAMPLE 11 Finding the Distance from a Point to a Plane

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution We find a point P in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector $\mathbf{n}$ normal to the plane (Figure 12.41). The coefficients in the equation $3x + 2y + 6z = 6$ give

$$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

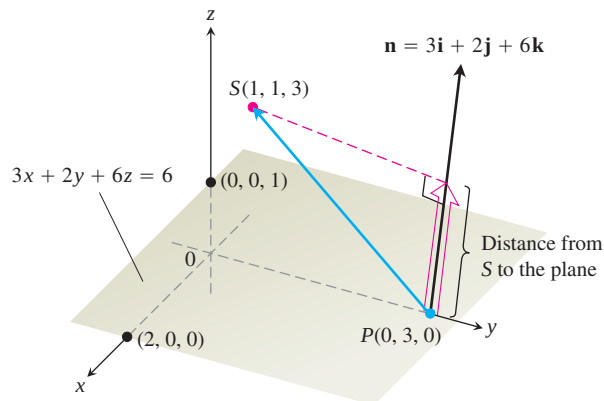


FIGURE 12.41 The distance from S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$ (Example 11).

The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y -intercept $(0, 3, 0)$, then

$$\vec{PS} = (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k}$$

$$= \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$

$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.$$

The distance from S to the plane is

$$\begin{aligned} d &= \left| \vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| && \text{length of proj}_{\mathbf{n}} \vec{PS} \\ &= \left| (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \right| \\ &= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}. \end{aligned}$$

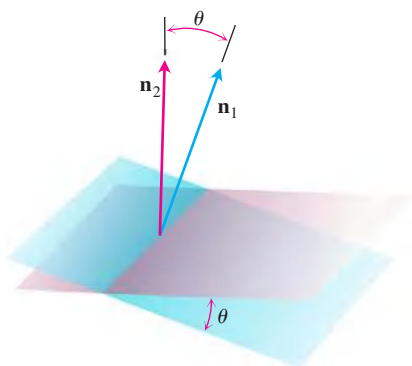


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.

Angles Between Planes

The angle between two intersecting planes is defined to be the (acute) angle determined by their normal vectors (Figure 12.42).



EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The vectors

$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \quad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

are normals to the planes. The angle between them is

$$\begin{aligned} \theta &= \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \right) \\ &= \cos^{-1} \left(\frac{4}{21} \right) \\ &\approx 1.38 \text{ radians.} \quad \text{About } 79 \text{ deg} \end{aligned}$$

EXERCISES 12.5

Lines and Line Segments

Find parametric equations for the lines in Exercises 1–12.



1. The line through the point $P(3, -4, -1)$ parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$
2. The line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$
3. The line through $P(-2, 0, 3)$ and $Q(3, 5, -2)$
4. The line through $P(1, 2, 0)$ and $Q(1, 1, -1)$
5. The line through the origin parallel to the vector $2\mathbf{j} + \mathbf{k}$
6. The line through the point $(3, -2, 1)$ parallel to the line $x = 1 + 2t, y = 2 - t, z = 3t$
7. The line through $(1, 1, 1)$ parallel to the z -axis
8. The line through $(2, 4, 5)$ perpendicular to the plane $3x + 7y - 5z = 21$
9. The line through $(0, -7, 0)$ perpendicular to the plane $x + 2y + 2z = 13$



10. The line through $(2, 3, 0)$ perpendicular to the vectors $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$
11. The x -axis 12. The z -axis

Find parametrizations for the line segments joining the points in Exercises 13–20. Draw coordinate axes and sketch each segment, indicating the direction of increasing t for your parametrization.



13. $(0, 0, 0)$, $(1, 1, 3/2)$ 14. $(0, 0, 0)$, $(1, 0, 0)$
15. $(1, 0, 0)$, $(1, 1, 0)$ 16. $(1, 1, 0)$, $(1, 1, 1)$
17. $(0, 1, 1)$, $(0, -1, 1)$ 18. $(0, 2, 0)$, $(3, 0, 0)$
19. $(2, 0, 2)$, $(0, 2, 0)$ 20. $(1, 0, -1)$, $(0, 3, 0)$

Planes

Find equations for the planes in Exercises 21–26.



21. The plane through $P_0(0, 2, -1)$ normal to $\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$
22. The plane through $(1, -1, 3)$ parallel to the plane

$$3x + y + z = 7$$

23. The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$
24. The plane through $(2, 4, 5)$, $(1, 5, 7)$, and $(-1, 6, 8)$
25. The plane through $P_0(2, 4, 5)$ perpendicular to the line

$$x = 5 + t, \quad y = 1 + 3t, \quad z = 4t$$

26. The plane through $A(1, -2, 1)$ perpendicular to the vector from the origin to A
27. Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and then find the plane determined by these lines.
28. Find the point of intersection of the lines $x = t$, $y = -t + 2$, $z = t + 1$, and $x = 2s + 2$, $y = s + 3$, $z = 5s + 6$, and then find the plane determined by these lines.

In Exercises 29 and 30, find the plane determined by the intersecting lines.



29. $L1: x = -1 + t, y = 2 + t, z = 1 - t; -\infty < t < \infty$
 $L2: x = 1 - 4s, y = 1 + 2s, z = 2 - 2s; -\infty < s < \infty$
30. $L1: x = t, y = 3 - 3t, z = -2 - t; -\infty < t < \infty$
 $L2: x = 1 + s, y = 4 + s, z = -1 + s; -\infty < s < \infty$
31. Find a plane through $P_0(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$, $x + 2y + z = 2$.
32. Find a plane through the points $P_1(1, 2, 3)$, $P_2(3, 2, 1)$ and perpendicular to the plane $4x - y + 2z = 7$.

Distances

In Exercises 33–38, find the distance from the point to the line.



33. $(0, 0, 12); x = 4t, y = -2t, z = 2t$
34. $(0, 0, 0); x = 5 + 3t, y = 5 + 4t, z = -3 - 5t$
35. $(2, 1, 3); x = 2 + 2t, y = 1 + 6t, z = 3$

36. $(2, 1, -1); x = 2t, y = 1 + 2t, z = 2t$
37. $(3, -1, 4); x = 4 - t, y = 3 + 2t, z = -5 + 3t$
38. $(-1, 4, 3); x = 10 + 4t, y = -3, z = 4t$

In Exercises 39–44, find the distance from the point to the plane.

39. $(2, -3, 4), x + 2y + 2z = 13$
40. $(0, 0, 0), 3x + 2y + 6z = 6$
41. $(0, 1, 1), 4y + 3z = -12$
42. $(2, 2, 3), 2x + y + 2z = 4$
43. $(0, -1, 0), 2x + y + 2z = 4$
44. $(1, 0, -1), -4x + y + z = 4$
45. Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.
46. Find the distance from the line $x = 2 + t, y = 1 + t, z = -(1/2) - (1/2)t$ to the plane $x + 2y + 6z = 10$.

Angles

Find the angles between the planes in Exercises 47 and 48.

47. $x + y = 1, 2x + y - 2z = 2$
48. $5x + y - z = 10, x - 2y + 3z = -1$

T Use a calculator to find the acute angles between the planes in Exercises 49–52 to the nearest hundredth of a radian.

49. $2x + 2y + 2z = 3, 2x - 2y - z = 5$
50. $x + y + z = 1, z = 0$ (the xy -plane)
51. $2x + 2y - z = 3, x + 2y + z = 2$
52. $4y + 3z = -12, 3x + 2y + 6z = 6$

Intersecting Lines and Planes

In Exercises 53–56, find the point in which the line meets the plane.

53. $x = 1 - t, y = 3t, z = 1 + t; 2x - y + 3z = 6$
54. $x = 2, y = 3 + 2t, z = -2 - 2t; 6x + 3y - 4z = -12$
55. $x = 1 + 2t, y = 1 + 5t, z = 3t; x + y + z = 2$
56. $x = -1 + 3t, y = -2, z = 5t; 2x - 3z = 7$

Find parametrizations for the lines in which the planes in Exercises 57–60 intersect.

57. $x + y + z = 1, x + y = 2$
58. $3x - 6y - 2z = 3, 2x + y - 2z = 2$
59. $x - 2y + 4z = 2, x + y - 2z = 5$
60. $5x - 2y = 11, 4y - 5z = -17$

Given two lines in space, either they are parallel, or they intersect, or they are skew (imagine, for example, the flight paths of two planes in the sky). Exercises 61 and 62 each give three lines. In each exercise, determine whether the lines, taken two at a time, are parallel, intersect, or are skew. If they intersect, find the point of intersection.

61. $L_1: x = 3 + 2t, y = -1 + 4t, z = 2 - t; -\infty < t < \infty$
 $L_2: x = 1 + 4s, y = 1 + 2s, z = -3 + 4s; -\infty < s < \infty$
 $L_3: x = 3 + 2r, y = 2 + r, z = -2 + 2r; -\infty < r < \infty$
62. $L_1: x = 1 + 2t, y = -1 - t, z = 3t; -\infty < t < \infty$
 $L_2: x = 2 - s, y = 3s, z = 1 + s; -\infty < s < \infty$
 $L_3: x = 5 + 2r, y = 1 - r, z = 8 + 3r; -\infty < r < \infty$

Theory and Examples

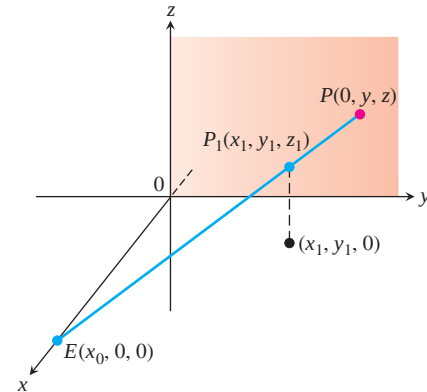
63. Use Equations (3) to generate a parametrization of the line through $P(2, -4, 7)$ parallel to $\mathbf{v}_1 = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Then generate another parametrization of the line using the point $P_2(-2, -2, 1)$ and the vector $\mathbf{v}_2 = -\mathbf{i} + (1/2)\mathbf{j} - (3/2)\mathbf{k}$.
64. Use the component form to generate an equation for the plane through $P_1(4, 1, 5)$ normal to $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Then generate another equation for the same plane using the point $P_2(3, -2, 0)$ and the normal vector $\mathbf{n}_2 = -\sqrt{2}\mathbf{i} + 2\sqrt{2}\mathbf{j} - \sqrt{2}\mathbf{k}$.
65. Find the points in which the line $x = 1 + 2t, y = -1 - t, z = 3t$ meets the coordinate planes. Describe the reasoning behind your answer.
66. Find equations for the line in the plane $z = 3$ that makes an angle of $\pi/6$ rad with $\mathbf{i}$ and an angle of $\pi/3$ rad with $\mathbf{j}$. Describe the reasoning behind your answer.
67. Is the line $x = 1 - 2t, y = 2 + 5t, z = -3t$ parallel to the plane $2x + y - z = 8$? Give reasons for your answer.
68. How can you tell when two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel? Perpendicular? Give reasons for your answer.
69. Find two different planes whose intersection is the line $x = 1 + t, y = 2 - t, z = 3 + 2t$. Write equations for each plane in the form $Ax + By + Cz = D$.
70. Find a plane through the origin that meets the plane $M: 2x + 3y + z = 12$ in a right angle. How do you know that your plane is perpendicular to M ?
71. For any nonzero numbers $a, b,$ and c , the graph of $(x/a) + (y/b) + (z/c) = 1$ is a plane. Which planes have an equation of this form?

72. Suppose L_1 and L_2 are disjoint (nonintersecting) nonparallel lines. Is it possible for a nonzero vector to be perpendicular to both L_1 and L_2 ? Give reasons for your answer.

Computer Graphics

73. **Perspective in computer graphics** In computer graphics and perspective drawing, we need to represent objects seen by the eye in space as images on a two-dimensional plane. Suppose that the eye is at $E(x_0, 0, 0)$ as shown here and that we want to represent a point $P_1(x_1, y_1, z_1)$ as a point on the yz -plane. We do this by projecting P_1 onto the plane with a ray from E . The point P_1 will be portrayed as the point $P(0, y, z)$. The problem for us as graphics designers is to find y and z given E and P_1 .

- a. Write a vector equation that holds between $\vec{EP}$ and $\vec{EP}_1$. Use the equation to express y and z in terms of $x_0, x_1, y_1,$ and z_1 .
- b. Test the formulas obtained for y and z in part (a) by investigating their behavior at $x_1 = 0$ and $x_1 = x_0$ and by seeing what happens as $x_0 \rightarrow \infty$. What do you find?



74. **Hidden lines** Here is another typical problem in computer graphics. Your eye is at $(4, 0, 0)$. You are looking at a triangular plate whose vertices are at $(1, 0, 1), (1, 1, 0),$ and $(-2, 2, 2)$. The line segment from $(1, 0, 0)$ to $(0, 2, 2)$ passes through the plate. What portion of the line segment is hidden from your view by the plate? (This is an exercise in finding intersections of lines and planes.)

12.6

Cylinders and Quadric Surfaces



Up to now, we have studied two special types of surfaces: spheres and planes. In this section, we extend our inventory to include a variety of cylinders and quadric surfaces. Quadric surfaces are surfaces defined by second-degree equations in x , y , and z . Spheres are quadric surfaces, but there are others of equal interest.

Cylinders

A **cylinder** is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a **generating curve** for the cylinder (Figure 12.43). In solid geometry, where *cylinder* means *circular*

cylinder, the generating curves are circles, but now we allow generating curves of any kind. The cylinder in our first example is generated by a parabola.

When graphing a cylinder or other surface by hand or analyzing one generated by a computer, it helps to look at the curves formed by intersecting the surface with planes parallel to the coordinate planes. These curves are called **cross-sections** or **traces**.



EXAMPLE 1 The Parabolic Cylinder $y = x^2$

Find an equation for the cylinder made by the lines parallel to the z -axis that pass through the parabola $y = x^2, z = 0$ (Figure 12.44).

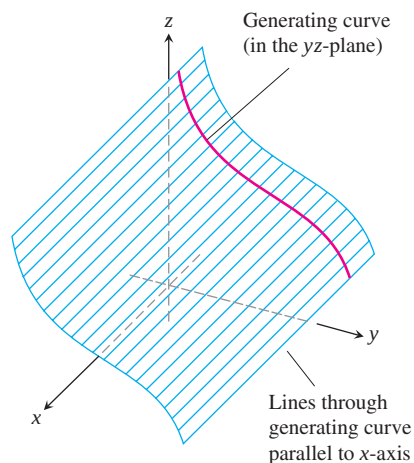


FIGURE 12.43 A cylinder and generating curve.

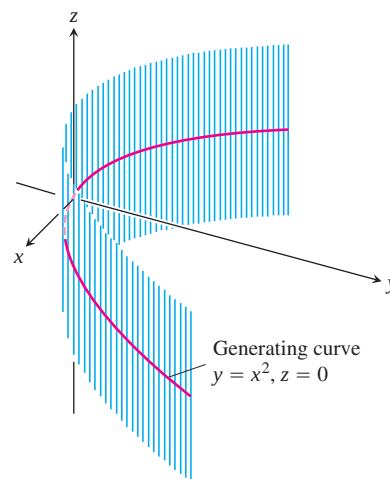


FIGURE 12.44 The cylinder of lines passing through the parabola $y = x^2$ in the xy -plane parallel to the z -axis (Example 1).

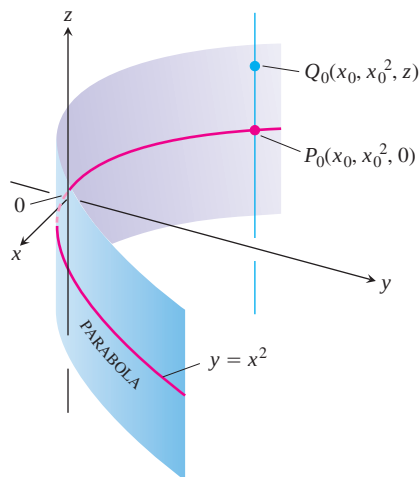


FIGURE 12.45 Every point of the cylinder in Figure 12.44 has coordinates of the form (x_0, x_0^2, z) . We call it “the cylinder $y = x^2$.”

Solution Suppose that the point $P_0(x_0, x_0^2, 0)$ lies on the parabola $y = x^2$ in the xy -plane. Then, for any value of z , the point $Q(x_0, x_0^2, z)$ will lie on the cylinder because it lies on the line $x = x_0, y = x_0^2$ through P_0 parallel to the z -axis. Conversely, any point $Q(x_0, x_0^2, z)$ whose y -coordinate is the square of its x -coordinate lies on the cylinder because it lies on the line $x = x_0, y = x_0^2$ through P_0 parallel to the z -axis (Figure 12.45).

Regardless of the value of z , therefore, the points on the surface are the points whose coordinates satisfy the equation $y = x^2$. This makes $y = x^2$ an equation for the cylinder. Because of this, we call the cylinder “the cylinder $y = x^2$.”

As Example 1 suggests, any curve $f(x, y) = c$ in the xy -plane defines a cylinder parallel to the z -axis whose equation is also $f(x, y) = c$. The equation $x^2 + y^2 = 1$ defines the circular cylinder made by the lines parallel to the z -axis that pass through the circle $x^2 + y^2 = 1$ in the xy -plane. The equation $x^2 + 4y^2 = 9$ defines the elliptical cylinder made by the lines parallel to the z -axis that pass through the ellipse $x^2 + 4y^2 = 9$ in the xy -plane.

In a similar way, any curve $g(x, z) = c$ in the xz -plane defines a cylinder parallel to the y -axis whose space equation is also $g(x, z) = c$ (Figure 12.46). Any curve $h(y, z) = c$

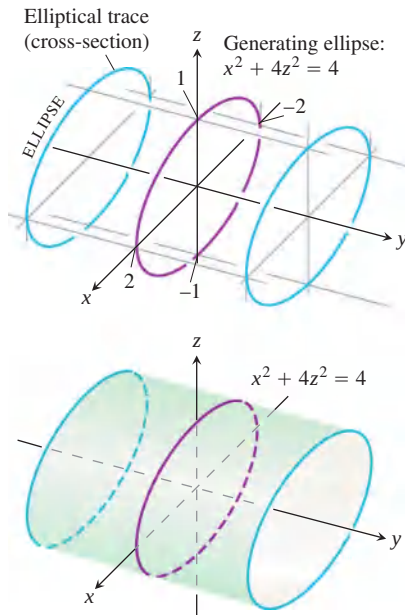


FIGURE 12.46 The elliptical cylinder $x^2 + 4z^2 = 4$ is made of lines parallel to the y -axis and passing through the ellipse $x^2 + 4z^2 = 4$ in the xz -plane. The cross-sections or “traces” of the cylinder in planes perpendicular to the y -axis are ellipses congruent to the generating ellipse. The cylinder extends along the entire y -axis.

defines a cylinder parallel to the x -axis whose space equation is also $h(y, z) = c$ (Figure 12.47). The axis of a cylinder need not be parallel to a coordinate axis, however.

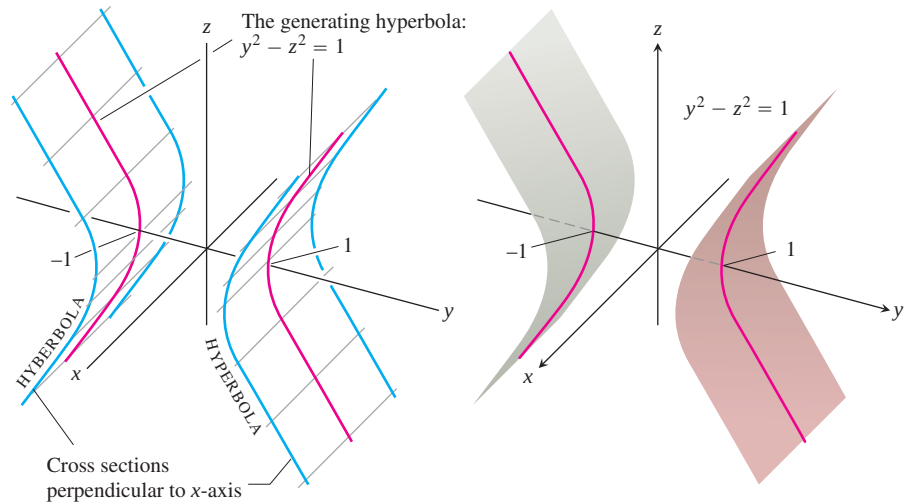


FIGURE 12.47 The hyperbolic cylinder $y^2 - z^2 = 1$ is made of lines parallel to the x -axis and passing through the hyperbola $y^2 - z^2 = 1$ in the yz -plane. The cross-sections of the cylinder in planes perpendicular to the x -axis are hyperbolas congruent to the generating hyperbola.

Quadric Surfaces

The next type of surface we examine is a *quadric* surface. These surfaces are the three-dimensional analogues of ellipses, parabolas, and hyperbolas.

A **quadric surface** is the graph in space of a second-degree equation in x , y , and z . The most general form is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0,$$

where A , B , C , and so on are constants. However, this equation can be simplified by translation and rotation, as in the two-dimensional case. We will study only the simpler equations. Although defined differently, the cylinders in Figures 12.45 through 12.47 were also examples of quadric surfaces. The basic quadric surfaces are **ellipsoids**, **paraboloids**, **elliptical cones**, and **hyperboloids**. (We think of spheres as special ellipsoids.) We now present examples of each type.



EXAMPLE 2 Ellipsoids

The **ellipsoid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (1)$$

(Figure 12.48) cuts the coordinate axes at $(\pm a, 0, 0)$, $(0, \pm b, 0)$, and $(0, 0, \pm c)$. It lies within the rectangular box defined by the inequalities $|x| \leq a$, $|y| \leq b$, and $|z| \leq c$. The surface is symmetric with respect to each of the coordinate planes because each variable in the defining equation is squared.

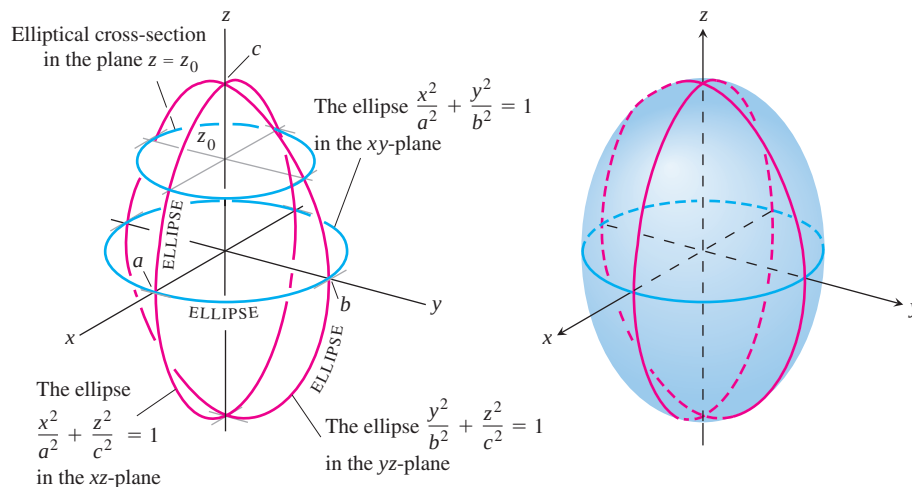


FIGURE 12.48 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in Example 2 has elliptical cross-sections in each of the three coordinate planes.

The curves in which the three coordinate planes cut the surface are ellipses. For example,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{when} \quad z = 0.$$

The section cut from the surface by the plane $z = z_0$, $|z_0| < c$, is the ellipse

$$\frac{x^2}{a^2(1 - (z_0/c)^2)} + \frac{y^2}{b^2(1 - (z_0/c)^2)} = 1.$$

If any two of the semiaxes a , b , and c are equal, the surface is an **ellipsoid of revolution**. If all three are equal, the surface is a sphere. ■



EXAMPLE 3 Paraboloids

The elliptical paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} \quad (2)$$

is symmetric with respect to the planes $x = 0$ and $y = 0$ (Figure 12.49). The only intercept on the axes is the origin. Except for this point, the surface lies above (if $c > 0$) or entirely below (if $c < 0$) the xy -plane, depending on the sign of c . The sections cut by the coordinate planes are

$$x = 0: \quad \text{the parabola } z = \frac{c}{b^2}y^2$$

$$y = 0: \quad \text{the parabola } z = \frac{c}{a^2}x^2$$

$$z = 0: \quad \text{the point } (0, 0, 0).$$

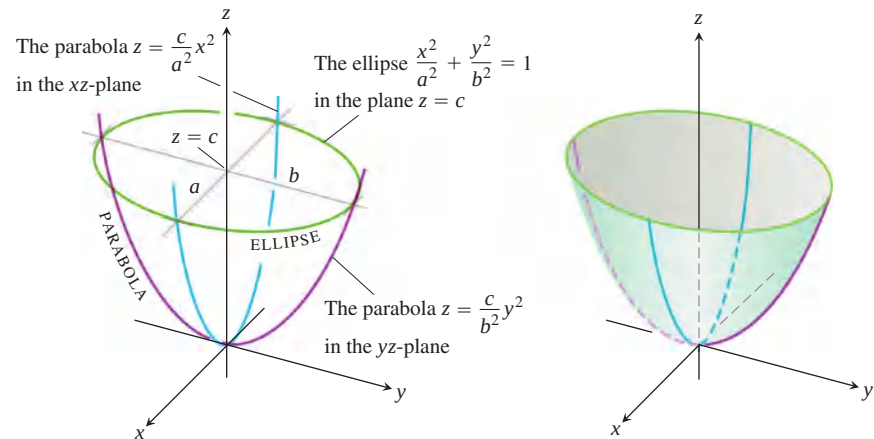


FIGURE 12.49 The elliptical paraboloid $(x^2/a^2) + (y^2/b^2) = z/c$ in Example 3, shown for $c > 0$. The cross-sections perpendicular to the z -axis above the xy -plane are ellipses. The cross-sections in the planes that contain the z -axis are parabolas.

Each plane $z = z_0$ above the xy -plane cuts the surface in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z_0}{c}.$$



EXAMPLE 4 Cones

The **elliptical cone**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \quad (3)$$

is symmetric with respect to the three coordinate planes (Figure 12.50). The sections cut

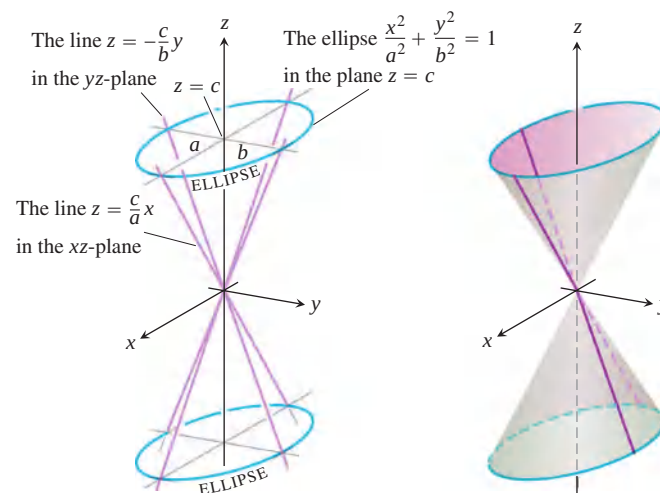


FIGURE 12.50 The elliptical cone $(x^2/a^2) + (y^2/b^2) = (z^2/c^2)$ in Example 4. Planes perpendicular to the z -axis cut the cone in ellipses above and below the xy -plane. Vertical planes that contain the z -axis cut it in pairs of intersecting lines.

by the coordinate planes are

$$x = 0: \text{ the lines } z = \pm \frac{c}{b}y$$

$$y = 0: \text{ the lines } z = \pm \frac{c}{a}x$$

$$z = 0: \text{ the point } (0, 0, 0).$$

The sections cut by planes $z = z_0$ above and below the xy -plane are ellipses whose centers lie on the z -axis and whose vertices lie on the lines given above.

If $a = b$, the cone is a right circular cone. ■



EXAMPLE 5 Hyperboloids

The **hyperboloid of one sheet**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad (4)$$

is symmetric with respect to each of the three coordinate planes (Figure 12.51).

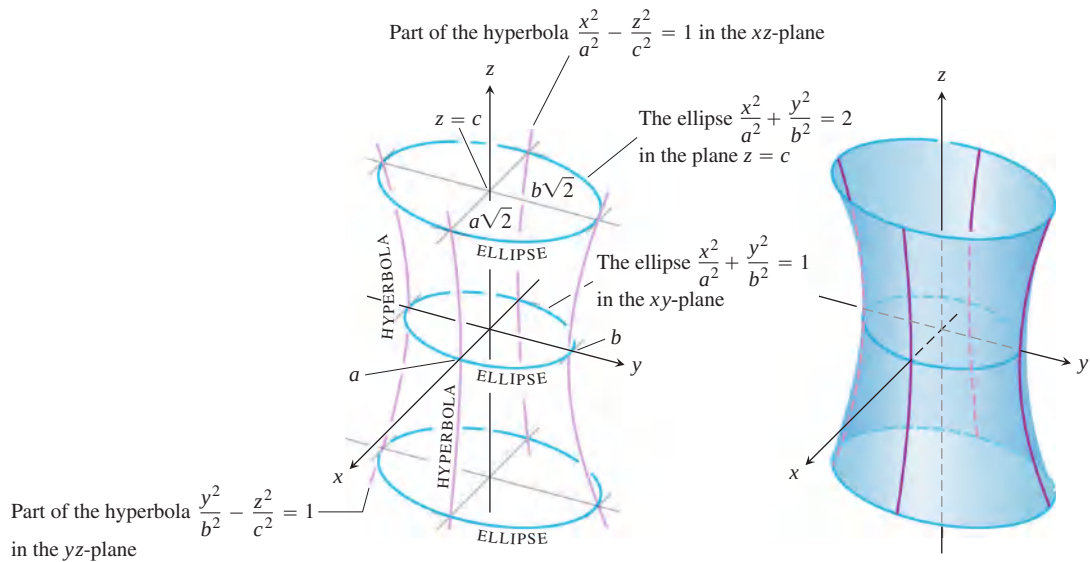


FIGURE 12.51 The hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ in Example 5. Planes perpendicular to the z -axis cut it in ellipses. Vertical planes containing the z -axis cut it in hyperbolas.

The sections cut out by the coordinate planes are

$$x = 0: \text{ the hyperbola } \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$y = 0: \text{ the hyperbola } \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$$

$$z = 0: \text{ the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The plane $z = z_0$ cuts the surface in an ellipse with center on the z -axis and vertices on one of the hyperbolic sections above.

The surface is connected, meaning that it is possible to travel from one point on it to any other without leaving the surface. For this reason, it is said to have *one* sheet, in contrast to the hyperboloid in the next example, which has two sheets.

If $a = b$, the hyperboloid is a surface of revolution. ■



EXAMPLE 6 Hyperboloids

The hyperboloid of two sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (5)$$

is symmetric with respect to the three coordinate planes (Figure 12.52). The plane $z = 0$ does not intersect the surface; in fact, for a horizontal plane to intersect the surface, we must have $|z| \geq c$. The hyperbolic sections

$$x = 0: \quad \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$$

$$y = 0: \quad \frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$$

have their vertices and foci on the z -axis. The surface is separated into two portions, one above the plane $z = c$ and the other below the plane $z = -c$. This accounts for its name.

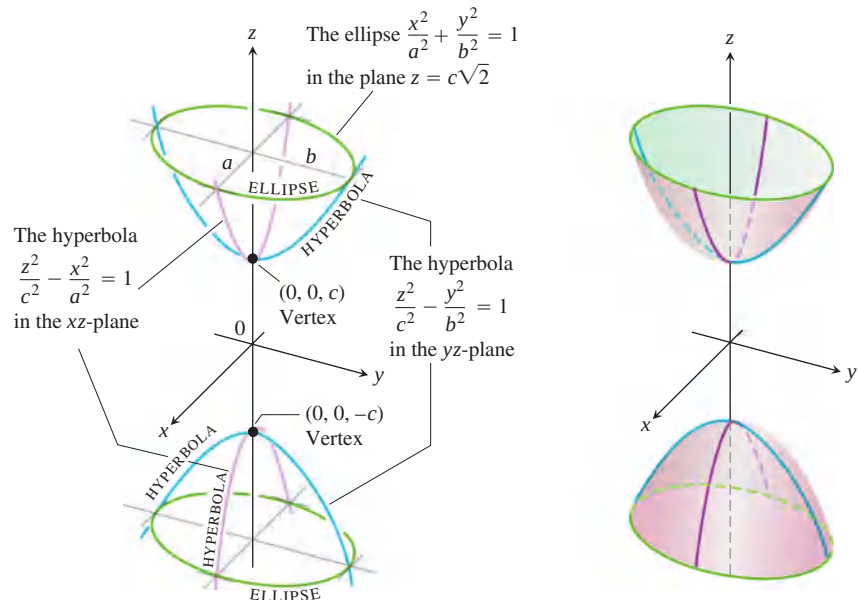


FIGURE 12.52 The hyperboloid $(z^2/c^2) - (x^2/a^2) - (y^2/b^2) = 1$ in Example 6. Planes perpendicular to the z -axis above and below the vertices cut it in ellipses. Vertical planes containing the z -axis cut it in hyperbolas.

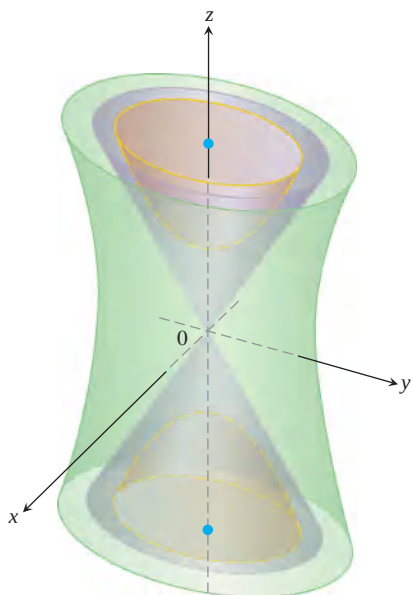


FIGURE 12.53 Both hyperboloids are asymptotic to the cone (Example 6).

Equations (4) and (5) have different numbers of negative terms. The number in each case is the same as the number of sheets of the hyperboloid. If we replace the 1 on the right side of either Equation (4) or Equation (5) by 0, we obtain the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

for an elliptical cone (Equation 3). The hyperboloids are asymptotic to this cone (Figure 12.53) in the same way that the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$$

are asymptotic to the lines

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

in the xy -plane. ■

EXAMPLE 7 A Saddle Point

The **hyperbolic paraboloid**

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0 \quad (6)$$

has symmetry with respect to the planes $x = 0$ and $y = 0$ (Figure 12.54). The sections in these planes are

$$x = 0: \quad \text{the parabola } z = \frac{c}{b^2}y^2. \quad (7)$$

$$y = 0: \quad \text{the parabola } z = -\frac{c}{a^2}x^2. \quad (8)$$

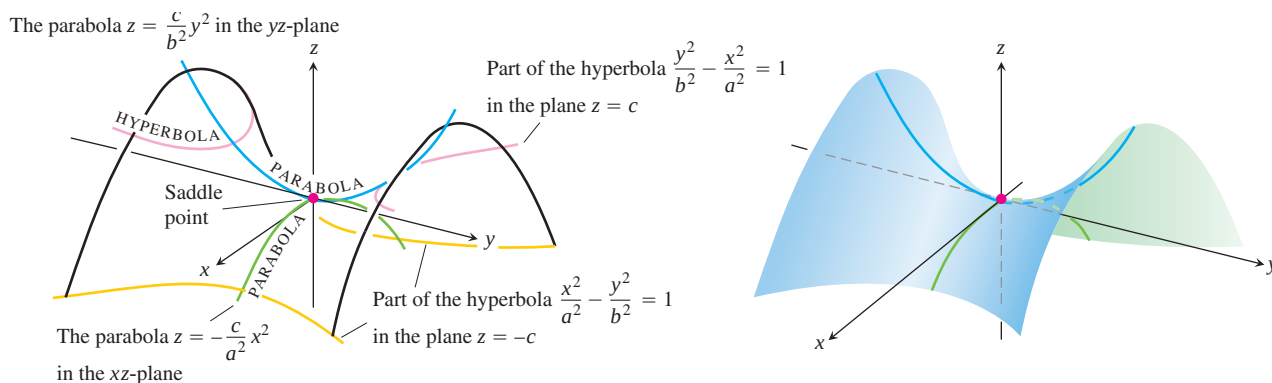


FIGURE 12.54 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$, $c > 0$. The cross-sections in planes perpendicular to the z -axis above and below the xy -plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

In the plane $x = 0$, the parabola opens upward from the origin. The parabola in the plane $y = 0$ opens downward.

If we cut the surface by a plane $z = z_0 > 0$, the section is a hyperbola,

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z_0}{c},$$

with its focal axis parallel to the y -axis and its vertices on the parabola in Equation (7). If z_0 is negative, the focal axis is parallel to the x -axis and the vertices lie on the parabola in Equation (8).

Near the origin, the surface is shaped like a saddle or mountain pass. To a person traveling along the surface in the yz -plane the origin looks like a minimum. To a person traveling in the xz -plane the origin looks like a maximum. Such a point is called a **saddle point** of a surface. ■

USING TECHNOLOGY Visualizing in Space

A CAS or other graphing utility can help in visualizing surfaces in space. It can draw traces in different planes, and many computer graphing systems can rotate a figure so you can see it as if it were a physical model you could turn in your hand. Hidden-line algorithms (see Exercise 74, Section 12.5) are used to block out portions of the surface that you would not see from your current viewing angle. A system may require surfaces to be entered in parametric form, as discussed in Section 16.6 (see also CAS Exercises 57 through 60 in Section 14.1). Sometimes you may have to manipulate the grid mesh to see all portions of a surface.

EXERCISES 12.6

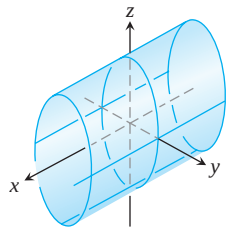
Matching Equations with Surfaces

In Exercises 1–12, match the equation with the surface it defines. Also, identify each surface by type (paraboloid, ellipsoid, etc.) The surfaces are labeled (a)–(f).

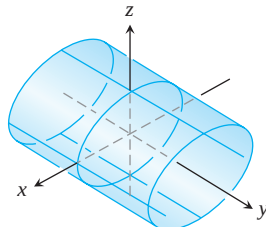


1. $x^2 + y^2 + 4z^2 = 10$
2. $z^2 + 4y^2 - 4x^2 = 4$
3. $9y^2 + z^2 = 16$
4. $y^2 + z^2 = x^2$
5. $x = y^2 - z^2$
6. $x = -y^2 - z^2$
7. $x^2 + 2z^2 = 8$
8. $z^2 + x^2 - y^2 = 1$
9. $x = z^2 - y^2$
10. $z = -4x^2 - y^2$
11. $x^2 + 4z^2 = y^2$
12. $9x^2 + 4y^2 + 2z^2 = 36$

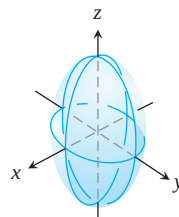
a.



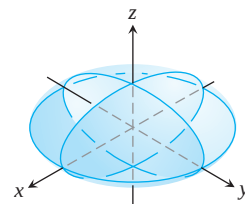
b.



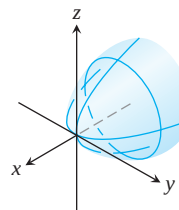
c.



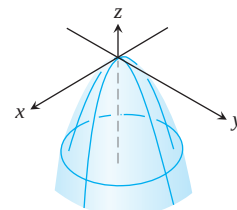
d.

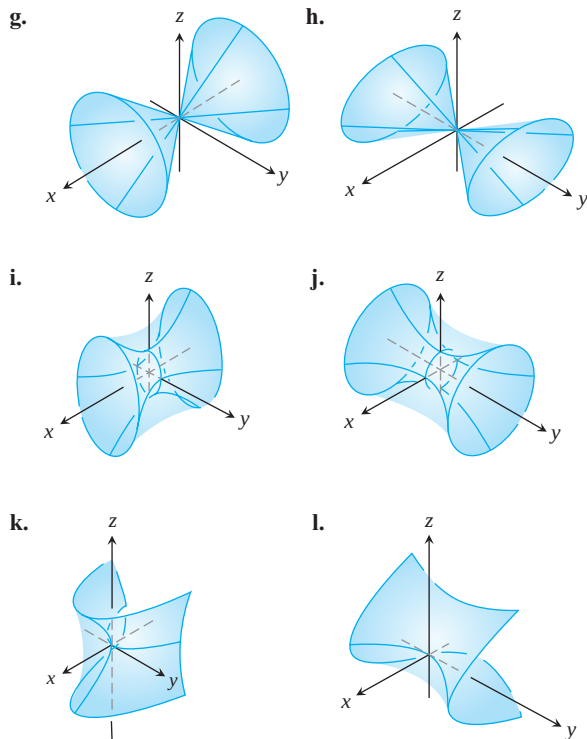


e.



f.





Drawing

Sketch the surfaces in Exercises 13–76.



CYLINDERS

13. $x^2 + y^2 = 4$	14. $x^2 + z^2 = 4$
15. $z = y^2 - 1$	16. $x = y^2$
17. $x^2 + 4z^2 = 16$	18. $4x^2 + y^2 = 36$
19. $z^2 - y^2 = 1$	20. $yz = 1$

ELLIPSOIDS

21. $9x^2 + y^2 + z^2 = 9$	22. $4x^2 + 4y^2 + z^2 = 16$
23. $4x^2 + 9y^2 + 4z^2 = 36$	24. $9x^2 + 4y^2 + 36z^2 = 36$

PARABOLOIDS

25. $z = x^2 + 4y^2$	26. $z = x^2 + 9y^2$
27. $z = 8 - x^2 - y^2$	28. $z = 18 - x^2 - 9y^2$
29. $x = 4 - 4y^2 - z^2$	30. $y = 1 - x^2 - z^2$

CONES

31. $x^2 + y^2 = z^2$	32. $y^2 + z^2 = x^2$
33. $4x^2 + 9z^2 = 9y^2$	34. $9x^2 + 4y^2 = 36z^2$

HYPERBOLOIDS

35. $x^2 + y^2 - z^2 = 1$	36. $y^2 + z^2 - x^2 = 1$

$$37. (y^2/4) + (z^2/9) - (x^2/4) = 1$$

$$38. (x^2/4) + (y^2/4) - (z^2/9) = 1$$

$$39. z^2 - x^2 - y^2 = 1$$

$$41. x^2 - y^2 - (z^2/4) = 1$$

$$40. (y^2/4) - (x^2/4) - z^2 = 1$$

$$42. (x^2/4) - y^2 - (z^2/4) = 1$$

HYPERBOLIC PARABOLOIDS

$$43. y^2 - x^2 = z$$

$$44. x^2 - y^2 = z$$

ASSORTED

$$45. x^2 + y^2 + z^2 = 4$$

$$47. z = 1 + y^2 - x^2$$

$$49. y = -(x^2 + z^2)$$

$$51. 16x^2 + 4y^2 = 1$$

$$53. x^2 + y^2 - z^2 = 4$$

$$55. x^2 + z^2 = y$$

$$57. x^2 + z^2 = 1$$

$$59. 16y^2 + 9z^2 = 4x^2$$

$$61. 9x^2 + 4y^2 + z^2 = 36$$

$$63. x^2 + y^2 - 16z^2 = 16$$

$$65. z = -(x^2 + y^2)$$

$$67. x^2 - 4y^2 = 1$$

$$69. 4y^2 + z^2 - 4x^2 = 4$$

$$71. x^2 + y^2 = z$$

$$73. yz = 1$$

$$75. 9x^2 + 16y^2 = 4z^2$$

$$46. 4x^2 + 4y^2 = z^2$$

$$48. y^2 - z^2 = 4$$

$$50. z^2 - 4x^2 - 4y^2 = 4$$

$$52. z = x^2 + y^2 + 1$$

$$54. x = 4 - y^2$$

$$56. z^2 - (x^2/4) - y^2 = 1$$

$$58. 4x^2 + 4y^2 + z^2 = 4$$

$$60. z = x^2 - y^2 - 1$$

$$62. 4x^2 + 9z^2 = y^2$$

$$64. z^2 + 4y^2 = 9$$

$$66. y^2 - x^2 - z^2 = 1$$

$$68. z = 4x^2 + y^2 - 4$$

$$70. z = 1 - x^2$$

$$72. (x^2/4) + y^2 - z^2 = 1$$

$$74. 36x^2 + 9y^2 + 4z^2 = 36$$

$$76. 4z^2 - x^2 - y^2 = 4$$

Theory and Examples

77. a. Express the area A of the cross-section cut from the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

by the plane $z = c$ as a function of c . (The area of an ellipse with semiaxes a and b is πab .)

- b. Use slices perpendicular to the z -axis to find the volume of the ellipsoid in part (a).

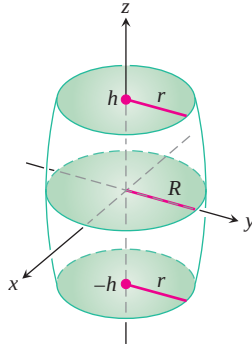
- c. Now find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Does your formula give the volume of a sphere of radius a if $a = b = c$?

78. The barrel shown here is shaped like an ellipsoid with equal pieces cut from the ends by planes perpendicular to the z -axis. The cross-sections perpendicular to the z -axis are circular. The

barrel is $2h$ units high, its midsection radius is R , and its end radii are both r . Find a formula for the barrel's volume. Then check two things. First, suppose the sides of the barrel are straightened to turn the barrel into a cylinder of radius R and height $2h$. Does your formula give the cylinder's volume? Second, suppose $r = 0$ and $h = R$ so the barrel is a sphere. Does your formula give the sphere's volume?



79. Show that the volume of the segment cut from the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

by the plane $z = h$ equals half the segment's base times its altitude. (Figure 12.49 shows the segment for the special case $h = c$.)

80. a. Find the volume of the solid bounded by the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and the planes $z = 0$ and $z = h$, $h > 0$.

- b. Express your answer in part (a) in terms of h and the areas A_0 and A_h of the regions cut by the hyperboloid from the planes $z = 0$ and $z = h$.

- c. Show that the volume in part (a) is also given by the formula

$$V = \frac{h}{6}(A_0 + 4A_m + A_h),$$

where A_m is the area of the region cut by the hyperboloid from the plane $z = h/2$.

81. If the hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$ is cut by the plane $y = y_1$, the resulting curve is a parabola. Find its vertex and focus.

82. Suppose you set $z = 0$ in the equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Jz + K = 0$$

to obtain a curve in the xy -plane. What will the curve be like? Give reasons for your answer.

83. Every time we found the trace of a quadric surface in a plane parallel to one of the coordinate planes, it turned out to be a conic section. Was this mere coincidence? Did it have to happen? Give reasons for your answer.

84. Suppose you intersect a quadric surface with a plane that is *not* parallel to one of the coordinate planes. What will the trace in the plane be like? Give reasons for your answer.

T Computer Grapher Explorations

Plot the surfaces in Exercises 85–88 over the indicated domains. If you can, rotate the surface into different viewing positions.

85. $z = y^2$, $-2 \leq x \leq 2$, $-0.5 \leq y \leq 2$
 86. $z = 1 - y^2$, $-2 \leq x \leq 2$, $-2 \leq y \leq 2$
 87. $z = x^2 + y^2$, $-3 \leq x \leq 3$, $-3 \leq y \leq 3$
 88. $z = x^2 + 2y^2$ over
 a. $-3 \leq x \leq 3$, $-3 \leq y \leq 3$
 b. $-1 \leq x \leq 1$, $-2 \leq y \leq 3$
 c. $-2 \leq x \leq 2$, $-2 \leq y \leq 2$
 d. $-2 \leq x \leq 2$, $-1 \leq y \leq 1$

COMPUTER EXPLORATIONS

Surface Plots

Use a CAS to plot the surfaces in Exercises 89–94. Identify the type of quadric surface from your graph.

89. $\frac{x^2}{9} + \frac{y^2}{36} = 1 - \frac{z^2}{25}$ 90. $\frac{x^2}{9} - \frac{z^2}{9} = 1 - \frac{y^2}{16}$
 91. $5x^2 = z^2 - 3y^2$ 92. $\frac{y^2}{16} = 1 - \frac{x^2}{9} + z$
 93. $\frac{x^2}{9} - 1 = \frac{y^2}{16} + \frac{z^2}{2}$ 94. $y - \sqrt{4 - z^2} = 0$

Chapter 12

Questions to Guide Your Review

1. When do directed line segments in the plane represent the same vector?
2. How are vectors added and subtracted geometrically? Algebraically?
3. How do you find a vector's magnitude and direction?
4. If a vector is multiplied by a positive scalar, how is the result related to the original vector? What if the scalar is zero? Negative?

5. Define the *dot product* (*scalar product*) of two vectors. Which algebraic laws are satisfied by dot products? Give examples. When is the dot product of two vectors equal to zero?
6. What geometric interpretation does the dot product have? Give examples.
7. What is the vector projection of a vector $\mathbf{u}$ onto a vector $\mathbf{v}$? How do you write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$?
8. Define the *cross product* (*vector product*) of two vectors. Which algebraic laws are satisfied by cross products, and which are not? Give examples. When is the cross product of two vectors equal to zero?
9. What geometric or physical interpretations do cross products have? Give examples.
10. What is the determinant formula for calculating the cross product of two vectors relative to the Cartesian $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ -coordinate system? Use it in an example.
11. How do you find equations for lines, line segments, and planes in space? Give examples. Can you express a line in space by a single equation? A plane?
12. How do you find the distance from a point to a line in space? From a point to a plane? Give examples.
13. What are box products? What significance do they have? How are they evaluated? Give an example.
14. How do you find equations for spheres in space? Give examples.
15. How do you find the intersection of two lines in space? A line and a plane? Two planes? Give examples.
16. What is a cylinder? Give examples of equations that define cylinders in Cartesian coordinates.
17. What are quadric surfaces? Give examples of different kinds of ellipsoids, paraboloids, cones, and hyperboloids (equations and sketches).

Chapter 12

Practice Exercises

Vector Calculations in Two Dimensions

In Exercises 1–4, let $\mathbf{u} = \langle -3, 4 \rangle$ and $\mathbf{v} = \langle 2, -5 \rangle$. Find (a) the component form of the vector and (b) its magnitude.

1. $3\mathbf{u} - 4\mathbf{v}$
2. $\mathbf{u} + \mathbf{v}$
3. $-2\mathbf{u}$
4. $5\mathbf{v}$

In Exercises 5–8, find the component form of the vector.

5. The vector obtained by rotating $\langle 0, 1 \rangle$ through an angle of $2\pi/3$ radians
6. The unit vector that makes an angle of $\pi/6$ radian with the positive x -axis
7. The vector 2 units long in the direction $4\mathbf{i} - \mathbf{j}$
8. The vector 5 units long in the direction opposite to the direction of $(3/5)\mathbf{i} + (4/5)\mathbf{j}$

Express the vectors in Exercises 9–12 in terms of their lengths and directions.

9. $\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$
10. $-\mathbf{i} - \mathbf{j}$
11. Velocity vector $\mathbf{v} = (-2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j}$ when $t = \pi/2$.
12. Velocity vector $\mathbf{v} = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j}$ when $t = \ln 2$.

Vector Calculations in Three Dimensions

Express the vectors in Exercises 13 and 14 in terms of their lengths and directions.

13. $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$
14. $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
15. Find a vector 2 units long in the direction of $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

16. Find a vector 5 units long in the direction opposite to the direction of $\mathbf{v} = (3/5)\mathbf{i} + (4/5)\mathbf{k}$.

In Exercises 17 and 18, find $|\mathbf{v}|$, $|\mathbf{u}|$, $\mathbf{v} \cdot \mathbf{u}$, $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{v} \times \mathbf{u}$, $\mathbf{u} \times \mathbf{v}$, $|\mathbf{v} \times \mathbf{u}|$, the angle between $\mathbf{v}$ and $\mathbf{u}$, the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$, and the vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

17. $\mathbf{v} = \mathbf{i} + \mathbf{j}$
 $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
18. $\mathbf{v} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$
 $\mathbf{u} = -\mathbf{i} - \mathbf{k}$

In Exercises 19 and 20, write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{v}$ and a vector orthogonal to $\mathbf{v}$.

19. $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$
 $\mathbf{u} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$
20. $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$
 $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

In Exercises 21 and 22, draw coordinate axes and then sketch $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ as vectors at the origin.

21. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$
22. $\mathbf{u} = \mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$
23. If $|\mathbf{v}| = 2$, $|\mathbf{w}| = 3$, and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/3$, find $|\mathbf{v} - 2\mathbf{w}|$.
24. For what value or values of a will the vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} - 8\mathbf{j} + a\mathbf{k}$ be parallel?

In Exercises 25 and 26, find (a) the area of the parallelogram determined by vectors $\mathbf{u}$ and $\mathbf{v}$ and (b) the volume of the parallelepiped determined by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

25. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{w} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$
26. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j}$, $\mathbf{w} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Lines, Planes, and Distances

27. Suppose that $\mathbf{n}$ is normal to a plane and that $\mathbf{v}$ is parallel to the plane. Describe how you would find a vector $\mathbf{n}$ that is both perpendicular to $\mathbf{v}$ and parallel to the plane.

28. Find a vector in the plane parallel to the line $ax + by = c$.

In Exercises 29 and 30, find the distance from the point to the line.

29. $(2, 2, 0)$; $x = -t$, $y = t$, $z = -1 + t$

30. $(0, 4, 1)$; $x = 2 + t$, $y = 2 + t$, $z = t$

31. Parametrize the line that passes through the point $(1, 2, 3)$ parallel to the vector $\mathbf{v} = -3\mathbf{i} + 7\mathbf{k}$.

32. Parametrize the line segment joining the points $P(1, 2, 0)$ and $Q(1, 3, -1)$.

In Exercises 33 and 34, find the distance from the point to the plane.

33. $(6, 0, -6)$, $x - y = 4$

34. $(3, 0, 10)$, $2x + 3y + z = 2$

35. Find an equation for the plane that passes through the point $(3, -2, 1)$ normal to the vector $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

36. Find an equation for the plane that passes through the point $(-1, 6, 0)$ perpendicular to the line $x = -1 + t$, $y = 6 - 2t$, $z = 3t$.

In Exercises 37 and 38, find an equation for the plane through points P , Q , and R .

37. $P(1, -1, 2)$, $Q(2, 1, 3)$, $R(-1, 2, -1)$

38. $P(1, 0, 0)$, $Q(0, 1, 0)$, $R(0, 0, 1)$

39. Find the points in which the line $x = 1 + 2t$, $y = -1 - t$, $z = 3t$ meets the three coordinate planes.

40. Find the point in which the line through the origin perpendicular to the plane $2x - y - z = 4$ meets the plane $3x - 5y + 2z = 6$.

41. Find the acute angle between the planes $x = 7$ and $x + y + \sqrt{2}z = -3$.

42. Find the acute angle between the planes $x + y = 1$ and $y + z = 1$.

43. Find parametric equations for the line in which the planes $x + 2y + z = 1$ and $x - y + 2z = -8$ intersect.

44. Show that the line in which the planes

$$x + 2y - 2z = 5 \quad \text{and} \quad 5x - 2y - z = 0$$

intersect is parallel to the line

$$x = -3 + 2t, \quad y = 3t, \quad z = 1 + 4t.$$

45. The planes $3x + 6z = 1$ and $2x + 2y - z = 3$ intersect in a line.

a. Show that the planes are orthogonal.

b. Find equations for the line of intersection.

46. Find an equation for the plane that passes through the point $(1, 2, 3)$ parallel to $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

47. Is $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ related in any special way to the plane $2x + y = 5$? Give reasons for your answer.

48. The equation $\mathbf{n} \cdot \vec{P_0P} = 0$ represents the plane through P_0 normal to $\mathbf{n}$. What set does the inequality $\mathbf{n} \cdot \vec{P_0P} > 0$ represent?

49. Find the distance from the point $P(1, 4, 0)$ to the plane through $A(0, 0, 0)$, $B(2, 0, -1)$ and $C(2, -1, 0)$.

50. Find the distance from the point $(2, 2, 3)$ to the plane $2x + 3y + 5z = 0$.

51. Find a vector parallel to the plane $2x - y - z = 4$ and orthogonal to $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

52. Find a unit vector orthogonal to $\mathbf{A}$ in the plane of $\mathbf{B}$ and $\mathbf{C}$ if $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, and $\mathbf{C} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

53. Find a vector of magnitude 2 parallel to the line of intersection of the planes $x + 2y + z - 1 = 0$ and $x - y + 2z + 7 = 0$.

54. Find the point in which the line through the origin perpendicular to the plane $2x - y - z = 4$ meets the plane $3x - 5y + 2z = 6$.

55. Find the point in which the line through $P(3, 2, 1)$ normal to the plane $2x - y + 2z = -2$ meets the plane.

56. What angle does the line of intersection of the planes $2x + y - z = 0$ and $x + y + 2z = 0$ make with the positive x -axis?

57. The line

$$L: \quad x = 3 + 2t, \quad y = 2t, \quad z = t$$

intersects the plane $x + 3y - z = -4$ in a point P . Find the coordinates of P and find equations for the line in the plane through P perpendicular to L .

58. Show that for every real number k the plane

$$x - 2y + z + 3 + k(2x - y - z + 1) = 0$$

contains the line of intersection of the planes

$$x - 2y + z + 3 = 0 \quad \text{and} \quad 2x - y - z + 1 = 0.$$

59. Find an equation for the plane through $A(-2, 0, -3)$ and $B(1, -2, 1)$ that lies parallel to the line through $C(-2, -13/5, 26/5)$ and $D(16/5, -13/5, 0)$.

60. Is the line $x = 1 + 2t$, $y = -2 + 3t$, $z = -5t$ related in any way to the plane $-4x - 6y + 10z = 9$? Give reasons for your answer.

61. Which of the following are equations for the plane through the points $P(1, 1, -1)$, $Q(3, 0, 2)$, and $R(-2, 1, 0)$?

a. $(2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \cdot ((x + 2)\mathbf{i} + (y - 1)\mathbf{j} + z\mathbf{k}) = 0$

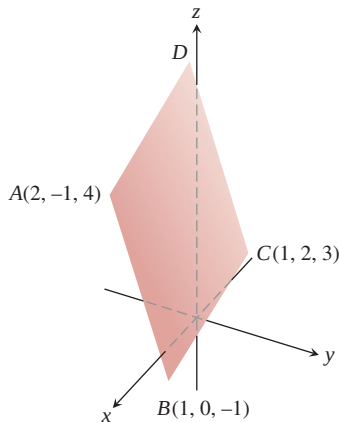
b. $x = 3 - t$, $y = -11t$, $z = 2 - 3t$

c. $(x + 2) + 11(y - 1) = 3z$

d. $(2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \times ((x + 2)\mathbf{i} + (y - 1)\mathbf{j} + z\mathbf{k}) = \mathbf{0}$

e. $(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (-3\mathbf{i} + \mathbf{k}) \cdot ((x + 2)\mathbf{i} + (y - 1)\mathbf{j} + z\mathbf{k}) = 0$

62. The parallelogram shown on page 902 has vertices at $A(2, -1, 4)$, $B(1, 0, -1)$, $C(1, 2, 3)$, and D . Find



- a. the coordinates of D ,
- b. the cosine of the interior angle at B ,
- c. the vector projection of $\vec{BA}$ onto $\vec{BC}$,
- d. the area of the parallelogram,
- e. an equation for the plane of the parallelogram,

f. the areas of the orthogonal projections of the parallelogram on the three coordinate planes.

63. **Distance between lines** Find the distance between the line L_1 through the points $A(1, 0, -1)$ and $B(-1, 1, 0)$ and the line L_2 through the points $C(3, 1, -1)$ and $D(4, 5, -2)$. The distance is to be measured along the line perpendicular to the two lines. First find a vector $\mathbf{n}$ perpendicular to both lines. Then project $\vec{AC}$ onto $\mathbf{n}$.
64. (Continuation of Exercise 63.) Find the distance between the line through $A(4, 0, 2)$ and $B(2, 4, 1)$ and the line through $C(1, 3, 2)$ and $D(2, 2, 4)$.

Quadric Surfaces

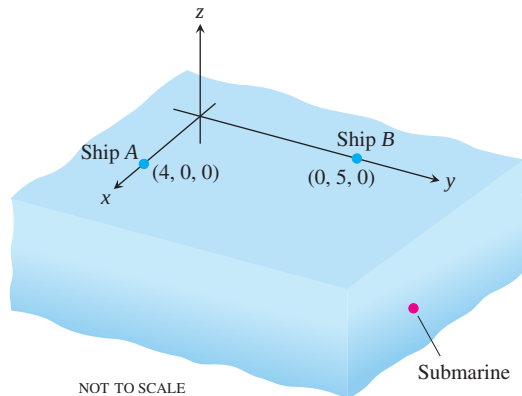
Identify and sketch the surfaces in Exercises 65–76.

65. $x^2 + y^2 + z^2 = 4$	66. $x^2 + (y - 1)^2 + z^2 = 1$
67. $4x^2 + 4y^2 + z^2 = 4$	68. $36x^2 + 9y^2 + 4z^2 = 36$
69. $z = -(x^2 + y^2)$	70. $y = -(x^2 + z^2)$
71. $x^2 + y^2 = z^2$	72. $x^2 + z^2 = y^2$
73. $x^2 + y^2 - z^2 = 4$	74. $4y^2 + z^2 - 4x^2 = 4$
75. $y^2 - x^2 - z^2 = 1$	76. $z^2 - x^2 - y^2 = 1$

Chapter 12

Additional and Advanced Exercises

- 1. Submarine hunting** Two surface ships on maneuvers are trying to determine a submarine's course and speed to prepare for an aircraft intercept. As shown here, ship A is located at $(4, 0, 0)$, whereas ship B is located at $(0, 5, 0)$. All coordinates are given in thousands of feet. Ship A locates the submarine in the direction of the vector $2\mathbf{i} + 3\mathbf{j} - (1/3)\mathbf{k}$, and ship B locates it in the direction of the vector $18\mathbf{i} - 6\mathbf{j} - \mathbf{k}$. Four minutes ago, the submarine was located at $(2, -1, -1/3)$. The aircraft is due in 20 min. Assuming that the submarine moves in a straight line at a constant speed, to what position should the surface ships direct the aircraft?



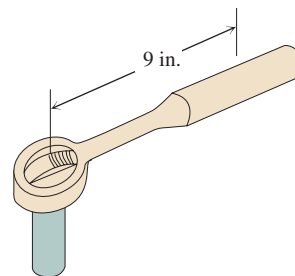
- 2. A helicopter rescue** Two helicopters, H_1 and H_2 , are traveling together. At time $t = 0$, they separate and follow different straight-line paths given by

$$H_1: x = 6 + 40t, \quad y = -3 + 10t, \quad z = -3 + 2t$$

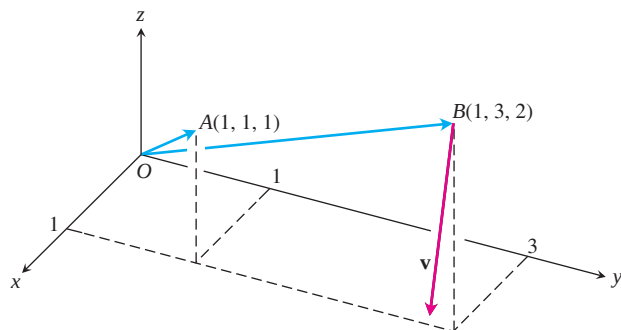
$$H_2: x = 6 + 110t, \quad y = -3 + 4t, \quad z = -3 + t.$$

Time t is measured in hours and all coordinates are measured in miles. Due to system malfunctions, H_2 stops its flight at $(446, 13, 1)$ and, in a negligible amount of time, lands at $(446, 13, 0)$. Two hours later, H_1 is advised of this fact and heads toward H_2 at 150 mph. How long will it take H_1 to reach H_2 ?

- 3. Torque** The operator's manual for the Toro[®] 21 in. lawnmower says "tighten the spark plug to 15 ft-lb ($20.4 \text{ N} \cdot \text{m}$).'" If you are installing the plug with a 10.5-in. socket wrench that places the center of your hand 9 in. from the axis of the spark plug, about how hard should you pull? Answer in pounds.



4. **Rotating body** The line through the origin and the point $A(1, 1, 1)$ is the axis of rotation of a right body rotating with a constant angular speed of $3/2$ rad/sec. The rotation appears to be clockwise when we look toward the origin from A . Find the velocity $\mathbf{v}$ of the point of the body that is at the position $B(1, 3, 2)$.



5. Determinants and planes

- a. Show that

$$\begin{vmatrix} x_1 - x & y_1 - y & z_1 - z \\ x_2 - x & y_2 - y & z_2 - z \\ x_3 - x & y_3 - y & z_3 - z \end{vmatrix} = 0$$

is an equation for the plane through the three noncollinear points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$.

- b. What set of points in space is described by the equation

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0?$$

6. Determinants and lines

Show that the lines

$$x = a_1s + b_1, y = a_2s + b_2, z = a_3s + b_3, -\infty < s < \infty,$$

and

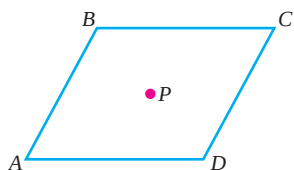
$$x = c_1t + d_1, y = c_2t + d_2, z = c_3t + d_3, -\infty < t < \infty,$$

intersect or are parallel if and only if

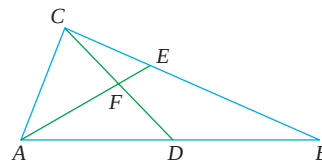
$$\begin{vmatrix} a_1 & c_1 & b_1 - d_1 \\ a_2 & c_2 & b_2 - d_2 \\ a_3 & c_3 & b_3 - d_3 \end{vmatrix} = 0.$$

7. **Parallelogram** The accompanying figure shows parallelogram $ABCD$ and the midpoint P of diagonal BD .

- Express $\vec{BD}$ in terms of $\vec{AB}$ and $\vec{AD}$.
- Express $\vec{AP}$ in terms of $\vec{AB}$ and $\vec{AD}$.
- Prove that P is also the midpoint of diagonal AC .



8. In the figure here, D is the midpoint of side AB of triangle ABC , and E is one-third of the way between C and B . Use vectors to prove that F is the midpoint of line segment CD .



9. Use vectors to show that the distance from $P_1(x_1, y_1)$ to the line $ax + by = c$ is

$$d = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}.$$

10. a. Use vectors to show that the distance from $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz = D$ is

$$d = \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

- b. Find an equation for the sphere that is tangent to the planes $x + y + z = 3$ and $x + y + z = 9$ if the planes $2x - y = 0$ and $3x - z = 0$ pass through the center of the sphere.

11. a. Show that the distance between the parallel planes $Ax + By + Cz = D_1$ and $Ax + By + Cz = D_2$ is

$$d = \frac{|D_1 - D_2|}{|A\mathbf{i} + B\mathbf{j} + C\mathbf{k}|}.$$

- b. Find the distance between the planes $2x + 3y - z = 6$ and $2x + 3y - z = 12$.

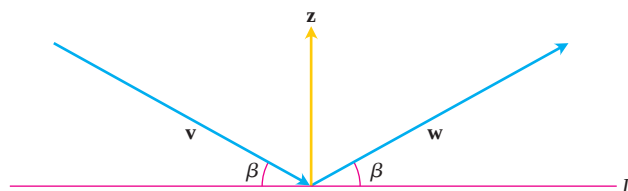
- c. Find an equation for the plane parallel to the plane $2x - y + 2z = -4$ if the point $(3, 2, -1)$ is equidistant from the two planes.

- d. Write equations for the planes that lie parallel to and 5 units away from the plane $x - 2y + z = 3$.

12. Prove that four points A , B , C , and D are coplanar (lie in a common plane) if and only if $\vec{AD} \cdot (\vec{AB} \times \vec{BC}) = 0$.

13. **The projection of a vector on a plane** Let P be a plane in space and let $\mathbf{v}$ be a vector. The vector projection of $\mathbf{v}$ onto the plane P , $\text{proj}_P \mathbf{v}$, can be defined informally as follows. Suppose the sun is shining so that its rays are normal to the plane P . Then $\text{proj}_P \mathbf{v}$ is the “shadow” of $\mathbf{v}$ onto P . If P is the plane $x + 2y + 6z = 6$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, find $\text{proj}_P \mathbf{v}$.

14. The accompanying figure shows nonzero vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$, with $\mathbf{z}$ orthogonal to the line L , and $\mathbf{v}$ and $\mathbf{w}$ making equal angles β with L . Assuming $|\mathbf{v}| = |\mathbf{w}|$, find $\mathbf{w}$ in terms of $\mathbf{v}$ and $\mathbf{z}$.



- 15. Triple vector products** The *triple vector products* $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ are usually not equal, although the formulas for evaluating them from components are similar:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}.$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

Verify each formula for the following vectors by evaluating its two sides and comparing the results.

$\mathbf{u}$	$\mathbf{v}$	$\mathbf{w}$
a. $2\mathbf{i}$	$2\mathbf{j}$	$2\mathbf{k}$
b. $\mathbf{i} - \mathbf{j} + \mathbf{k}$	$2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
c. $2\mathbf{i} + \mathbf{j}$	$2\mathbf{i} - \mathbf{j} + \mathbf{k}$	$\mathbf{i} + 2\mathbf{k}$
d. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$	$-\mathbf{i} - \mathbf{k}$	$2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

- 16. Cross and dot products** Show that if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{r}$ are any vectors, then

a. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$

b. $\mathbf{u} \times \mathbf{v} = (\mathbf{u} \cdot \mathbf{v} \times \mathbf{i})\mathbf{i} + (\mathbf{u} \cdot \mathbf{v} \times \mathbf{j})\mathbf{j} + (\mathbf{u} \cdot \mathbf{v} \times \mathbf{k})\mathbf{k}$

c. $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{r}) = \begin{vmatrix} \mathbf{u} \cdot \mathbf{w} & \mathbf{v} \cdot \mathbf{w} \\ \mathbf{u} \cdot \mathbf{r} & \mathbf{v} \cdot \mathbf{r} \end{vmatrix}.$

- 17. Cross and dot products** Prove or disprove the formula

$$\mathbf{u} \times (\mathbf{u} \times (\mathbf{u} \times \mathbf{v})) \cdot \mathbf{w} = -|\mathbf{u}|^2 \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}.$$

- 18.** By forming the cross product of two appropriate vectors, derive the trigonometric identity

$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

- 19.** Use vectors to prove that

$$(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$$

for any four numbers a , b , c , and d . (Hint: Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$.)

- 20.** Suppose that vectors $\mathbf{u}$ and $\mathbf{v}$ are not parallel and that $\mathbf{u} = \mathbf{w} + \mathbf{r}$, where $\mathbf{w}$ is parallel to $\mathbf{v}$ and $\mathbf{r}$ is orthogonal to $\mathbf{v}$. Express $\mathbf{w}$ and $\mathbf{r}$ in terms of $\mathbf{u}$ and $\mathbf{v}$.

- 21.** Show that $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

- 22.** Show that $\mathbf{w} = |\mathbf{v}||\mathbf{u}| + |\mathbf{u}||\mathbf{v}|$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

- 23.** Show that $|\mathbf{v}||\mathbf{u}| + |\mathbf{u}||\mathbf{v}|$ and $|\mathbf{v}||\mathbf{u}| - |\mathbf{u}||\mathbf{v}|$ are orthogonal.

- 24. Dot multiplication is positive definite** Show that dot multiplication of vectors is *positive definite*; that is, show that $\mathbf{u} \cdot \mathbf{u} \geq 0$ for every vector $\mathbf{u}$ and that $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

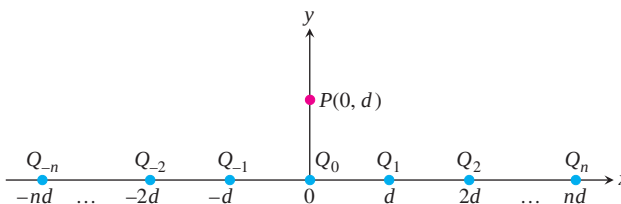
- 25. Point masses and gravitation** In physics, the law of gravitation says that if P and Q are (point) masses with mass M and m , respectively, then P is attracted to Q by the force

$$\mathbf{F} = \frac{GMm\mathbf{r}}{|\mathbf{r}|^3},$$

where $\mathbf{r}$ is the vector from P to Q and G is the universal gravitational constant. Moreover, if $Q_1, \dots, Q_k$ are (point) masses with mass $m_1, \dots, m_k$, respectively, then the force on P due to all the Q_i 's is

$$\mathbf{F} = \sum_{i=1}^k \frac{GMm_i}{|\mathbf{r}_i|^3} \mathbf{r}_i,$$

where $\mathbf{r}_i$ is the vector from P to Q_i .



- a. Let point P with mass M be located at the point $(0, d)$, $d > 0$, in the coordinate plane. For $i = -n, -n + 1, \dots, -1, 0, 1, \dots, n$, let Q_i be located at the point $(id, 0)$ and have mass mi . Find the magnitude of the gravitational force on P due to all the Q_i 's.

- b. Is the limit as $n \rightarrow \infty$ of the magnitude of the force on P finite? Why, or why not?

- 26. Relativistic sums** Einstein's special theory of relativity roughly says that with respect to a reference frame (coordinate system) no material object can travel as fast as c , the speed of light. So, if $\vec{x}$ and $\vec{y}$ are two velocities such that $|\vec{x}| < c$ and $|\vec{y}| < c$, then the *relativistic sum* $\vec{x} \oplus \vec{y}$ of $\vec{x}$ and $\vec{y}$ must have length less than c . Einstein's special theory of relativity says that

$$\vec{x} \oplus \vec{y} = \frac{\vec{x} + \vec{y}}{1 + \frac{\vec{x} \cdot \vec{y}}{c^2}} + \frac{1}{c^2} \cdot \frac{\gamma_x}{\gamma_x + 1} \cdot \frac{\vec{x} \times (\vec{x} \times \vec{y})}{1 + \frac{\vec{x} \cdot \vec{y}}{c^2}},$$

where

$$\gamma_x = \frac{1}{\sqrt{1 - \frac{\vec{x} \cdot \vec{x}}{c^2}}}.$$

It can be shown that if $|\vec{x}| < c$ and $|\vec{y}| < c$, then $|\vec{x} \oplus \vec{y}| < c$. This exercise deals with two special cases.

- a. Prove that if $\vec{x}$ and $\vec{y}$ are orthogonal, $|\vec{x}| < c$, $|\vec{y}| < c$, then $|\vec{x} \oplus \vec{y}| < c$.
- b. Prove that if $\vec{x}$ and $\vec{y}$ are parallel, $|\vec{x}| < c$, $|\vec{y}| < c$, then $|\vec{x} \oplus \vec{y}| < c$.
- c. Compute $\lim_{c \rightarrow \infty} \vec{x} \oplus \vec{y}$.

Chapter 12 Technology Application Projects

Mathematica/Maple Module

Using Vectors to Represent Lines and Find Distances

Parts I and II: Learn the advantages of interpreting lines as vectors.

Part III: Use vectors to find the distance from a point to a line.

Mathematica/Maple Module

Putting a Scene in Three Dimensions onto a Two-Dimensional Canvas

Use the concept of planes in space to obtain a two-dimensional image.

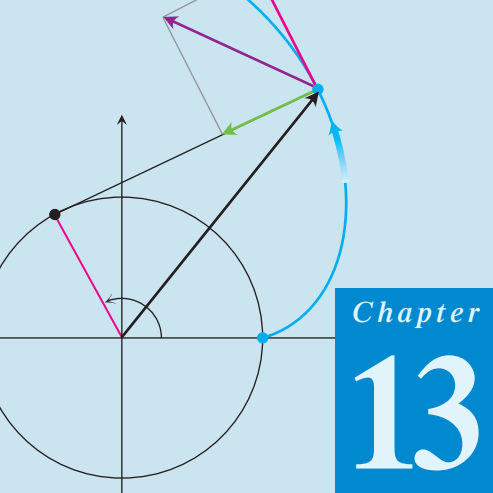
Mathematica/Maple Module

Getting Started in Plotting in 3D

Part I: Use the vector definition of lines and planes to generate graphs and equations, and to compare different forms for the equations of a single line.

Part II: Plot functions that are defined implicitly.





Chapter

13

VECTOR-VALUED FUNCTIONS AND MOTION IN SPACE

OVERVIEW When a body (or object) travels through space, the equations $x = f(t)$, $y = g(t)$, and $z = h(t)$ that give the body's coordinates as functions of time serve as parametric equations for the body's motion and path. With vector notation, we can condense these into a single equation $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ that gives the body's position as a vector function of time. For an object moving in the xy -plane, the component function $h(t)$ is zero for all time (that is, identically zero).

In this chapter, we use calculus to study the paths, velocities, and accelerations of moving bodies. As we go along, we will see how our work answers the standard questions about the paths and motions of projectiles, planets, and satellites. In the final section, we use our new vector calculus to derive Kepler's laws of planetary motion from Newton's laws of motion and gravitation.

13.1

Vector Functions



When a particle moves through space during a time interval I , we think of the particle's coordinates as functions defined on I :

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I. \quad (1)$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the **curve** in space that we call the particle's **path**. The equations and interval in Equation (1) **parametrize** the curve. A curve in space can also be represented in vector form. The vector

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad (2)$$

from the origin to the particle's **position** $P(f(t), g(t), h(t))$ at time t is the particle's **position vector** (Figure 13.1). The functions f , g , and h are the **component functions (components)** of the position vector. We think of the particle's path as the **curve traced by $\mathbf{r}$** during the time interval I . Figure 13.2 displays several space curves generated by a computer graphing program. It would not be easy to plot these curves by hand.

Equation (2) defines $\mathbf{r}$ as a vector function of the real variable t on the interval I . More generally, a **vector function** or **vector-valued function** on a domain set D is a rule that assigns a vector in space to each element in D . For now, the domains will be intervals of real numbers resulting in a space curve. Later, in Chapter 16, the domains will be regions

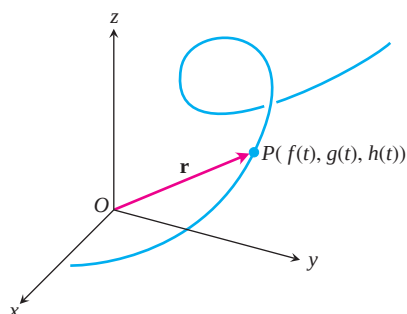


FIGURE 13.1 The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

in the plane. Vector functions will then represent surfaces in space. Vector functions on a domain in the plane or space also give rise to “vector fields,” which are important to the study of the flow of a fluid, gravitational fields, and electromagnetic phenomena. We investigate vector fields and their applications in Chapter 16.

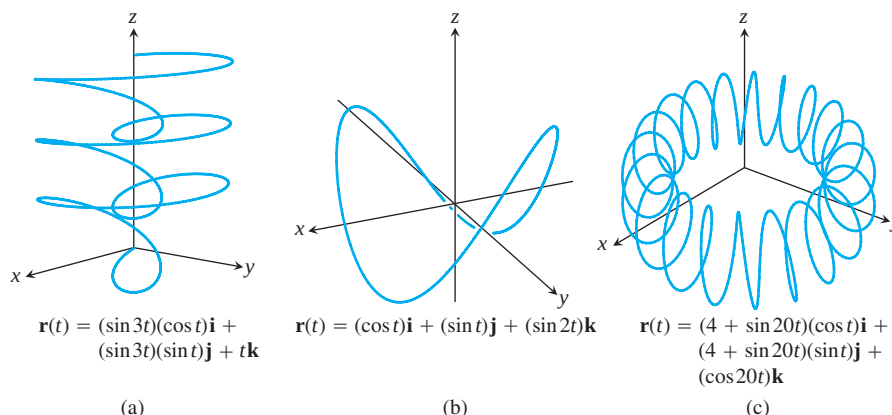


FIGURE 13.2 Computer-generated space curves are defined by the position vectors $\mathbf{r}(t)$.

We refer to real-valued functions as **scalar functions** to distinguish them from vector functions. The components of $\mathbf{r}$ are scalar functions of t . When we define a vector-valued function by giving its component functions, we assume the vector function's domain to be the common domain of the components.

EXAMPLE 1 Graphing a Helix

Graph the vector function

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

Solution The vector function

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

is defined for all real values of t . The curve traced by $\mathbf{r}$ is a helix (from an old Greek word for “spiral”) that winds around the circular cylinder $x^2 + y^2 = 1$ (Figure 13.3). The curve lies on the cylinder because the $\mathbf{i}$ - and $\mathbf{j}$ -components of $\mathbf{r}$, being the x - and y -coordinates of the tip of $\mathbf{r}$, satisfy the cylinder's equation:

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1.$$

The curve rises as the $\mathbf{k}$ -component $z = t$ increases. Each time t increases by 2π , the curve completes one turn around the cylinder. The equations

$$x = \cos t, \quad y = \sin t, \quad z = t$$

parametrize the helix, the interval $-\infty < t < \infty$ being understood. You will find more helices in Figure 13.4. ■

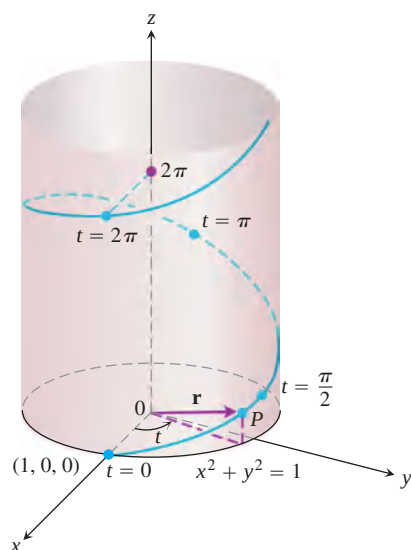


FIGURE 13.3 The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ (Example 1).

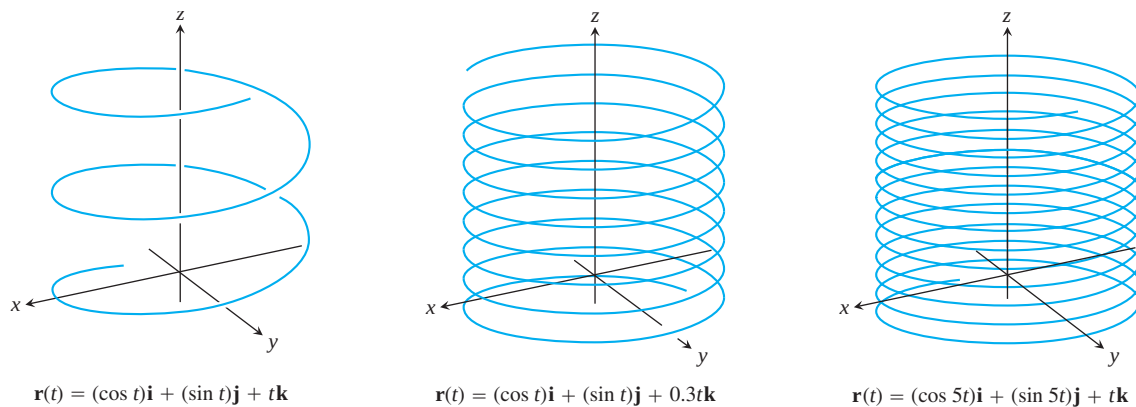


FIGURE 13.4 Helices drawn by computer.

Limits and Continuity

The way we define limits of vector-valued functions is similar to the way we define limits of real-valued functions.

DEFINITION Limit of Vector Functions

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has **limit** $\mathbf{L}$ as t approaches t_0 and write

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all t

$$0 < |t - t_0| < \delta \quad \Rightarrow \quad |\mathbf{r}(t) - \mathbf{L}| < \epsilon.$$

If $\mathbf{L} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}$, then $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$ precisely when

$$\lim_{t \rightarrow t_0} f(t) = L_1, \quad \lim_{t \rightarrow t_0} g(t) = L_2, \quad \text{and} \quad \lim_{t \rightarrow t_0} h(t) = L_3.$$

The equation

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \left(\lim_{t \rightarrow t_0} f(t) \right) \mathbf{i} + \left(\lim_{t \rightarrow t_0} g(t) \right) \mathbf{j} + \left(\lim_{t \rightarrow t_0} h(t) \right) \mathbf{k} \quad (3)$$

provides a practical way to calculate limits of vector functions.

EXAMPLE 2 Finding Limits of Vector Functions

If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\begin{aligned} \lim_{t \rightarrow \pi/4} \mathbf{r}(t) &= \left(\lim_{t \rightarrow \pi/4} \cos t \right) \mathbf{i} + \left(\lim_{t \rightarrow \pi/4} \sin t \right) \mathbf{j} + \left(\lim_{t \rightarrow \pi/4} t \right) \mathbf{k} \\ &= \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} + \frac{\pi}{4} \mathbf{k}. \end{aligned}$$

We define continuity for vector functions the same way we define continuity for scalar functions. ■

DEFINITION Continuous at a Point

A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

From Equation (3), we see that $\mathbf{r}(t)$ is continuous at $t = t_0$ if and only if each component function is continuous there.

EXAMPLE 3 Continuity of Space Curves

- (a) All the space curves shown in Figures 13.2 and 13.4 are continuous because their component functions are continuous at every value of t in $(-\infty, \infty)$.
 (b) The function

$$\mathbf{g}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \lfloor t \rfloor \mathbf{k}$$

is discontinuous at every integer, where the greatest integer function $\lfloor t \rfloor$ is discontinuous. ■

Derivatives and Motion

Suppose that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is the position vector of a particle moving along a curve in space and that f , g , and h are differentiable functions of t . Then the difference between the particle's positions at time t and time $t + \Delta t$ is

$$\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

(Figure 13.5a). In terms of components,

$$\begin{aligned} \Delta \mathbf{r} &= \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \\ &= [f(t + \Delta t)\mathbf{i} + g(t + \Delta t)\mathbf{j} + h(t + \Delta t)\mathbf{k}] \\ &\quad - [f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}] \\ &= [f(t + \Delta t) - f(t)]\mathbf{i} + [g(t + \Delta t) - g(t)]\mathbf{j} + [h(t + \Delta t) - h(t)]\mathbf{k}. \end{aligned}$$

As Δt approaches zero, three things seem to happen simultaneously. First, Q approaches P along the curve. Second, the secant line PQ seems to approach a limiting position tangent to the curve at P . Third, the quotient $\Delta \mathbf{r} / \Delta t$ (Figure 13.5b) approaches the limit

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} &= \left[\lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \right] \mathbf{i} + \left[\lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t} \right] \mathbf{j} \\ &\quad + \left[\lim_{\Delta t \rightarrow 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \right] \mathbf{k} \\ &= \left[\frac{df}{dt} \right] \mathbf{i} + \left[\frac{dg}{dt} \right] \mathbf{j} + \left[\frac{dh}{dt} \right] \mathbf{k}. \end{aligned}$$

We are therefore led by past experience to the following definition.

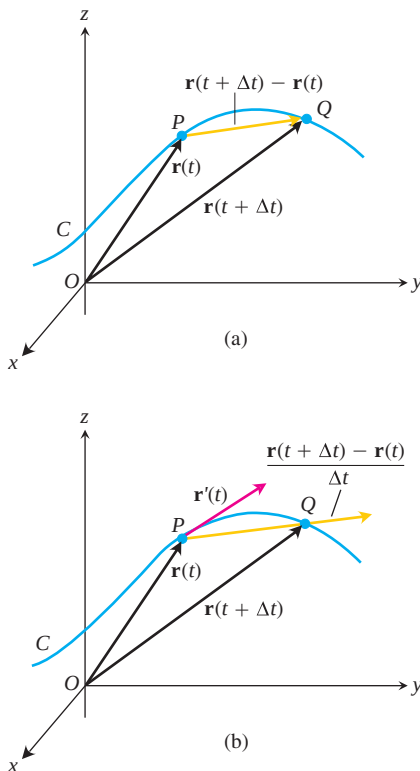


FIGURE 13.5 As $\Delta t \rightarrow 0$, the point Q approaches the point P along the curve C . In the limit, the vector $\overrightarrow{PQ} / \Delta t$ becomes the tangent vector $\mathbf{r}'(t)$.

DEFINITION Derivative

The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at t** if f , g , and h have derivatives at t . The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

A vector function $\mathbf{r}$ is **differentiable** if it is differentiable at every point of its domain. The curve traced by $\mathbf{r}$ is **smooth** if $d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$, that is, if f , g , and h have continuous first derivatives that are not simultaneously 0.

The geometric significance of the definition of derivative is shown in Figure 13.5. The points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + \Delta t)$, and the vector $\overrightarrow{PQ}$ is represented by $\mathbf{r}(t + \Delta t) - \mathbf{r}(t)$. For $\Delta t > 0$, the scalar multiple $(1/\Delta t)(\mathbf{r}(t + \Delta t) - \mathbf{r}(t))$ points in the same direction as the vector $\overrightarrow{PQ}$. As $\Delta t \rightarrow 0$, this vector approaches a vector that is tangent to the curve at P (Figure 13.5b). The vector $\mathbf{r}'(t)$, when different from $\mathbf{0}$, is defined to be the vector **tangent** to the curve at P . The **tangent line** to the curve at a point $(f(t_0), g(t_0), h(t_0))$ is defined to be the line through the point parallel to $\mathbf{r}'(t_0)$. We require $d\mathbf{r}/dt \neq \mathbf{0}$ for a smooth curve to make sure the curve has a continuously turning tangent at each point. On a smooth curve, there are no sharp corners or cusps.

A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piecewise smooth** (Figure 13.6).

Look once again at Figure 13.5. We drew the figure for Δt positive, so $\Delta \mathbf{r}$ points forward, in the direction of the motion. The vector $\Delta \mathbf{r}/\Delta t$, having the same direction as $\Delta \mathbf{r}$, points forward too. Had Δt been negative, $\Delta \mathbf{r}$ would have pointed backward, against the direction of motion. The quotient $\Delta \mathbf{r}/\Delta t$, however, being a negative scalar multiple of $\Delta \mathbf{r}$, would once again have pointed forward. No matter how $\Delta \mathbf{r}$ points, $\Delta \mathbf{r}/\Delta t$ points forward and we expect the vector $d\mathbf{r}/dt = \lim_{\Delta t \rightarrow 0} \Delta \mathbf{r}/\Delta t$, when different from $\mathbf{0}$, to do the same. This means that the derivative $d\mathbf{r}/dt$ is just what we want for modeling a particle's velocity. It points in the direction of motion and gives the rate of change of position with respect to time. For a smooth curve, the velocity is never zero; the particle does not stop or reverse direction.

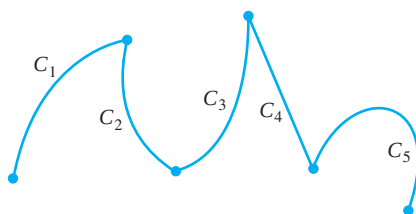


FIGURE 13.6 A piecewise smooth curve made up of five smooth curves connected end to end in continuous fashion.

**DEFINITIONS** Velocity, Direction, Speed, Acceleration

If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t , the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \frac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: $\text{Speed} = |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t .

We can express the velocity of a moving particle as the product of its speed and direction:

$$\text{Velocity} = |\mathbf{v}| \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) = (\text{speed})(\text{direction}).$$

In Section 12.5, Example 4 we found this expression for velocity useful in locating, for example, the position of a helicopter moving along a straight line in space. Now let's look at an example of an object moving along a (nonlinear) space curve.

EXAMPLE 4 Flight of a Hang Glider

A person on a hang glider is spiraling upward due to rapidly rising air on a path having position vector $\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t^2\mathbf{k}$. The path is similar to that of a helix (although it's *not* a helix, as you will see in Section 13.4) and is shown in Figure 13.7 for $0 \leq t \leq 4\pi$. Find

- the velocity and acceleration vectors,
- the glider's speed at any time t ,
- the times, if any, when the glider's acceleration is orthogonal to its velocity.

Solution

(a) $\mathbf{r} = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t^2\mathbf{k}$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -(3 \cos t)\mathbf{i} - (3 \sin t)\mathbf{j} + 2\mathbf{k}$$

- (b) Speed is the magnitude of $\mathbf{v}$:

$$\begin{aligned} |\mathbf{v}(t)| &= \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (2t)^2} \\ &= \sqrt{9 \sin^2 t + 9 \cos^2 t + 4t^2} \\ &= \sqrt{9 + 4t^2}. \end{aligned}$$

The glider is moving faster and faster as it rises along its path.

- (c) To find the times when $\mathbf{v}$ and $\mathbf{a}$ are orthogonal, we look for values of t for which

$$\mathbf{v} \cdot \mathbf{a} = 9 \sin t \cos t - 9 \cos t \sin t + 4t = 4t = 0.$$

Thus, the only time the acceleration vector is orthogonal to $\mathbf{v}$ is when $t = 0$. We study acceleration for motions along paths in more detail in Section 13.5. There we discover how the acceleration vector reveals the curving nature and tendency of the path to “twist” out of a certain plane containing the velocity vector. ■

Differentiation Rules

Because the derivatives of vector functions may be computed component by component, the rules for differentiating vector functions have the same form as the rules for differentiating scalar functions.

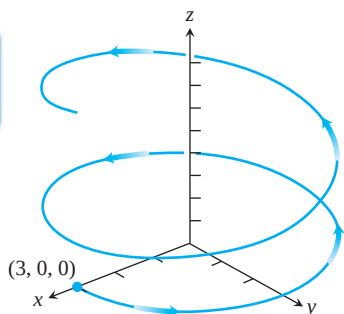


FIGURE 13.7 The path of a hang glider with position vector $\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t^2\mathbf{k}$ (Example 4).



Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t , $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt} \mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt} [c\mathbf{u}(t)] = c\mathbf{u}'(t)$
 $\frac{d}{dt} [f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt} [\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt} [\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt} [\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

When you use the Cross Product Rule, remember to preserve the order of the factors. If $\mathbf{u}$ comes first on the left side of the equation, it must also come first on the right or the signs will be wrong.

We will prove the product rules and Chain Rule but leave the rules for constants, scalar multiples, sums, and differences as exercises.

Proof of the Dot Product Rule Suppose that

$$\mathbf{u} = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$$

and

$$\mathbf{v} = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}.$$

Then

$$\begin{aligned} \frac{d}{dt} (\mathbf{u} \cdot \mathbf{v}) &= \frac{d}{dt} (u_1 v_1 + u_2 v_2 + u_3 v_3) \\ &= \underbrace{u'_1 v_1 + u'_2 v_2 + u'_3 v_3}_{\mathbf{u}' \cdot \mathbf{v}} + \underbrace{u_1 v'_1 + u_2 v'_2 + u_3 v'_3}_{\mathbf{u} \cdot \mathbf{v}'} \end{aligned}$$

Proof of the Cross Product Rule We model the proof after the proof of the Product Rule for scalar functions. According to the definition of derivative,

$$\frac{d}{dt} (\mathbf{u} \times \mathbf{v}) = \lim_{h \rightarrow 0} \frac{\mathbf{u}(t+h) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t)}{h}.$$

To change this fraction into an equivalent one that contains the difference quotients for the derivatives of $\mathbf{u}$ and $\mathbf{v}$, we subtract and add $\mathbf{u}(t) \times \mathbf{v}(t + h)$ in the numerator. Then

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= \lim_{h \rightarrow 0} \frac{\mathbf{u}(t+h) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t+h) + \mathbf{u}(t) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} \times \mathbf{v}(t+h) + \mathbf{u}(t) \times \frac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} \times \lim_{h \rightarrow 0} \mathbf{v}(t+h) + \lim_{h \rightarrow 0} \mathbf{u}(t) \times \lim_{h \rightarrow 0} \frac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h}. \end{aligned}$$

The last of these equalities holds because the limit of the cross product of two vector functions is the cross product of their limits if the latter exist (Exercise 52). As h approaches zero, $\mathbf{v}(t+h)$ approaches $\mathbf{v}(t)$ because $\mathbf{v}$, being differentiable at t , is continuous at t (Exercise 53). The two fractions approach the values of $d\mathbf{u}/dt$ and $d\mathbf{v}/dt$ at t . In short,

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt}. \quad \blacksquare$$

As an algebraic convenience, we sometimes write the product of a scalar c and a vector $\mathbf{v}$ as $c\mathbf{v}$ instead of $\mathbf{v}c$. This permits us, for instance, to write the Chain Rule in a familiar form:

$$\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u}}{ds} \frac{ds}{dt},$$

where $s = f(t)$.

Proof of the Chain Rule Suppose that $\mathbf{u}(s) = a(s)\mathbf{i} + b(s)\mathbf{j} + c(s)\mathbf{k}$ is a differentiable vector function of s and that $s = f(t)$ is a differentiable scalar function of t . Then a , b , and c are differentiable functions of t , and the Chain Rule for differentiable real-valued functions gives

$$\begin{aligned} \frac{d}{dt}[\mathbf{u}(s)] &= \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k} \\ &= \frac{da}{ds} \frac{ds}{dt}\mathbf{i} + \frac{db}{ds} \frac{ds}{dt}\mathbf{j} + \frac{dc}{ds} \frac{ds}{dt}\mathbf{k} \\ &= \frac{ds}{dt} \left(\frac{da}{ds}\mathbf{i} + \frac{db}{ds}\mathbf{j} + \frac{dc}{ds}\mathbf{k} \right) \\ &= \frac{ds}{dt} \frac{d\mathbf{u}}{ds} \\ &= f'(t)\mathbf{u}'(f(t)). \end{aligned} \quad s = f(t) \quad \blacksquare$$

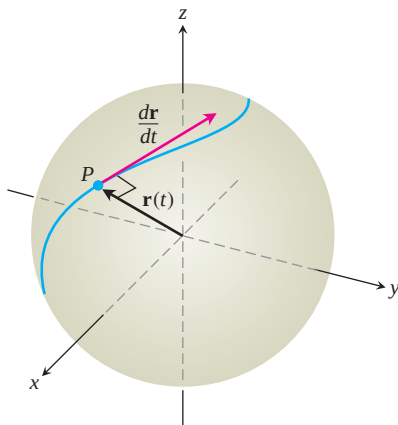


FIGURE 13.8 If a particle moves on a sphere in such a way that its position $\mathbf{r}$ is a differentiable function of time, then $\mathbf{r} \cdot (d\mathbf{r}/dt) = 0$.

Vector Functions of Constant Length

When we track a particle moving on a sphere centered at the origin (Figure 13.8), the position vector has a constant length equal to the radius of the sphere. The velocity vector $d\mathbf{r}/dt$, tangent to the path of motion, is tangent to the sphere and hence perpendicular to $\mathbf{r}$. This is always the case for a differentiable vector function of constant length: The vector and its first derivative are orthogonal. With the length constant, the change in the function is a change in direction only, and direction changes take place at right angles. We can also obtain this result by direct calculation:

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = c^2 \quad |\mathbf{r}(t)| = c \text{ is constant.}$$

$$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = 0 \quad \text{Differentiate both sides.}$$

$$\mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \quad \text{Rule 5 with } \mathbf{r}(t) = \mathbf{u}(t) = \mathbf{v}(t)$$

$$2\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0.$$

The vectors $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ are orthogonal because their dot product is 0. In summary,

If $\mathbf{r}$ is a differentiable vector function of t of constant length, then

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0. \quad (4)$$

We will use this observation repeatedly in Section 13.4.

EXAMPLE 5 Supporting Equation (4)

Show that $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \sqrt{3}\mathbf{k}$ has constant length and is orthogonal to its derivative.

Solution

$$\begin{aligned} \mathbf{r}(t) &= (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \sqrt{3}\mathbf{k} \\ |\mathbf{r}(t)| &= \sqrt{(\sin t)^2 + (\cos t)^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2 \\ \frac{d\mathbf{r}}{dt} &= (\cos t)\mathbf{i} - (\sin t)\mathbf{j} \\ \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} &= \sin t \cos t - \sin t \cos t = 0 \end{aligned}$$

Integrals of Vector Functions

A differentiable vector function $\mathbf{R}(t)$ is an **antiderivative** of a vector function $\mathbf{r}(t)$ on an interval I if $d\mathbf{R}/dt = \mathbf{r}$ at each point of I . If $\mathbf{R}$ is an antiderivative of $\mathbf{r}$ on I , it can be shown, working one component at a time, that every antiderivative of $\mathbf{r}$ on I has the form $\mathbf{R} + \mathbf{C}$ for some constant vector $\mathbf{C}$ (Exercise 56). The set of all antiderivatives of $\mathbf{r}$ on I is the **indefinite integral** of $\mathbf{r}$ on I .

DEFINITION Indefinite Integral

The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t) dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C}.$$

The usual arithmetic rules for indefinite integrals apply.

EXAMPLE 6 Finding Indefinite Integrals

$$\int ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k}) dt = \left(\int \cos t dt \right) \mathbf{i} + \left(\int dt \right) \mathbf{j} - \left(\int 2t dt \right) \mathbf{k} \quad (5)$$

$$= (\sin t + C_1)\mathbf{i} + (t + C_2)\mathbf{j} - (t^2 + C_3)\mathbf{k} \quad (6)$$

$$= (\sin t)\mathbf{i} + t\mathbf{j} - t^2\mathbf{k} + \mathbf{C} \quad \mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} - C_3\mathbf{k}$$

As in the integration of scalar functions, we recommend that you skip the steps in Equations (5) and (6) and go directly to the final form. Find an antiderivative for each component and add a constant vector at the end. ■

Definite integrals of vector functions are best defined in terms of components.

DEFINITION Definite Integral

If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}.$$



EXAMPLE 7 Evaluating Definite Integrals

$$\begin{aligned} \int_0^\pi ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k}) dt &= \left(\int_0^\pi \cos t dt \right) \mathbf{i} + \left(\int_0^\pi dt \right) \mathbf{j} - \left(\int_0^\pi 2t dt \right) \mathbf{k} \\ &= [\sin t]_0^\pi \mathbf{i} + [t]_0^\pi \mathbf{j} - [t^2]_0^\pi \mathbf{k} \\ &= [0 - 0]\mathbf{i} + [\pi - 0]\mathbf{j} - [\pi^2 - 0^2]\mathbf{k} \\ &= \pi\mathbf{j} - \pi^2\mathbf{k} \end{aligned}$$

The Fundamental Theorem of Calculus for continuous vector functions says that

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(t) \Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, so that $\mathbf{R}'(t) = \mathbf{r}(t)$ (Exercise 57).



EXAMPLE 8 Revisiting the Flight of a Glider

Suppose that we did not know the path of the glider in Example 4, but only its acceleration vector $\mathbf{a}(t) = -(3 \cos t)\mathbf{i} - (3 \sin t)\mathbf{j} + 2\mathbf{k}$. We also know that initially (at time $t = 0$), the glider departed from the point $(3, 0, 0)$ with velocity $\mathbf{v}(0) = 3\mathbf{j}$. Find the glider's position as a function of t .

Solution Our goal is to find $\mathbf{r}(t)$ knowing

The differential equation: $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -(3 \cos t)\mathbf{i} - (3 \sin t)\mathbf{j} + 2\mathbf{k}$

The initial conditions: $\mathbf{v}(0) = 3\mathbf{j}$ and $\mathbf{r}(0) = 3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$.

Integrating both sides of the differential equation with respect to t gives

$$\mathbf{v}(t) = -(3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 2t\mathbf{k} + \mathbf{C}_1.$$

We use $\mathbf{v}(0) = 3\mathbf{j}$ to find $\mathbf{C}_1$:

$$3\mathbf{j} = -(3 \sin 0)\mathbf{i} + (3 \cos 0)\mathbf{j} + (0)\mathbf{k} + \mathbf{C}_1$$

$$3\mathbf{j} = 3\mathbf{j} + \mathbf{C}_1$$

$$\mathbf{C}_1 = \mathbf{0}.$$

The glider's velocity as a function of time is

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = -(3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 2t\mathbf{k}.$$

Integrating both sides of this last differential equation gives

$$\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t^2\mathbf{k} + \mathbf{C}_2.$$

We then use the initial condition $\mathbf{r}(0) = 3\mathbf{i}$ to find $\mathbf{C}_2$:

$$3\mathbf{i} = (3 \cos 0)\mathbf{i} + (3 \sin 0)\mathbf{j} + (0^2)\mathbf{k} + \mathbf{C}_2$$

$$3\mathbf{i} = 3\mathbf{i} + (0)\mathbf{j} + (0)\mathbf{k} + \mathbf{C}_2$$

$$\mathbf{C}_2 = \mathbf{0}.$$

The glider's position as a function of t is

$$\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t^2\mathbf{k}.$$

This is the path of the glider we know from Example 4 and is shown in Figure 13.7.

Note: It was peculiar to this example that both of the constant vectors of integration, $\mathbf{C}_1$ and $\mathbf{C}_2$, turned out to be $\mathbf{0}$. Exercises 31 and 32 give different results for these constants. ■

EXERCISES 13.1

Motion in the xy -plane

In Exercises 1–4, $\mathbf{r}(t)$ is the position of a particle in the xy -plane at time t . Find an equation in x and y whose graph is the path of the particle. Then find the particle's velocity and acceleration vectors at the given value of t .

1. $\mathbf{r}(t) = (t + 1)\mathbf{i} + (t^2 - 1)\mathbf{j}, \quad t = 1$

2. $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (2t - 1)\mathbf{j}, \quad t = 1/2$

3. $\mathbf{r}(t) = e^t\mathbf{i} + \frac{2}{9}e^{2t}\mathbf{j}, \quad t = \ln 3$

4. $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (3 \sin 2t)\mathbf{j}, \quad t = 0$

Exercises 5–8 give the position vectors of particles moving along various curves in the xy -plane. In each case, find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve.

5. Motion on the circle $x^2 + y^2 = 1$

$$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \quad t = \pi/4 \text{ and } \pi/2$$

6. Motion on the circle $x^2 + y^2 = 16$

$$\mathbf{r}(t) = \left(4 \cos \frac{t}{2}\right)\mathbf{i} + \left(4 \sin \frac{t}{2}\right)\mathbf{j}; \quad t = \pi \text{ and } 3\pi/2$$

7. Motion on the cycloid $x = t - \sin t, y = 1 - \cos t$

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}; \quad t = \pi \text{ and } 3\pi/2$$

8. Motion on the parabola $y = x^2 + 1$

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j}; \quad t = -1, 0, \text{ and } 1$$

Velocity and Acceleration in Space

In Exercises 9–14, $\mathbf{r}(t)$ is the position of a particle in space at time t . Find the particle's velocity and acceleration vectors. Then find the particle's speed and direction of motion at the given value of t . Write the particle's velocity at that time as the product of its speed and direction.

9. $\mathbf{r}(t) = (t + 1)\mathbf{i} + (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, \quad t = 1$

10. $\mathbf{r}(t) = (1 + t)\mathbf{i} + \frac{t^2}{\sqrt{2}}\mathbf{j} + \frac{t^3}{3}\mathbf{k}, \quad t = 1$

11. $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + 4t\mathbf{k}, \quad t = \pi/2$

12. $\mathbf{r}(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j} + \frac{4}{3}t\mathbf{k}, \quad t = \pi/6$

13. $\mathbf{r}(t) = (2 \ln(t + 1))\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}, \quad t = 1$

14. $\mathbf{r}(t) = (e^{-t})\mathbf{i} + (2 \cos 3t)\mathbf{j} + (2 \sin 3t)\mathbf{k}, \quad t = 0$



In Exercises 15–18, $\mathbf{r}(t)$ is the position of a particle in space at time t . Find the angle between the velocity and acceleration vectors at time $t = 0$.

15. $\mathbf{r}(t) = (3t + 1)\mathbf{i} + \sqrt{3}t\mathbf{j} + t^2\mathbf{k}$

16. $\mathbf{r}(t) = \left(\frac{\sqrt{2}}{2}t\right)\mathbf{i} + \left(\frac{\sqrt{2}}{2}t - 16t^2\right)\mathbf{j}$

17. $\mathbf{r}(t) = (\ln(t^2 + 1))\mathbf{i} + (\tan^{-1}t)\mathbf{j} + \sqrt{t^2 + 1}\mathbf{k}$

18. $\mathbf{r}(t) = \frac{4}{9}(1 + t)^{3/2}\mathbf{i} + \frac{4}{9}(1 - t)^{3/2}\mathbf{j} + \frac{1}{3}t\mathbf{k}$

In Exercises 19 and 20, $\mathbf{r}(t)$ is the position vector of a particle in space at time t . Find the time or times in the given time interval when the velocity and acceleration vectors are orthogonal.

19. $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}, \quad 0 \leq t \leq 2\pi$

20. $\mathbf{r}(t) = (\sin t)\mathbf{i} + t\mathbf{j} + (\cos t)\mathbf{k}, \quad t \geq 0$

Integrating Vector-Valued Functions

Evaluate the integrals in Exercises 21–26.

21. $\int_0^1 [t^3\mathbf{i} + 7\mathbf{j} + (t + 1)\mathbf{k}] dt$

22. $\int_1^2 \left[(6 - 6t)\mathbf{i} + 3\sqrt{t}\mathbf{j} + \left(\frac{4}{t^2}\right)\mathbf{k} \right] dt$

23. $\int_{-\pi/4}^{\pi/4} [(\sin t)\mathbf{i} + (1 + \cos t)\mathbf{j} + (\sec^2 t)\mathbf{k}] dt$

24. $\int_0^{\pi/3} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2 \sin t \cos t)\mathbf{k}] dt$

25. $\int_1^4 \left[\frac{1}{t}\mathbf{i} + \frac{1}{5-t}\mathbf{j} + \frac{1}{2t}\mathbf{k} \right] dt$

26. $\int_0^1 \left[\frac{2}{\sqrt{1-t^2}}\mathbf{i} + \frac{\sqrt{3}}{1+t^2}\mathbf{k} \right] dt$

Initial Value Problems for Vector-Valued Functions

Solve the initial value problems in Exercises 27–32 for $\mathbf{r}$ as a vector function of t .

27. Differential equation: $\frac{d\mathbf{r}}{dt} = -t\mathbf{i} - t\mathbf{j} - t\mathbf{k}$

Initial condition: $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

28. Differential equation: $\frac{d\mathbf{r}}{dt} = (180t)\mathbf{i} + (180t - 16t^2)\mathbf{j}$

Initial condition: $\mathbf{r}(0) = 100\mathbf{j}$

29. Differential equation: $\frac{d\mathbf{r}}{dt} = \frac{3}{2}(t + 1)^{1/2}\mathbf{i} + e^{-t}\mathbf{j} + \frac{1}{t+1}\mathbf{k}$

Initial condition: $\mathbf{r}(0) = \mathbf{k}$

30. Differential equation: $\frac{d\mathbf{r}}{dt} = (t^3 + 4t)\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}$

Initial condition: $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$

31. Differential equation: $\frac{d^2\mathbf{r}}{dt^2} = -32\mathbf{k}$

Initial conditions: $\mathbf{r}(0) = 100\mathbf{k}$ and

$\left. \frac{d\mathbf{r}}{dt} \right|_{t=0} = 8\mathbf{i} + 8\mathbf{j}$

32. Differential equation: $\frac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Initial conditions: $\mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k}$ and

$\left. \frac{d\mathbf{r}}{dt} \right|_{t=0} = \mathbf{0}$

Tangent Lines to Smooth Curves

As mentioned in the text, the tangent line to a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ at $t = t_0$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $\mathbf{v}(t_0)$, the curve's velocity vector at t_0 . In Exercises 33–36, find parametric equations for the line that is tangent to the given curve at the given parameter value $t = t_0$.

33. $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}, \quad t_0 = 0$

34. $\mathbf{r}(t) = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 5t\mathbf{k}, \quad t_0 = 4\pi$

35. $\mathbf{r}(t) = (a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + bt\mathbf{k}, \quad t_0 = 2\pi$

36. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}, \quad t_0 = \frac{\pi}{2}$

Motion on Circular Paths

37. Each of the following equations in parts (a)–(e) describes the motion of a particle having the same path, namely the unit circle $x^2 + y^2 = 1$. Although the path of each particle in parts (a)–(e) is the same, the behavior, or “dynamics,” of each particle is different. For each particle, answer the following questions.

i. Does the particle have constant speed? If so, what is its constant speed?

ii. Is the particle's acceleration vector always orthogonal to its velocity vector?

iii. Does the particle move clockwise or counterclockwise around the circle?

iv. Does the particle begin at the point $(1, 0)$?

a. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad t \geq 0$

b. $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}, \quad t \geq 0$

c. $\mathbf{r}(t) = \cos(t - \pi/2)\mathbf{i} + \sin(t - \pi/2)\mathbf{j}, \quad t \geq 0$

d. $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}, \quad t \geq 0$

e. $\mathbf{r}(t) = \cos(t^2)\mathbf{i} + \sin(t^2)\mathbf{j}, \quad t \geq 0$

38. Show that the vector-valued function

$\mathbf{r}(t) = (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

+ $\cos t \left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} \right) + \sin t \left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \right)$ describes the motion of a particle moving in the circle of radius 1 centered at the point $(2, 2, 1)$ and lying in the plane $x + y - 2z = 2$.



Motion Along a Straight Line

39. At time $t = 0$, a particle is located at the point $(1, 2, 3)$. It travels in a straight line to the point $(4, 1, 4)$, has speed 2 at $(1, 2, 3)$ and constant acceleration $3\mathbf{i} - \mathbf{j} + \mathbf{k}$. Find an equation for the position vector $\mathbf{r}(t)$ of the particle at time t .
40. A particle traveling in a straight line is located at the point $(1, -1, 2)$ and has speed 2 at time $t = 0$. The particle moves toward the point $(3, 0, 3)$ with constant acceleration $2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find its position vector $\mathbf{r}(t)$ at time t .

Theory and Examples

41. **Motion along a parabola** A particle moves along the top of the parabola $y^2 = 2x$ from left to right at a constant speed of 5 units per second. Find the velocity of the particle as it moves through the point $(2, 2)$.
42. **Motion along a cycloid** A particle moves in the xy -plane in such a way that its position at time t is

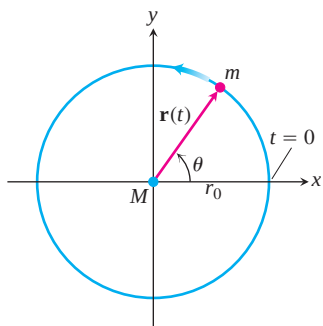
$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}.$$

- T** a. Graph $\mathbf{r}(t)$. The resulting curve is a cycloid.
- b. Find the maximum and minimum values of $|\mathbf{v}|$ and $|\mathbf{a}|$. (Hint: Find the extreme values of $|\mathbf{v}|^2$ and $|\mathbf{a}|^2$ first and take square roots later.)
43. **Motion along an ellipse** A particle moves around the ellipse $(y/3)^2 + (z/2)^2 = 1$ in the yz -plane in such a way that its position at time t is

$$\mathbf{r}(t) = (3 \cos t)\mathbf{j} + (2 \sin t)\mathbf{k}.$$

Find the maximum and minimum values of $|\mathbf{v}|$ and $|\mathbf{a}|$. (Hint: Find the extreme values of $|\mathbf{v}|^2$ and $|\mathbf{a}|^2$ first and take square roots later.)

44. **A satellite in circular orbit** A satellite of mass m is revolving at a constant speed v around a body of mass M (Earth, for example) in a circular orbit of radius r_0 (measured from the body's center of mass). Determine the satellite's orbital period T (the time to complete one full orbit), as follows:
- a. Coordinatize the orbital plane by placing the origin at the body's center of mass, with the satellite on the x -axis at $t = 0$ and moving counterclockwise, as in the accompanying figure.



Let $\mathbf{r}(t)$ be the satellite's position vector at time t . Show that $\theta = vt/r_0$ and hence that

$$\mathbf{r}(t) = \left(r_0 \cos \frac{vt}{r_0}\right)\mathbf{i} + \left(r_0 \sin \frac{vt}{r_0}\right)\mathbf{j}.$$

- b. Find the acceleration of the satellite.
- c. According to Newton's law of gravitation, the gravitational force exerted on the satellite is directed toward M and is given by

$$\mathbf{F} = \left(-\frac{GmM}{r_0^2}\right)\frac{\mathbf{r}}{r_0},$$

where G is the universal constant of gravitation. Using Newton's second law, $\mathbf{F} = m\mathbf{a}$, show that $v^2 = GM/r_0$.

- d. Show that the orbital period T satisfies $vT = 2\pi r_0$.
- e. From parts (c) and (d), deduce that

$$T^2 = \frac{4\pi^2}{GM}r_0^3.$$

That is, the square of the period of a satellite in circular orbit is proportional to the cube of the radius from the orbital center.

45. Let $\mathbf{v}$ be a differentiable vector function of t . Show that if $\mathbf{v} \cdot (d\mathbf{v}/dt) = 0$ for all t , then $|\mathbf{v}|$ is constant.
46. **Derivatives of triple scalar products**

- a. Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are differentiable vector functions of t , then

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} \times \mathbf{w} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} \times \mathbf{w} + \mathbf{u} \cdot \mathbf{v} \times \frac{d\mathbf{w}}{dt}. \quad (7)$$

- b. Show that Equation (7) is equivalent to

$$\begin{aligned} \frac{d}{dt} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} &= \begin{vmatrix} \frac{du_1}{dt} & \frac{du_2}{dt} & \frac{du_3}{dt} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &+ \begin{vmatrix} u_1 & u_2 & u_3 \\ \frac{dv_1}{dt} & \frac{dv_2}{dt} & \frac{dv_3}{dt} \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &+ \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \frac{dw_1}{dt} & \frac{dw_2}{dt} & \frac{dw_3}{dt} \end{vmatrix}. \quad (8) \end{aligned}$$

Equation (8) says that the derivative of a 3 by 3 determinant of differentiable functions is the sum of the three determinants obtained from the original by differentiating one row at a time. The result extends to determinants of any order.

47. (Continuation of Exercise 46.) Suppose that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ and that f , g , and h have derivatives through order three. Use Equation (7) or (8) to show that

$$\frac{d}{dt} \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} \right) = \mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \frac{d^3\mathbf{r}}{dt^3} \right). \quad (9)$$

(Hint: Differentiate on the left and look for vectors whose products are zero.)

48. **Constant Function Rule** Prove that if $\mathbf{u}$ is the vector function with the constant value $\mathbf{C}$, then $d\mathbf{u}/dt = \mathbf{0}$.

49. **Scalar Multiple Rules**

- a. Prove that if $\mathbf{u}$ is a differentiable function of t and c is any real number, then

$$\frac{d(c\mathbf{u})}{dt} = c \frac{d\mathbf{u}}{dt}.$$

- b. Prove that if $\mathbf{u}$ is a differentiable function of t and f is a differentiable scalar function of t , then

$$\frac{d}{dt}(f\mathbf{u}) = \frac{df}{dt}\mathbf{u} + f \frac{d\mathbf{u}}{dt}.$$

50. **Sum and Difference Rules** Prove that if $\mathbf{u}$ and $\mathbf{v}$ are differentiable functions of t , then

$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt}$$

and

$$\frac{d}{dt}(\mathbf{u} - \mathbf{v}) = \frac{d\mathbf{u}}{dt} - \frac{d\mathbf{v}}{dt}.$$

51. **Component Test for Continuity at a Point** Show that the vector function $\mathbf{r}$ defined by $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous at $t = t_0$ if and only if f , g , and h are continuous at t_0 .

52. **Limits of cross products of vector functions** Suppose that $\mathbf{r}_1(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$, $\mathbf{r}_2(t) = g_1(t)\mathbf{i} + g_2(t)\mathbf{j} + g_3(t)\mathbf{k}$, $\lim_{t \rightarrow t_0} \mathbf{r}_1(t) = \mathbf{A}$, and $\lim_{t \rightarrow t_0} \mathbf{r}_2(t) = \mathbf{B}$. Use the determinant formula for cross products and the Limit Product Rule for scalar functions to show that

$$\lim_{t \rightarrow t_0} (\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \mathbf{A} \times \mathbf{B}$$

53. **Differentiable vector functions are continuous** Show that if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is differentiable at $t = t_0$, then it is continuous at t_0 as well.

54. Establish the following properties of integrable vector functions.

- a. The *Constant Scalar Multiple Rule*:

$$\int_a^b k\mathbf{r}(t) dt = k \int_a^b \mathbf{r}(t) dt \quad (\text{any scalar } k)$$

The *Rule for Negatives*,

$$\int_a^b (-\mathbf{r}(t)) dt = - \int_a^b \mathbf{r}(t) dt,$$

is obtained by taking $k = -1$.

- b. The *Sum and Difference Rules*:

$$\int_a^b (\mathbf{r}_1(t) \pm \mathbf{r}_2(t)) dt = \int_a^b \mathbf{r}_1(t) dt \pm \int_a^b \mathbf{r}_2(t) dt$$

- c. The *Constant Vector Multiple Rules*:

$$\int_a^b \mathbf{C} \cdot \mathbf{r}(t) dt = \mathbf{C} \cdot \int_a^b \mathbf{r}(t) dt \quad (\text{any constant vector } \mathbf{C})$$

and

$$\int_a^b \mathbf{C} \times \mathbf{r}(t) dt = \mathbf{C} \times \int_a^b \mathbf{r}(t) dt \quad (\text{any constant vector } \mathbf{C})$$

55. **Products of scalar and vector functions** Suppose that the scalar function $u(t)$ and the vector function $\mathbf{r}(t)$ are both defined for $a \leq t \leq b$.

- a. Show that $u\mathbf{r}$ is continuous on $[a, b]$ if u and $\mathbf{r}$ are continuous on $[a, b]$.
- b. If u and $\mathbf{r}$ are both differentiable on $[a, b]$, show that $u\mathbf{r}$ is differentiable on $[a, b]$ and that

$$\frac{d}{dt}(u\mathbf{r}) = u \frac{d\mathbf{r}}{dt} + \mathbf{r} \frac{du}{dt}.$$

56. **Antiderivatives of vector functions**

- a. Use Corollary 2 of the Mean Value Theorem for scalar functions to show that if two vector functions $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ have identical derivatives on an interval I , then the functions differ by a constant vector value throughout I .
- b. Use the result in part (a) to show that if $\mathbf{R}(t)$ is any antiderivative of $\mathbf{r}(t)$ on I , then any other antiderivative of $\mathbf{r}$ on I equals $\mathbf{R}(t) + \mathbf{C}$ for some constant vector $\mathbf{C}$.

57. **The Fundamental Theorem of Calculus** The Fundamental Theorem of Calculus for scalar functions of a real variable holds for vector functions of a real variable as well. Prove this by using the theorem for scalar functions to show first that if a vector function $\mathbf{r}(t)$ is continuous for $a \leq t \leq b$, then

$$\frac{d}{dt} \int_a^t \mathbf{r}(\tau) d\tau = \mathbf{r}(t)$$

at every point t of (a, b) . Then use the conclusion in part (b) of Exercise 56 to show that if $\mathbf{R}$ is any antiderivative of $\mathbf{r}$ on $[a, b]$ then

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(b) - \mathbf{R}(a).$$

COMPUTER EXPLORATIONS

Drawing Tangents to Space Curves

Use a CAS to perform the following steps in Exercises 58–61.

- a. Plot the space curve traced out by the position vector $\mathbf{r}$.
 - b. Find the components of the velocity vector $d\mathbf{r}/dt$.
 - c. Evaluate $d\mathbf{r}/dt$ at the given point t_0 and determine the equation of the tangent line to the curve at $\mathbf{r}(t_0)$.
 - d. Plot the tangent line together with the curve over the given interval.
58. $\mathbf{r}(t) = (\sin t - t \cos t)\mathbf{i} + (\cos t + t \sin t)\mathbf{j} + t^2\mathbf{k}$,
 $0 \leq t \leq 6\pi$, $t_0 = 3\pi/2$
59. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, $-2 \leq t \leq 3$, $t_0 = 1$
60. $\mathbf{r}(t) = (\sin 2t)\mathbf{i} + (\ln(1 + t))\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 4\pi$,
 $t_0 = \pi/4$
61. $\mathbf{r}(t) = (\ln(t^2 + 2))\mathbf{i} + (\tan^{-1} 3t)\mathbf{j} + \sqrt{t^2 + 1}\mathbf{k}$,
 $-3 \leq t \leq 5$, $t_0 = 3$

In Exercises 62 and 63, you will explore graphically the behavior of the helix

$$\mathbf{r}(t) = (\cos at)\mathbf{i} + (\sin at)\mathbf{j} + bt\mathbf{k}.$$

as you change the values of the constants a and b . Use a CAS to perform the steps in each exercise.

62. Set $b = 1$. Plot the helix $\mathbf{r}(t)$ together with the tangent line to the curve at $t = 3\pi/2$ for $a = 1, 2, 4$, and 6 over the interval $0 \leq t \leq 4\pi$. Describe in your own words what happens to the graph of the helix and the position of the tangent line as a increases through these positive values.
63. Set $a = 1$. Plot the helix $\mathbf{r}(t)$ together with the tangent line to the curve at $t = 3\pi/2$ for $b = 1/4, 1/2, 2$, and 4 over the interval $0 \leq t \leq 4\pi$. Describe in your own words what happens to the graph of the helix and the position of the tangent line as b increases through these positive values.

13.2

Modeling Projectile Motion

When we shoot a projectile into the air we usually want to know beforehand how far it will go (will it reach the target?), how high it will rise (will it clear the hill?), and when it will land (when do we get results?). We get this information from the direction and magnitude of the projectile's initial velocity vector, using Newton's second law of motion.

The Vector and Parametric Equations for Ideal Projectile Motion

To derive equations for projectile motion, we assume that the projectile behaves like a particle moving in a vertical coordinate plane and that the only force acting on the projectile during its flight is the constant force of gravity, which always points straight down. In practice, none of these assumptions really holds. The ground moves beneath the projectile as the earth turns, the air creates a frictional force that varies with the projectile's speed and altitude, and the force of gravity changes as the projectile moves along. All this must be taken into account by applying corrections to the predictions of the *ideal* equations we are about to derive. The corrections, however, are not the subject of this section.

We assume that the projectile is launched from the origin at time $t = 0$ into the first quadrant with an initial velocity $\mathbf{v}_0$ (Figure 13.9). If $\mathbf{v}_0$ makes an angle α with the horizontal, then

$$\mathbf{v}_0 = (|\mathbf{v}_0| \cos \alpha) \mathbf{i} + (|\mathbf{v}_0| \sin \alpha) \mathbf{j}.$$

If we use the simpler notation v_0 for the initial speed $|\mathbf{v}_0|$, then

$$\mathbf{v}_0 = (v_0 \cos \alpha) \mathbf{i} + (v_0 \sin \alpha) \mathbf{j}. \quad (1)$$

The projectile's initial position is

$$\mathbf{r}_0 = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}. \quad (2)$$

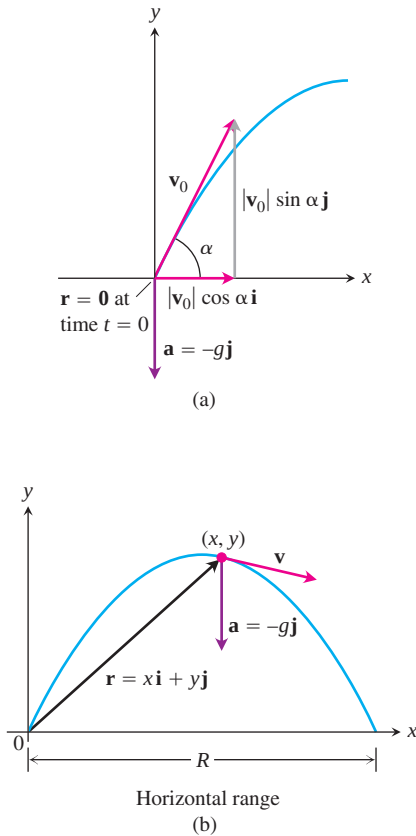


FIGURE 13.9 (a) Position, velocity, acceleration, and launch angle at $t = 0$. (b) Position, velocity, and acceleration at a later time t .

Newton's second law of motion says that the force acting on the projectile is equal to the projectile's mass m times its acceleration, or $m(d^2\mathbf{r}/dt^2)$ if $\mathbf{r}$ is the projectile's position vector and t is time. If the force is solely the gravitational force $-mg\mathbf{j}$, then

$$m \frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{j} \quad \text{and} \quad \frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j}.$$

We find $\mathbf{r}$ as a function of t by solving the following initial value problem.

$$\text{Differential equation:} \quad \frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j}$$

$$\text{Initial conditions:} \quad \mathbf{r} = \mathbf{r}_0 \quad \text{and} \quad \frac{d\mathbf{r}}{dt} = \mathbf{v}_0 \quad \text{when } t = 0$$

The first integration gives

$$\frac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0.$$

A second integration gives

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0t + \mathbf{r}_0.$$

Substituting the values of $\mathbf{v}_0$ and $\mathbf{r}_0$ from Equations (1) and (2) gives

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \underbrace{(v_0 \cos \alpha)t\mathbf{i} + (v_0 \sin \alpha)t\mathbf{j}}_{\mathbf{v}_0t} + \mathbf{0}$$

Collecting terms, we have

Ideal Projectile Motion Equation

$$\mathbf{r} = (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2 \right)\mathbf{j}. \quad (3)$$

Equation (3) is the *vector equation* for ideal projectile motion. The angle α is the projectile's **launch angle (firing angle, angle of elevation)**, and v_0 , as we said before, is the projectile's **initial speed**. The components of $\mathbf{r}$ give the parametric equations

$$x = (v_0 \cos \alpha)t \quad \text{and} \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2, \quad (4)$$

where x is the distance downrange and y is the height of the projectile at time $t \geq 0$.



EXAMPLE 1 Firing an Ideal Projectile

A projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of 60° . Where will the projectile be 10 sec later?

Solution We use Equation (3) with $v_0 = 500$, $\alpha = 60^\circ$, $g = 9.8$, and $t = 10$ to find the projectile's components 10 sec after firing.

$$\begin{aligned}\mathbf{r} &= (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2 \right)\mathbf{j} \\ &= (500)\left(\frac{1}{2}\right)(10)\mathbf{i} + \left((500)\left(\frac{\sqrt{3}}{2}\right)10 - \left(\frac{1}{2}\right)(9.8)(100) \right)\mathbf{j} \\ &\approx 2500\mathbf{i} + 3840\mathbf{j}.\end{aligned}$$

Ten seconds after firing, the projectile is about 3840 m in the air and 2500 m downrange. ■

Height, Flight Time, and Range

Equation (3) enables us to answer most questions about the ideal motion for a projectile fired from the origin.

The projectile reaches its highest point when its vertical velocity component is zero, that is, when

$$\frac{dy}{dt} = v_0 \sin \alpha - gt = 0, \quad \text{or} \quad t = \frac{v_0 \sin \alpha}{g}.$$

For this value of t , the value of y is

$$y_{\max} = (v_0 \sin \alpha)\left(\frac{v_0 \sin \alpha}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin \alpha}{g}\right)^2 = \frac{(v_0 \sin \alpha)^2}{2g}.$$

To find when the projectile lands when fired over horizontal ground, we set the vertical component equal to zero in Equation (3) and solve for t .

$$(v_0 \sin \alpha)t - \frac{1}{2}gt^2 = 0$$

$$t\left(v_0 \sin \alpha - \frac{1}{2}gt\right) = 0$$

$$t = 0, \quad t = \frac{2v_0 \sin \alpha}{g}$$

Since 0 is the time the projectile is fired, $(2v_0 \sin \alpha)/g$ must be the time when the projectile strikes the ground.

To find the projectile's **range** R , the distance from the origin to the point of impact on horizontal ground, we find the value of the horizontal component when $t = (2v_0 \sin \alpha)/g$.

$$x = (v_0 \cos \alpha)t$$

$$R = (v_0 \cos \alpha)\left(\frac{2v_0 \sin \alpha}{g}\right) = \frac{v_0^2}{g}(2 \sin \alpha \cos \alpha) = \frac{v_0^2}{g} \sin 2\alpha$$

The range is largest when $\sin 2\alpha = 1$ or $\alpha = 45^\circ$.

Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α :

$$\text{Maximum height: } y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g}$$

$$\text{Flight time: } t = \frac{2v_0 \sin \alpha}{g}$$

$$\text{Range: } R = \frac{v_0^2}{g} \sin 2\alpha.$$

**EXAMPLE 2** Investigating Ideal Projectile Motion

Find the maximum height, flight time, and range of a projectile fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of 60° (same projectile as Example 1).

Solution

$$\begin{aligned} \text{Maximum height: } y_{\max} &= \frac{(v_0 \sin \alpha)^2}{2g} \\ &= \frac{(500 \sin 60^\circ)^2}{2(9.8)} \approx 9566 \text{ m} \end{aligned}$$

$$\begin{aligned} \text{Flight time: } t &= \frac{2v_0 \sin \alpha}{g} \\ &= \frac{2(500) \sin 60^\circ}{9.8} \approx 88.4 \text{ sec} \end{aligned}$$

$$\begin{aligned} \text{Range: } R &= \frac{v_0^2}{g} \sin 2\alpha \\ &= \frac{(500)^2 \sin 120^\circ}{9.8} \approx 22,092 \text{ m} \end{aligned}$$

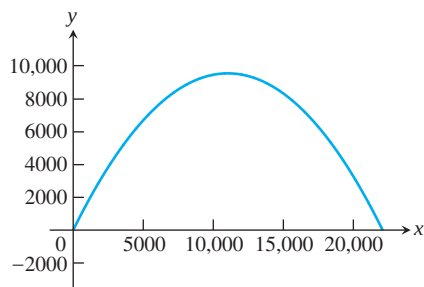


FIGURE 13.10 The graph of the projectile described in Example 2.

From Equation (3), the position vector of the projectile is

$$\begin{aligned} \mathbf{r} &= (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2 \right)\mathbf{j} \\ &= (500 \cos 60^\circ)t\mathbf{i} + \left((500 \sin 60^\circ)t - \frac{1}{2}(9.8)t^2 \right)\mathbf{j} \\ &= 250t\mathbf{i} + \left((250\sqrt{3})t - 4.9t^2 \right)\mathbf{j}. \end{aligned}$$

A graph of the projectile's path is shown in Figure 13.10. ■

Ideal Trajectories Are Parabolic

It is often claimed that water from a hose traces a parabola in the air, but anyone who looks closely enough will see this is not so. The air slows the water down, and its forward progress is too slow at the end to keep pace with the rate at which it falls.

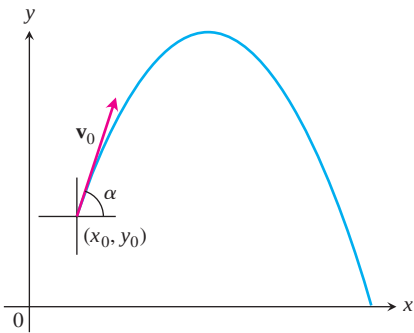


FIGURE 13.11 The path of a projectile fired from (x_0, y_0) with an initial velocity $\mathbf{v}_0$ at an angle of α degrees with the horizontal.

What is really being claimed is that ideal projectiles move along parabolas, and this we can see from Equations (4). If we substitute $t = x/(v_0 \cos \alpha)$ from the first equation into the second, we obtain the Cartesian-coordinate equation

$$y = -\left(\frac{g}{2v_0^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x.$$

This equation has the form $y = ax^2 + bx$, so its graph is a parabola.

Firing from (x_0, y_0)

If we fire our ideal projectile from the point (x_0, y_0) instead of the origin (Figure 13.11), the position vector for the path of motion is

$$\mathbf{r} = (x_0 + (v_0 \cos \alpha)t)\mathbf{i} + \left(y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}, \tag{5}$$

as you are asked to show in Exercise 19.



EXAMPLE 3 Firing a Flaming Arrow

To open the 1992 Summer Olympics in Barcelona, bronze medalist archer Antonio Rebollo lit the Olympic torch with a flaming arrow (Figure 13.12). Suppose that Rebollo shot the arrow at a height of 6 ft above ground level 90 ft from the 70-ft-high cauldron, and he wanted the arrow to reach maximum height exactly 4 ft above the center of the cauldron (Figure 13.12).

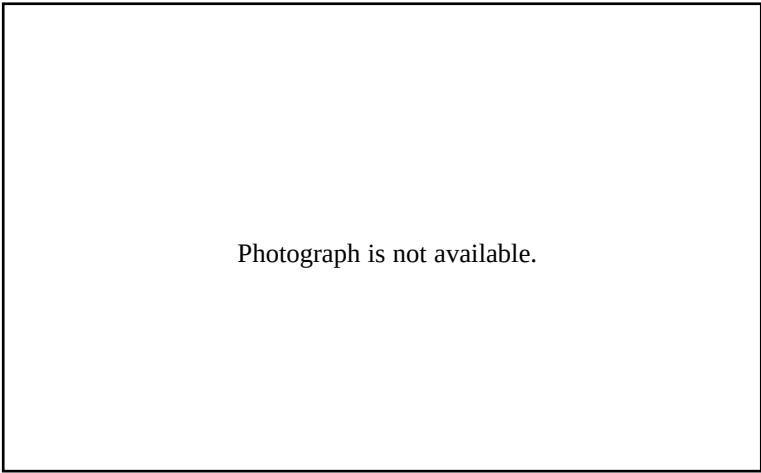


FIGURE 13.12 Spanish archer Antonio Rebollo lights the Olympic torch in Barcelona with a flaming arrow.

- (a) Express $y_{\max}$ in terms of the initial speed v_0 and firing angle α .
- (b) Use $y_{\max} = 74$ ft (Figure 13.13) and the result from part (a) to find the value of $v_0 \sin \alpha$.
- (c) Find the value of $v_0 \cos \alpha$.
- (d) Find the initial firing angle of the arrow.

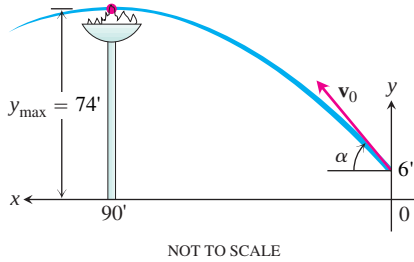


FIGURE 13.13 Ideal path of the arrow that lit the Olympic torch (Example 3).

Solution

- (a) We use a coordinate system in which the positive x -axis lies along the ground toward the left (to match the second photograph in Figure 13.12) and the coordinates of the flaming arrow at $t = 0$ are $x_0 = 0$ and $y_0 = 6$ (Figure 13.13). We have

$$y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \quad \text{Equation (5), j-component}$$

$$= 6 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2. \quad y_0 = 6$$

We find the time when the arrow reaches its highest point by setting $dy/dt = 0$ and solving for t , obtaining

$$t = \frac{v_0 \sin \alpha}{g}.$$

For this value of t , the value of y is

$$\begin{aligned} y_{\max} &= 6 + (v_0 \sin \alpha) \left(\frac{v_0 \sin \alpha}{g} \right) - \frac{1}{2}g \left(\frac{v_0 \sin \alpha}{g} \right)^2 \\ &= 6 + \frac{(v_0 \sin \alpha)^2}{2g}. \end{aligned}$$

- (b) Using $y_{\max} = 74$ and $g = 32$, we see from the preceding equation in part (a) that

$$74 = 6 + \frac{(v_0 \sin \alpha)^2}{2(32)}$$

or

$$v_0 \sin \alpha = \sqrt{(68)(64)}.$$

- (c) When the arrow reaches $y_{\max}$, the horizontal distance traveled to the center of the cauldron is $x = 90$ ft. We substitute the time to reach $y_{\max}$ from part (a) and the horizontal distance $x = 90$ ft into the i -component of Equation (5) to obtain

$$x = x_0 + (v_0 \cos \alpha)t \quad \text{Equation (5), i-component}$$

$$90 = 0 + (v_0 \cos \alpha)t \quad x = 90, x_0 = 0$$

$$= (v_0 \cos \alpha) \left(\frac{v_0 \sin \alpha}{g} \right). \quad t = (v_0 \sin \alpha)/g$$

Solving this equation for $v_0 \cos \alpha$ and using $g = 32$ and the result from part (b), we have

$$v_0 \cos \alpha = \frac{90g}{v_0 \sin \alpha} = \frac{(90)(32)}{\sqrt{(68)(64)}}.$$

- (d) Parts (b) and (c) together tell us that

$$\tan \alpha = \frac{v_0 \sin \alpha}{v_0 \cos \alpha} = \frac{(\sqrt{(68)(64)})^2}{(90)(32)} = \frac{68}{45}$$

or

$$\alpha = \tan^{-1} \left(\frac{68}{45} \right) \approx 56.5^\circ.$$

This is Rebollo's firing angle. ■

Projectile Motion with Wind Gusts

The next example shows how to account for another force acting on a projectile. We also assume that the path of the baseball in Example 4 lies in a vertical plane.



EXAMPLE 4 Hitting a Baseball

A baseball is hit when it is 3 ft above the ground. It leaves the bat with initial speed of 152 ft/sec, making an angle of 20° with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-8.8\mathbf{i}$ (ft/sec) to the ball's initial velocity (8.8 ft/sec = 6 mph).

- (a) Find a vector equation (position vector) for the path of the baseball.
- (b) How high does the baseball go, and when does it reach maximum height?
- (c) Assuming that the ball is not caught, find its range and flight time.

Solution

- (a) Using Equation (1) and accounting for the gust of wind, the initial velocity of the baseball is

$$\begin{aligned} \mathbf{v}_0 &= (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} - 8.8\mathbf{i} \\ &= (152 \cos 20^\circ)\mathbf{i} + (152 \sin 20^\circ)\mathbf{j} - (8.8)\mathbf{i} \\ &= (152 \cos 20^\circ - 8.8)\mathbf{i} + (152 \sin 20^\circ)\mathbf{j}. \end{aligned}$$

The initial position is $\mathbf{r}_0 = 0\mathbf{i} + 3\mathbf{j}$. Integration of $d^2\mathbf{r}/dt^2 = -g\mathbf{j}$ gives

$$\frac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0.$$

A second integration gives

$$\mathbf{r} = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0t + \mathbf{r}_0.$$

Substituting the values of $\mathbf{v}_0$ and $\mathbf{r}_0$ into the last equation gives the position vector of the baseball.

$$\begin{aligned} \mathbf{r} &= -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0t + \mathbf{r}_0 \\ &= -16t^2\mathbf{j} + (152 \cos 20^\circ - 8.8)t\mathbf{i} + (152 \sin 20^\circ)t\mathbf{j} + 3\mathbf{j} \\ &= (152 \cos 20^\circ - 8.8)t\mathbf{i} + (3 + (152 \sin 20^\circ)t - 16t^2)\mathbf{j}. \end{aligned}$$

- (b) The baseball reaches its highest point when the vertical component of velocity is zero, or

$$\frac{dy}{dt} = 152 \sin 20^\circ - 32t = 0.$$

Solving for t we find

$$t = \frac{152 \sin 20^\circ}{32} \approx 1.62 \text{ sec.}$$

Substituting this time into the vertical component for $\mathbf{r}$ gives the maximum height

$$\begin{aligned} y_{\max} &= 3 + (152 \sin 20^\circ)(1.62) - 16(1.62)^2 \\ &\approx 45.2 \text{ ft.} \end{aligned}$$

That is, the maximum height of the baseball is about 45.2 ft, reached about 1.6 sec after leaving the bat.

- (c) To find when the baseball lands, we set the vertical component for $\mathbf{r}$ equal to 0 and solve for t :

$$\begin{aligned} 3 + (152 \sin 20^\circ)t - 16t^2 &= 0 \\ 3 + (51.99)t - 16t^2 &= 0. \end{aligned}$$

The solution values are about $t = 3.3$ sec and $t = -0.06$ sec. Substituting the positive time into the horizontal component for $\mathbf{r}$, we find the range

$$\begin{aligned} R &= (152 \cos 20^\circ - 8.8)(3.3) \\ &\approx 442 \text{ ft.} \end{aligned}$$

Thus, the horizontal range is about 442 ft, and the flight time is about 3.3 sec. ■

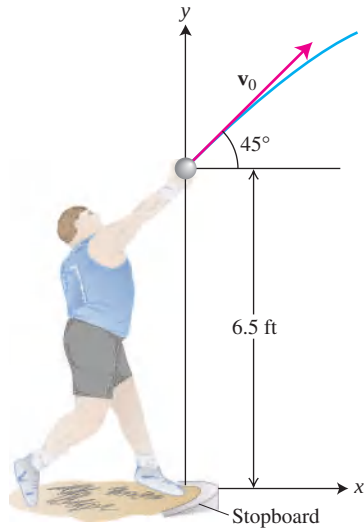
In Exercises 29 through 31, we consider projectile motion when there is air resistance slowing down the flight.

EXERCISES 13.2

Projectile flights in the following exercises are to be treated as ideal unless stated otherwise. All launch angles are assumed to be measured from the horizontal. All projectiles are assumed to be launched from the origin over a horizontal surface unless stated otherwise.

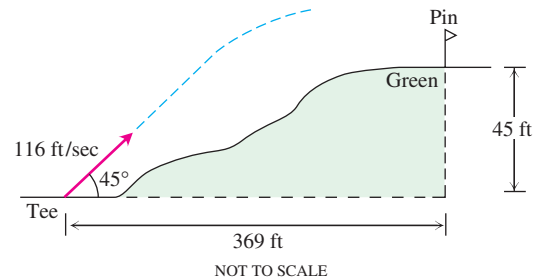


- 1. Travel time** A projectile is fired at a speed of 840 m/sec at an angle of 60° . How long will it take to get 21 km downrange?
- 2. Finding muzzle speed** Find the muzzle speed of a gun whose maximum range is 24.5 km.
- 3. Flight time and height** A projectile is fired with an initial speed of 500 m/sec at an angle of elevation of 45° .
 - a. When and how far away will the projectile strike?
 - b. How high overhead will the projectile be when it is 5 km downrange?
 - c. What is the greatest height reached by the projectile?
- 4. Throwing a baseball** A baseball is thrown from the stands 32 ft above the field at an angle of 30° up from the horizontal. When and how far away will the ball strike the ground if its initial speed is 32 ft/sec?
- 5. Shot put** An athlete puts a 16-lb shot at an angle of 45° to the horizontal from 6.5 ft above the ground at an initial speed of 44 ft/sec as suggested in the accompanying figure. How long after launch and how far from the inner edge of the stopboard does the shot land?

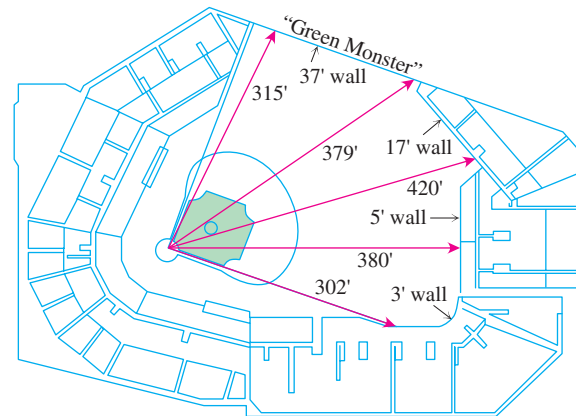


6. (Continuation of Exercise 5.) Because of its initial elevation, the shot in Exercise 5 would have gone slightly farther if it had been launched at a 40° angle. How much farther? Answer in inches.
7. **Firing golf balls** A spring gun at ground level fires a golf ball at an angle of 45° . The ball lands 10 m away.
 - a. What was the ball's initial speed?
 - b. For the same initial speed, find the two firing angles that make the range 6 m.
8. **Beaming electrons** An electron in a TV tube is beamed horizontally at a speed of 5×10^6 m/sec toward the face of the tube 40 cm away. About how far will the electron drop before it hits?
9. **Finding golf ball speed** Laboratory tests designed to find how far golf balls of different hardness go when hit with a driver showed that a 100-compression ball hit with a club-head speed of 100 mph at a launch angle of 9° carried 248.8 yd. What was the launch speed of the ball? (It was more than 100 mph. At the same time the club head was moving forward, the compressed ball was kicking away from the club face, adding to the ball's forward speed.)
10. A *human cannonball* is to be fired with an initial speed of $v_0 = 80\sqrt{10/3}$ ft/sec. The circus performer (of the right caliber, naturally) hopes to land on a special cushion located 200 ft downrange at the same height as the muzzle of the cannon. The circus is being held in a large room with a flat ceiling 75 ft higher than the muzzle. Can the performer be fired to the cushion without striking the ceiling? If so, what should the cannon's angle of elevation be?
11. A golf ball leaves the ground at a 30° angle at a speed of 90 ft/sec. Will it clear the top of a 30-ft tree that is in the way, 135 ft down the fairway? Explain.
12. **Elevated green** A golf ball is hit with an initial speed of 116 ft/sec at an angle of elevation of 45° from the tee to a green that is

elevated 45 ft above the tee as shown in the diagram. Assuming that the pin, 369 ft downrange, does not get in the way, where will the ball land in relation to the pin?



13. **The Green Monster** A baseball hit by a Boston Red Sox player at a 20° angle from 3 ft above the ground just cleared the left end of the "Green Monster," the left-field wall in Fenway Park. This wall is 37 ft high and 315 ft from home plate (see the accompanying figure).
 - a. What was the initial speed of the ball?
 - b. How long did it take the ball to reach the wall?



14. **Equal-range firing angles** Show that a projectile fired at an angle of α degrees, $0 < \alpha < 90$, has the same range as a projectile fired at the same speed at an angle of $(90 - \alpha)$ degrees. (In models that take air resistance into account, this symmetry is lost.)
15. **Equal-range firing angles** What two angles of elevation will enable a projectile to reach a target 16 km downrange on the same level as the gun if the projectile's initial speed is 400 m/sec?
16. **Range and height versus speed**
 - a. Show that doubling a projectile's initial speed at a given launch angle multiplies its range by 4.
 - b. By about what percentage should you increase the initial speed to double the height and range?
17. **Shot put** In Moscow in 1987, Natalya Lisouskaya set a women's world record by putting an 8 lb 13 oz shot 73 ft 10 in. Assuming that she launched the shot at a 40° angle to the horizontal from 6.5 ft above the ground, what was the shot's initial speed?

- 18. Height versus time** Show that a projectile attains three-quarters of its maximum height in half the time it takes to reach the maximum height.

- 19. Firing from (x_0, y_0)** Derive the equations

$$x = x_0 + (v_0 \cos \alpha)t,$$

$$y = y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2,$$

(see Equation (5) in the text) by solving the following initial value problem for a vector $\mathbf{r}$ in the plane.

Differential equation: $\frac{d^2 \mathbf{r}}{dt^2} = -g\mathbf{j}$

Initial conditions: $\mathbf{r}(0) = x_0\mathbf{i} + y_0\mathbf{j}$

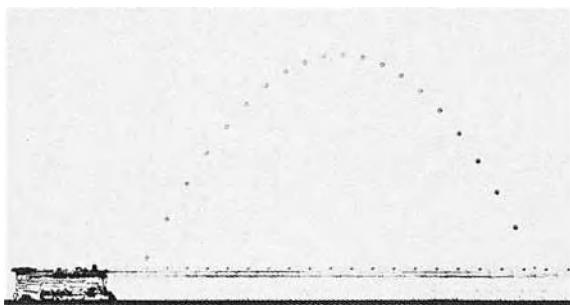
$$\frac{d\mathbf{r}}{dt}(0) = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j}$$

- 20. Flaming arrow** Using the firing angle found in Example 3, find the speed at which the flaming arrow left Rebollo's bow. See Figure 13.13.

- 21. Flaming arrow** The cauldron in Example 3 is 12 ft in diameter. Using Equation (5) and Example 3c, find how long it takes the flaming arrow to cover the horizontal distance to the rim. How high is the arrow at this time?

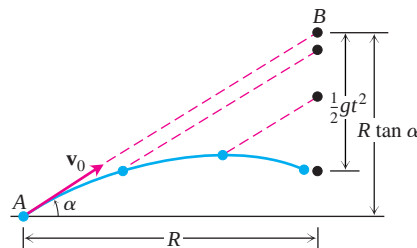
- 22.** Describe the path of a projectile given by Equations (4) when $\alpha = 90^\circ$.

- 23. Model train** The accompanying multiframe photograph shows a model train engine moving at a constant speed on a straight horizontal track. As the engine moved along, a marble was fired into the air by a spring in the engine's smokestack. The marble, which continued to move with the same forward speed as the engine, rejoined the engine 1 sec after it was fired. Measure the angle the marble's path made with the horizontal and use the information to find how high the marble went and how fast the engine was moving.



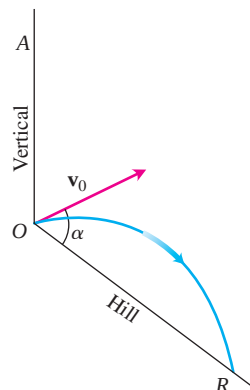
- 24. Colliding marbles** The figure shows an experiment with two marbles. Marble A was launched toward marble B with launch angle α and initial speed v_0 . At the same instant, marble B was released to fall from rest at $R \tan \alpha$ units directly above a spot R units downrange from A . The marbles were found to collide

regardless of the value of v_0 . Was this mere coincidence, or must this happen? Give reasons for your answer.



- 25. Launching downhill** An ideal projectile is launched straight down an inclined plane as shown in the accompanying figure.

- Show that the greatest downhill range is achieved when the initial velocity vector bisects angle AOR .
- If the projectile were fired uphill instead of down, what launch angle would maximize its range? Give reasons for your answer.



- 26. Hitting a baseball under a wind gust** A baseball is hit when it is 2.5 ft above the ground. It leaves the bat with an initial velocity of 145 ft/sec at a launch angle of 23° . At the instant the ball is hit, an instantaneous gust of wind blows against the ball, adding a component of $-14\mathbf{i}$ (ft/sec) to the ball's initial velocity. A 15-ft-high fence lies 300 ft from home plate in the direction of the flight.

- Find a vector equation for the path of the baseball.
- How high does the baseball go, and when does it reach maximum height?
- Find the range and flight time of the baseball, assuming that the ball is not caught.
- When is the baseball 20 ft high? How far (ground distance) is the baseball from home plate at that height?
- Has the batter hit a home run? Explain.

- 27. Volleyball** A volleyball is hit when it is 4 ft above the ground and 12 ft from a 6-ft-high net. It leaves the point of impact with an initial velocity of 35 ft/sec at an angle of 27° and slips by the opposing team untouched.

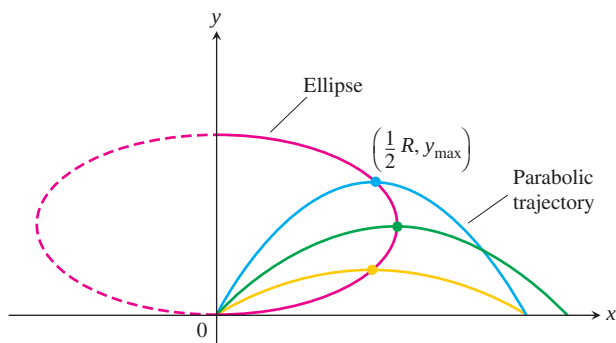




- Find a vector equation for the path of the volleyball.
 - How high does the volleyball go, and when does it reach maximum height?
 - Find its range and flight time.
 - When is the volleyball 7 ft above the ground? How far (ground distance) is the volleyball from where it will land?
 - Suppose that the net is raised to 8 ft. Does this change things? Explain.
- 28. Where trajectories crest** For a projectile fired from the ground at launch angle α with initial speed v_0 , consider α as a variable and v_0 as a fixed constant. For each α , $0 < \alpha < \pi/2$, we obtain a parabolic trajectory as shown in the accompanying figure. Show that the points in the plane that give the maximum heights of these parabolic trajectories all lie on the ellipse

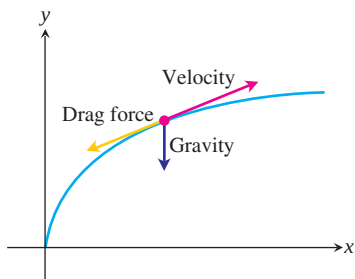
$$x^2 + 4\left(y - \frac{v_0^2}{4g}\right)^2 = \frac{v_0^4}{4g^2},$$

where $x \geq 0$.



Projectile Motion with Linear Drag

The main force affecting the motion of a projectile, other than gravity, is air resistance. This slowing down force is **drag force**, and it acts in a direction *opposite* to the velocity of the projectile (see accompanying figure). For projectiles moving through the air at relatively low speeds, however, the drag force is (very nearly) proportional to the speed (to the first power) and so is called **linear**.



- 29. Linear drag** Derive the equations

$$x = \frac{v_0}{k}(1 - e^{-kt}) \cos \alpha$$

$$y = \frac{v_0}{k}(1 - e^{-kt})(\sin \alpha) + \frac{g}{k^2}(1 - kt - e^{-kt})$$

by solving the following initial value problem for a vector $\mathbf{r}$ in the plane.

Differential equation: $\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j} - k\mathbf{v} = -g\mathbf{j} - k\frac{d\mathbf{r}}{dt}$

Initial conditions: $\mathbf{r}(0) = \mathbf{0}$

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{v}_0 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j}$$

The **drag coefficient** k is a positive constant representing resistance due to air density, v_0 and α are the projectile's initial speed and launch angle, and g is the acceleration of gravity.

- 30. Hitting a baseball with linear drag** Consider the baseball problem in Example 4 when there is linear drag (see Exercise 29). Assume a drag coefficient $k = 0.12$, but no gust of wind.

- From Exercise 29, find a vector form for the path of the baseball.
- How high does the baseball go, and when does it reach maximum height?
- Find the range and flight time of the baseball.
- When is the baseball 30 ft high? How far (ground distance) is the baseball from home plate at that height?
- A 10-ft-high outfield fence is 340 ft from home plate in the direction of the flight of the baseball. The outfielder can jump and catch any ball up to 11 ft off the ground to stop it from going over the fence. Has the batter hit a home run?

- 31. Hitting a baseball with linear drag under a wind gust** Consider again the baseball problem in Example 4. This time assume a drag coefficient of 0.08 *and* an instantaneous gust of wind that adds a component of $-17.6\mathbf{i}$ (ft/sec) to the initial velocity at the instant the baseball is hit.

- Find a vector equation for the path of the baseball.
- How high does the baseball go, and when does it reach maximum height?
- Find the range and flight time of the baseball.
- When is the baseball 35 ft high? How far (ground distance) is the baseball from home plate at that height?
- A 20-ft-high outfield fence is 380 ft from home plate in the direction of the flight of the baseball. Has the batter hit a home run? If "yes," what change in the horizontal component of the ball's initial velocity would have kept the ball in the park? If "no," what change would have allowed it to be a home run?



13.3

Arc Length and the Unit Tangent Vector **T**

Imagine the motions you might experience traveling at high speeds along a path through the air or space. Specifically, imagine the motions of turning to your left or right and the up-and-down motions tending to lift you from, or pin you down to, your seat. Pilots flying through the atmosphere, turning and twisting in flight acrobatics, certainly experience these motions. Turns that are too tight, descents or climbs that are too steep, or either one coupled with high and increasing speed can cause an aircraft to spin out of control, possibly even to break up in midair, and crash to Earth.

In this and the next two sections, we study the features of a curve's shape that describe mathematically the sharpness of its turning and its twisting perpendicular to the forward motion.

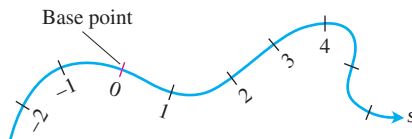


FIGURE 13.14 Smooth curves can be scaled like number lines, the coordinate of each point being its directed distance along the curve from a preselected base point.

Arc Length Along a Space Curve

One of the features of smooth space curves is that they have a measurable length. This enables us to locate points along these curves by giving their directed distance s along the curve from some **base point**, the way we locate points on coordinate axes by giving their directed distance from the origin (Figure 13.14). Time is the natural parameter for describing a moving body's velocity and acceleration, but s is the natural parameter for studying a curve's shape. Both parameters appear in analyses of space flight.

To measure distance along a smooth curve in space, we add a z -term to the formula we use for curves in the plane.

DEFINITION Length of a Smooth Curve

The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt. \quad (1)$$

Just as for plane curves, we can calculate the length of a curve in space from any convenient parametrization that meets the stated conditions. We omit the proof.

The square root in Equation (1) is $|\mathbf{v}|$, the length of a velocity vector $d\mathbf{r}/dt$. This enables us to write the formula for length a shorter way.

Arc Length Formula

$$L = \int_a^b |\mathbf{v}| dt \quad (2)$$

EXAMPLE 1 Distance Traveled by a Glider

A glider is soaring upward along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. How far does the glider travel along its path from $t = 0$ to $t = 2\pi \approx 6.28$ sec?

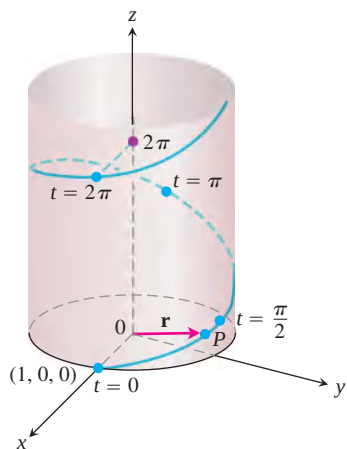


FIGURE 13.15 The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ in Example 1.

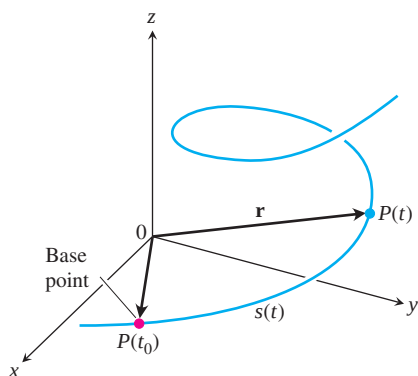


FIGURE 13.16 The directed distance along the curve from $P(t_0)$ to any point $P(t)$ is

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau.$$



Solution The path segment during this time corresponds to one full turn of the helix (Figure 13.15). The length of this portion of the curve is

$$\begin{aligned} L &= \int_a^b |\mathbf{v}| dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} dt \\ &= \int_0^{2\pi} \sqrt{2} dt = 2\pi\sqrt{2} \text{ units of length.} \end{aligned}$$

This is $\sqrt{2}$ times the length of the circle in the xy -plane over which the helix stands. ■

If we choose a base point $P(t_0)$ on a smooth curve C parametrized by t , each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a “directed distance”

$$s(t) = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau,$$

measured along C from the base point (Figure 13.16). If $t > t_0$, $s(t)$ is the distance from $P(t_0)$ to $P(t)$. If $t < t_0$, $s(t)$ is the negative of the distance. Each value of s determines a point on C and this parametrizes C with respect to s . We call s an **arc length parameter** for the curve. The parameter’s value increases in the direction of increasing t . The arc length parameter is particularly effective for investigating the turning and twisting nature of a space curve.

We use the Greek letter τ (“tau”) as the variable of integration because the letter t is already in use as the upper limit.

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^t \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2} d\tau = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau \quad (3)$$

If a curve $\mathbf{r}(t)$ is already given in terms of some parameter t and $s(t)$ is the arc length function given by Equation (3), then we may be able to solve for t as a function of s : $t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for t : $\mathbf{r} = \mathbf{r}(t(s))$.

EXAMPLE 2 Finding an Arc Length Parametrization

If $t_0 = 0$, the arc length parameter along the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

from t_0 to t is

$$\begin{aligned} s(t) &= \int_{t_0}^t |\mathbf{v}(\tau)| d\tau && \text{Equation (3)} \\ &= \int_0^t \sqrt{2} d\tau && \text{Value from Example 1} \\ &= \sqrt{2} t. \end{aligned}$$

Solving this equation for t gives $t = s/\sqrt{2}$. Substituting into the position vector $\mathbf{r}$ gives the following arc length parametrization for the helix:

$$\mathbf{r}(t(s)) = \left(\cos \frac{s}{\sqrt{2}} \right) \mathbf{i} + \left(\sin \frac{s}{\sqrt{2}} \right) \mathbf{j} + \frac{s}{\sqrt{2}} \mathbf{k}. \quad \blacksquare$$

Unlike Example 2, the arc length parametrization is generally difficult to find analytically for a curve already given in terms of some other parameter t . Fortunately, however, we rarely need an exact formula for $s(t)$ or its inverse $t(s)$.

EXAMPLE 3 Distance Along a Line

Show that if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a unit vector, then the arc length parameter along the line

$$\mathbf{r}(t) = (x_0 + tu_1)\mathbf{i} + (y_0 + tu_2)\mathbf{j} + (z_0 + tu_3)\mathbf{k}$$

from the point $P_0(x_0, y_0, z_0)$ where $t = 0$ is t itself.

Solution

$$\mathbf{v} = \frac{d}{dt}(x_0 + tu_1)\mathbf{i} + \frac{d}{dt}(y_0 + tu_2)\mathbf{j} + \frac{d}{dt}(z_0 + tu_3)\mathbf{k} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = \mathbf{u},$$

so

$$s(t) = \int_0^t |\mathbf{v}| d\tau = \int_0^t |\mathbf{u}| d\tau = \int_0^t 1 d\tau = t. \quad \blacksquare$$

HISTORICAL BIOGRAPHY

Josiah Willard Gibbs
(1839–1903)



Speed on a Smooth Curve

Since the derivatives beneath the radical in Equation (3) are continuous (the curve is smooth), the Fundamental Theorem of Calculus tells us that s is a differentiable function of t with derivative

$$\frac{ds}{dt} = |\mathbf{v}(t)|. \quad (4)$$

As we already knew, the speed with which a particle moves along its path is the magnitude of $\mathbf{v}$.

Notice that although the base point $P(t_0)$ plays a role in defining s in Equation (3), it plays no role in Equation (4). The rate at which a moving particle covers distance along its path is independent of how far away it is from the base point.

Notice also that $ds/dt > 0$ since, by definition, $|\mathbf{v}|$ is never zero for a smooth curve. We see once again that s is an increasing function of t .

Unit Tangent Vector **T**

We already know the velocity vector $\mathbf{v} = d\mathbf{r}/dt$ is tangent to the curve and that the vector

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

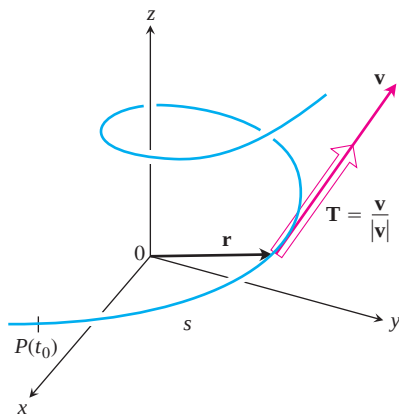


FIGURE 13.17 We find the unit tangent vector $\mathbf{T}$ by dividing $\mathbf{v}$ by $|\mathbf{v}|$.

is therefore a unit vector tangent to the (smooth) curve. Since $ds/dt > 0$ for the curves we are considering, s is one-to-one and has an inverse that gives t as a differentiable function of s (Section 7.1). The derivative of the inverse is

$$\frac{dt}{ds} = \frac{1}{ds/dt} = \frac{1}{|\mathbf{v}|}.$$

This makes $\mathbf{r}$ a differentiable function of s whose derivative can be calculated with the Chain Rule to be

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \frac{dt}{ds} = \mathbf{v} \frac{1}{|\mathbf{v}|} = \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T}.$$

This equation says that $d\mathbf{r}/ds$ is the unit tangent vector in the direction of the velocity vector $\mathbf{v}$ (Figure 13.17).

DEFINITION Unit Tangent Vector

The **unit tangent vector** of a smooth curve $\mathbf{r}(t)$ is

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{v}}{|\mathbf{v}|}. \quad (5)$$

The unit tangent vector $\mathbf{T}$ is a differentiable function of t whenever $\mathbf{v}$ is a differentiable function of t . As we see in Section 13.5, $\mathbf{T}$ is one of three unit vectors in a traveling reference frame that is used to describe the motion of space vehicles and other bodies traveling in three dimensions.



EXAMPLE 4 Finding the Unit Tangent Vector $\mathbf{T}$

Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t^2\mathbf{k}$$

representing the path of the glider in Example 4, Section 13.1.

Solution In that example, we found

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 2t\mathbf{k}$$

and

$$|\mathbf{v}| = \sqrt{9 + 4t^2}.$$

Thus,

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3 \sin t}{\sqrt{9 + 4t^2}}\mathbf{i} + \frac{3 \cos t}{\sqrt{9 + 4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9 + 4t^2}}\mathbf{k}. \quad \blacksquare$$

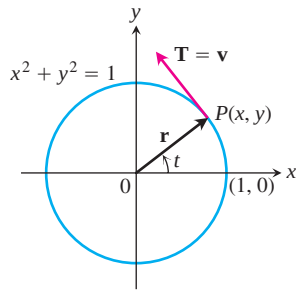


FIGURE 13.18 The motion $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$ (Example 5).

EXAMPLE 5 Motion on the Unit Circle

For the counterclockwise motion

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$$

around the unit circle,

$$\mathbf{v} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$$

is already a unit vector, so $\mathbf{T} = \mathbf{v}$ (Figure 13.18).

■

EXERCISES 13.3

Finding Unit Tangent Vectors and Lengths of Curves

In Exercises 1–8, find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

- $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}, \quad 0 \leq t \leq \pi$
- $\mathbf{r}(t) = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k}, \quad 0 \leq t \leq \pi$
- $\mathbf{r}(t) = t\mathbf{i} + (2/3)t^{3/2}\mathbf{k}, \quad 0 \leq t \leq 8$
- $\mathbf{r}(t) = (2 + t)\mathbf{i} - (t + 1)\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 3$
- $\mathbf{r}(t) = (\cos^3 t)\mathbf{j} + (\sin^3 t)\mathbf{k}, \quad 0 \leq t \leq \pi/2$
- $\mathbf{r}(t) = 6t^3\mathbf{i} - 2t^3\mathbf{j} - 3t^3\mathbf{k}, \quad 1 \leq t \leq 2$
- $\mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + (2\sqrt{2}/3)t^{3/2}\mathbf{k}, \quad 0 \leq t \leq \pi$
- $\mathbf{r}(t) = (t \sin t + \cos t)\mathbf{i} + (t \cos t - \sin t)\mathbf{j}, \quad \sqrt{2} \leq t \leq 2$
- Find the point on the curve

$$\mathbf{r}(t) = (5 \sin t)\mathbf{i} + (5 \cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the origin in the direction of increasing arc length.

10. Find the point on the curve

$$\mathbf{r}(t) = (12 \sin t)\mathbf{i} - (12 \cos t)\mathbf{j} + 5t\mathbf{k}$$

at a distance 13π units along the curve from the origin in the direction opposite to the direction of increasing arc length.

Arc Length Parameter

In Exercises 11–14, find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s = \int_0^t |\mathbf{v}(\tau)| d\tau$$

from Equation (3). Then find the length of the indicated portion of the curve.

- $\mathbf{r}(t) = (4 \cos t)\mathbf{i} + (4 \sin t)\mathbf{j} + 3t\mathbf{k}, \quad 0 \leq t \leq \pi/2$
- $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad \pi/2 \leq t \leq \pi$
- $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}, \quad -\ln 4 \leq t \leq 0$
- $\mathbf{r}(t) = (1 + 2t)\mathbf{i} + (1 + 3t)\mathbf{j} + (6 - 6t)\mathbf{k}, \quad -1 \leq t \leq 0$

Theory and Examples

15. **Arc length** Find the length of the curve

$$\mathbf{r}(t) = (\sqrt{2}t)\mathbf{i} + (\sqrt{2}t)\mathbf{j} + (1 - t^2)\mathbf{k}$$

from $(0, 0, 1)$ to $(\sqrt{2}, \sqrt{2}, 0)$.

16. **Length of helix** The length $2\pi\sqrt{2}$ of the turn of the helix in Example 1 is also the length of the diagonal of a square 2π units on a side. Show how to obtain this square by cutting away and flattening a portion of the cylinder around which the helix winds.

17. **Ellipse**

- Show that the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1 - \cos t)\mathbf{k}$, $0 \leq t \leq 2\pi$, is an ellipse by showing that it is the intersection of a right circular cylinder and a plane. Find equations for the cylinder and plane.
- Sketch the ellipse on the cylinder. Add to your sketch the unit tangent vectors at $t = 0, \pi/2, \pi$, and $3\pi/2$.
- Show that the acceleration vector always lies parallel to the plane (orthogonal to a vector normal to the plane). Thus, if you draw the acceleration as a vector attached to the ellipse, it will lie in the plane of the ellipse. Add the acceleration vectors for $t = 0, \pi/2, \pi$, and $3\pi/2$ to your sketch.
- Write an integral for the length of the ellipse. Do not try to evaluate the integral; it is nonelementary.

- T e. Numerical integrator** Estimate the length of the ellipse to two decimal places.

18. **Length is independent of parametrization** To illustrate that the length of a smooth space curve does not depend on



the parametrization you use to compute it, calculate the length of one turn of the helix in Example 1 with the following parametrizations.

a. $\mathbf{r}(t) = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + 4t\mathbf{k}, \quad 0 \leq t \leq \pi/2$

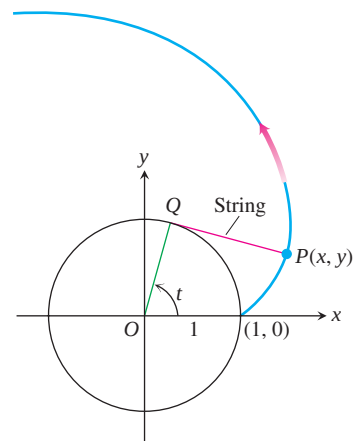
b. $\mathbf{r}(t) = [\cos(t/2)]\mathbf{i} + [\sin(t/2)]\mathbf{j} + (t/2)\mathbf{k}, \quad 0 \leq t \leq 4\pi$

c. $\mathbf{r}(t) = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - t\mathbf{k}, \quad -2\pi \leq t \leq 0$

- 19. The involute of a circle** If a string wound around a fixed circle is unwound while held taut in the plane of the circle, its end P traces an *involute* of the circle. In the accompanying figure, the circle in question is the circle $x^2 + y^2 = 1$ and the tracing point starts at $(1, 0)$. The unwound portion of the string is tangent to the circle at Q , and t is the radian measure of the angle from the positive x -axis to segment OQ . Derive the parametric equations

$$x = \cos t + t \sin t, \quad y = \sin t - t \cos t, \quad t > 0$$

of the point $P(x, y)$ for the involute.



- 20.** (Continuation of Exercise 19.) Find the unit tangent vector to the involute of the circle at the point $P(x, y)$.

13.4 Curvature and the Unit Normal Vector $\mathbf{N}$

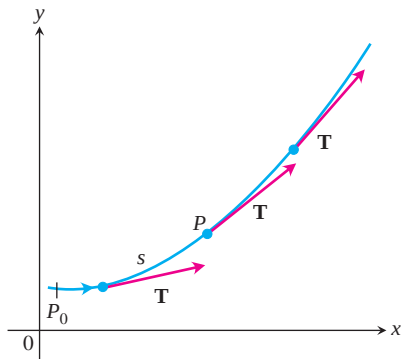


FIGURE 13.19 As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at P is called the *curvature* of the curve at P .

In this section we study how a curve turns or bends. We look first at curves in the coordinate plane, and then at curves in space.

Curvature of a Plane Curve

As a particle moves along a smooth curve in the plane, $\mathbf{T} = d\mathbf{r}/ds$ turns as the curve bends. Since $\mathbf{T}$ is a unit vector, its length remains constant and only its direction changes as the particle moves along the curve. The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the *curvature* (Figure 13.19). The traditional symbol for the curvature function is the Greek letter κ (“kappa”).

DEFINITION Curvature

If $\mathbf{T}$ is the unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|.$$

If $|d\mathbf{T}/ds|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P , and the curvature at P is large. If $|d\mathbf{T}/ds|$ is close to zero, $\mathbf{T}$ turns more slowly and the curvature at P is smaller.

If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length parameter s , we can calculate the curvature as

$$\begin{aligned} \kappa &= \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}}{dt} \frac{dt}{ds} \right| && \text{Chain Rule} \\ &= \frac{1}{|ds/dt|} \left| \frac{d\mathbf{T}}{dt} \right| \\ &= \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right|. && \frac{ds}{dt} = |\mathbf{v}| \end{aligned}$$



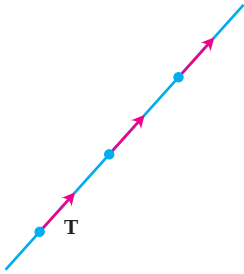


FIGURE 13.20 Along a straight line, $\mathbf{T}$ always points in the same direction. The curvature, $|d\mathbf{T}/ds|$, is zero (Example 1).

Formula for Calculating Curvature

If $\mathbf{r}(t)$ is a smooth curve, then the curvature is

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right|, \quad (1)$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Testing the definition, we see in Examples 1 and 2 below that the curvature is constant for straight lines and circles.

EXAMPLE 1 The Curvature of a Straight Line Is Zero

On a straight line, the unit tangent vector $\mathbf{T}$ always points in the same direction, so its components are constants. Therefore, $|d\mathbf{T}/ds| = |\mathbf{0}| = 0$ (Figure 13.20). ■

EXAMPLE 2 The Curvature of a Circle of Radius a is $1/a$

To see why, we begin with the parametrization

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$$

of a circle of radius a . Then,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{(-a \sin t)^2 + (a \cos t)^2} = \sqrt{a^2} = |a| = a. \quad \text{Since } a > 0, \text{ } |a| = a.$$

From this we find

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j}$$

$$\frac{d\mathbf{T}}{dt} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$$

$$\left| \frac{d\mathbf{T}}{dt} \right| = \sqrt{\cos^2 t + \sin^2 t} = 1.$$

Hence, for any value of the parameter t ,

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{a}(1) = \frac{1}{a}. \quad \blacksquare$$

Although the formula for calculating κ in Equation (1) is also valid for space curves, in the next section we find a computational formula that is usually more convenient to apply.

Among the vectors orthogonal to the unit tangent vector $\mathbf{T}$ is one of particular significance because it points in the direction in which the curve is turning. Since $\mathbf{T}$ has constant length (namely, 1), the derivative $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$ (Section 13.1). Therefore, if we divide $d\mathbf{T}/ds$ by its length κ , we obtain a *unit* vector $\mathbf{N}$ orthogonal to $\mathbf{T}$ (Figure 13.21).

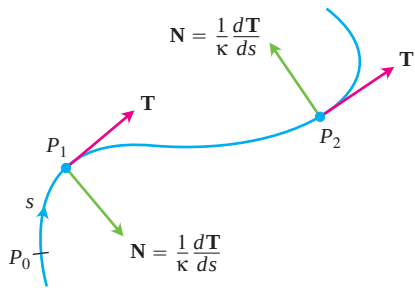


FIGURE 13.21 The vector $d\mathbf{T}/ds$, normal to the curve, always points in the direction in which $\mathbf{T}$ is turning. The unit normal vector $\mathbf{N}$ is the direction of $d\mathbf{T}/ds$.

DEFINITION Principal Unit Normal

At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}.$$

The vector $d\mathbf{T}/ds$ points in the direction in which $\mathbf{T}$ turns as the curve bends. Therefore, if we face in the direction of increasing arc length, the vector $d\mathbf{T}/ds$ points toward the right if $\mathbf{T}$ turns clockwise and toward the left if $\mathbf{T}$ turns counterclockwise. In other words, the principal normal vector $\mathbf{N}$ will point toward the concave side of the curve (Figure 13.21).

If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length parameter s , we can use the Chain Rule to calculate $\mathbf{N}$ directly:

$$\begin{aligned} \mathbf{N} &= \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} \\ &= \frac{(d\mathbf{T}/dt)(dt/ds)}{|d\mathbf{T}/dt||dt/ds|} \\ &= \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}. \quad \frac{dt}{ds} = \frac{1}{ds/dt} > 0 \text{ cancels} \end{aligned}$$

This formula enables us to find $\mathbf{N}$ without having to find κ and s first.

Formula for Calculating $\mathbf{N}$

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}, \quad (2)$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.



EXAMPLE 3 Finding $\mathbf{T}$ and $\mathbf{N}$

Find $\mathbf{T}$ and $\mathbf{N}$ for the circular motion

$$\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j}.$$

Solution We first find $\mathbf{T}$:

$$\mathbf{v} = -(2 \sin 2t)\mathbf{i} + (2 \cos 2t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{4 \sin^2 2t + 4 \cos^2 2t} = 2$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j}.$$

From this we find

$$\begin{aligned}\frac{d\mathbf{T}}{dt} &= -(2 \cos 2t)\mathbf{i} - (2 \sin 2t)\mathbf{j} \\ \left| \frac{d\mathbf{T}}{dt} \right| &= \sqrt{4 \cos^2 2t + 4 \sin^2 2t} = 2\end{aligned}$$

and

$$\begin{aligned}\mathbf{N} &= \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \\ &= -(\cos 2t)\mathbf{i} - (\sin 2t)\mathbf{j}. \quad \text{Equation (2)}\end{aligned}$$

Notice that $\mathbf{T} \cdot \mathbf{N} = 0$, verifying that $\mathbf{N}$ is orthogonal to $\mathbf{T}$. Notice too, that for the circular motion here, $\mathbf{N}$ points from $\mathbf{r}(t)$ towards the circle's center at the origin. ■

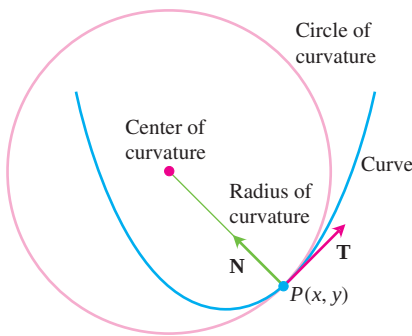


FIGURE 13.22 The osculating circle at $P(x, y)$ lies toward the inner side of the curve.

Circle of Curvature for Plane Curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. lies toward the concave or inner side of the curve (as in Figure 13.22).

The **radius of curvature** of the curve at P is the radius of the circle of curvature, which, according to Example 2, is

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$

To find ρ , we find κ and take the reciprocal. The **center of curvature** of the curve at P is the center of the circle of curvature.



EXAMPLE 4 Finding the Osculating Circle for a Parabola

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Solution We parametrize the parabola using the parameter $t = x$ (Section 10.4, Example 1)

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}.$$

First we find the curvature of the parabola at the origin, using Equation (1):

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \\ |\mathbf{v}| &= \sqrt{1 + 4t^2}\end{aligned}$$

so that

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (1 + 4t^2)^{-1/2}\mathbf{i} + 2t(1 + 4t^2)^{-1/2}\mathbf{j}.$$

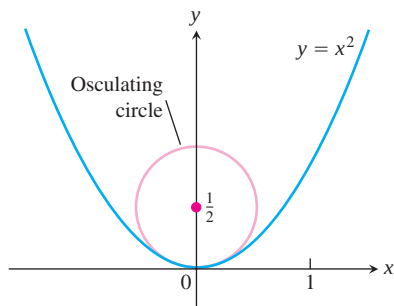


FIGURE 13.23 The osculating circle for the parabola $y = x^2$ at the origin (Example 4).

From this we find

$$\frac{d\mathbf{T}}{dt} = -4t(1 + 4t^2)^{-3/2}\mathbf{i} + [2(1 + 4t^2)^{-1/2} - 8t^2(1 + 4t^2)^{-3/2}]\mathbf{j}.$$

At the origin, $t = 0$, so the curvature is

$$\begin{aligned}\kappa(0) &= \frac{1}{|\mathbf{v}(0)|} \left| \frac{d\mathbf{T}}{dt}(0) \right| && \text{Equation (1)} \\ &= \frac{1}{\sqrt{1}} |0\mathbf{i} + 2\mathbf{j}| \\ &= (1)\sqrt{0^2 + 2^2} = 2.\end{aligned}$$

Therefore, the radius of curvature is $1/\kappa = 1/2$ and the center of the circle is $(0, 1/2)$ (see Figure 13.23). The equation of the osculating circle is

$$(x - 0)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

or

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

You can see from Figure 13.23 that the osculating circle is a better approximation to the parabola at the origin than is the tangent line approximation $y = 0$. ■

Curvature and Normal Vectors for Space Curves

If a smooth curve in space is specified by the position vector $\mathbf{r}(t)$ as a function of some parameter t , and if s is the arc length parameter of the curve, then the unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$. The **curvature** in space is then defined to be

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| \quad (3)$$

just as for plane curves. The vector $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$, and we define the **principal unit normal** to be

$$\mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}. \quad (4)$$

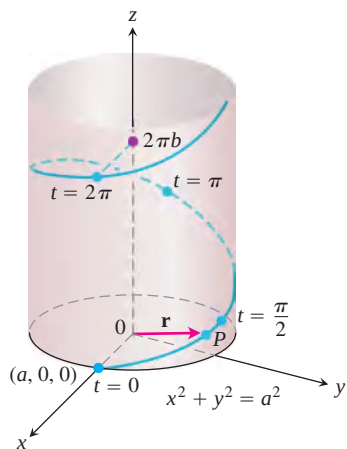


FIGURE 13.24 The helix

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k},$$

drawn with a and b positive and $t \geq 0$ (Example 5).

EXAMPLE 5 Finding Curvature

Find the curvature for the helix (Figure 13.24)

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k}, \quad a, b \geq 0, \quad a^2 + b^2 \neq 0.$$



Solution We calculate **T** from the velocity vector **v**:

$$\mathbf{v} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}$$

$$|\mathbf{v}| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2 + b^2}} [-(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}].$$

Then using Equation (3),

$$\begin{aligned} \kappa &= \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| \\ &= \frac{1}{\sqrt{a^2 + b^2}} \left| \frac{1}{\sqrt{a^2 + b^2}} [-(a \cos t)\mathbf{i} - (a \sin t)\mathbf{j}] \right| \\ &= \frac{a}{a^2 + b^2} |-(\cos t)\mathbf{i} - (\sin t)\mathbf{j}| \\ &= \frac{a}{a^2 + b^2} \sqrt{(\cos t)^2 + (\sin t)^2} = \frac{a}{a^2 + b^2}. \end{aligned}$$

From this equation, we see that increasing b for a fixed a decreases the curvature. Decreasing a for a fixed b eventually decreases the curvature as well. Stretching a spring tends to straighten it.

If $b = 0$, the helix reduces to a circle of radius a and its curvature reduces to $1/a$, as it should. If $a = 0$, the helix becomes the z -axis, and its curvature reduces to 0, again as it should. ■

EXAMPLE 6 Finding the Principal Unit Normal Vector **N**

Find **N** for the helix in Example 5.

Solution We have

$$\frac{d\mathbf{T}}{dt} = -\frac{1}{\sqrt{a^2 + b^2}} [(a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}]$$

Example 5

$$\left| \frac{d\mathbf{T}}{dt} \right| = \frac{1}{\sqrt{a^2 + b^2}} \sqrt{a^2 \cos^2 t + a^2 \sin^2 t} = \frac{a}{\sqrt{a^2 + b^2}}$$

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

Equation (4)

$$\begin{aligned} &= -\frac{\sqrt{a^2 + b^2}}{a} \cdot \frac{1}{\sqrt{a^2 + b^2}} [(a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}] \\ &= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}. \end{aligned}$$

■

EXERCISES 13.4

Plane Curves

Find $\mathbf{T}$, $\mathbf{N}$, and κ for the plane curves in Exercises 1–4.

1. $\mathbf{r}(t) = t\mathbf{i} + (\ln \cos t)\mathbf{j}$, $-\pi/2 < t < \pi/2$
2. $\mathbf{r}(t) = (\ln \sec t)\mathbf{i} + t\mathbf{j}$, $-\pi/2 < t < \pi/2$
3. $\mathbf{r}(t) = (2t + 3)\mathbf{i} + (5 - t^2)\mathbf{j}$
4. $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}$, $t > 0$
5. **A formula for the curvature of the graph of a function in the xy -plane**
 - a. The graph $y = f(x)$ in the xy -plane automatically has the parametrization $x = x$, $y = f(x)$, and the vector formula $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Use this formula to show that if f is a twice-differentiable function of x , then

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

- b. Use the formula for κ in part (a) to find the curvature of $y = \ln(\cos x)$, $-\pi/2 < x < \pi/2$. Compare your answer with the answer in Exercise 1.
 - c. Show that the curvature is zero at a point of inflection.
6. **A formula for the curvature of a parametrized plane curve**
 - a. Show that the curvature of a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ defined by twice-differentiable functions $x = f(t)$ and $y = g(t)$ is given by the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

Apply the formula to find the curvatures of the following curves.

- b. $\mathbf{r}(t) = t\mathbf{i} + (\ln \sin t)\mathbf{j}$, $0 < t < \pi$
 - c. $\mathbf{r}(t) = [\tan^{-1}(\sinh t)]\mathbf{i} + (\ln \cosh t)\mathbf{j}$.
7. **Normals to plane curves**
 - a. Show that $\mathbf{n}(t) = -g'(t)\mathbf{i} + f'(t)\mathbf{j}$ and $-\mathbf{n}(t) = g'(t)\mathbf{i} - f'(t)\mathbf{j}$ are both normal to the curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ at the point $(f(t), g(t))$.
To obtain $\mathbf{N}$ for a particular plane curve, we can choose the one of $\mathbf{n}$ or $-\mathbf{n}$ from part (a) that points toward the concave side of the curve, and make it into a unit vector. (See Figure 13.21.) Apply this method to find $\mathbf{N}$ for the following curves.
 - b. $\mathbf{r}(t) = t\mathbf{i} + e^{2t}\mathbf{j}$
 - c. $\mathbf{r}(t) = \sqrt{4 - t^2}\mathbf{i} + t\mathbf{j}$, $-2 \leq t \leq 2$
8. (Continuation of Exercise 7.)
 - a. Use the method of Exercise 7 to find $\mathbf{N}$ for the curve $\mathbf{r}(t) = t\mathbf{i} + (1/3)t^3\mathbf{j}$ when $t < 0$; when $t > 0$.

b. Calculate

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}, \quad t \neq 0,$$

for the curve in part (a). Does $\mathbf{N}$ exist at $t = 0$? Graph the curve and explain what is happening to $\mathbf{N}$ as t passes from negative to positive values.

Space Curves

Find $\mathbf{T}$, $\mathbf{N}$, and κ for the space curves in Exercises 9–16.

9. $\mathbf{r}(t) = (3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4t\mathbf{k}$
10. $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} + 3t\mathbf{k}$
11. $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + 2t\mathbf{k}$
12. $\mathbf{r}(t) = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k}$
13. $\mathbf{r}(t) = (t^3/3)\mathbf{i} + (t^2/2)\mathbf{j}$, $t > 0$
14. $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}$, $0 < t < \pi/2$
15. $\mathbf{r}(t) = t\mathbf{i} + (a \cosh(t/a))\mathbf{j}$, $a > 0$
16. $\mathbf{r}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k}$

More on Curvature

17. Show that the parabola $y = ax^2$, $a \neq 0$, has its largest curvature at its vertex and has no minimum curvature. (Note: Since the curvature of a curve remains the same if the curve is translated or rotated, this result is true for any parabola.)
18. Show that the ellipse $x = a \cos t$, $y = b \sin t$, $a > b > 0$, has its largest curvature on its major axis and its smallest curvature on its minor axis. (As in Exercise 17, the same is true for any ellipse.)
19. **Maximizing the curvature of a helix** In Example 5, we found the curvature of the helix $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k}$ ($a, b \geq 0$) to be $\kappa = a/(a^2 + b^2)$. What is the largest value κ can have for a given value of b ? Give reasons for your answer.
20. **Total curvature** We find the **total curvature** of the portion of a smooth curve that runs from $s = s_0$ to $s = s_1 > s_0$ by integrating κ from s_0 to s_1 . If the curve has some other parameter, say t , then the total curvature is

$$K = \int_{s_0}^{s_1} \kappa \, ds = \int_{t_0}^{t_1} \kappa \frac{ds}{dt} \, dt = \int_{t_0}^{t_1} \kappa |\mathbf{v}| \, dt,$$

where t_0 and t_1 correspond to s_0 and s_1 . Find the total curvatures of

- a. The portion of the helix $\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 4\pi$.
 - b. The parabola $y = x^2$, $-\infty < x < \infty$.
21. Find an equation for the circle of curvature of the curve $\mathbf{r}(t) = t\mathbf{i} + (\sin t)\mathbf{j}$ at the point $(\pi/2, 1)$. (The curve parametrizes the graph of $y = \sin x$ in the xy -plane.)



22. Find an equation for the circle of curvature of the curve $\mathbf{r}(t) = (2 \ln t)\mathbf{i} - [t + (1/t)]\mathbf{j}$, $e^{-2} \leq t \leq e^2$, at the point $(0, -2)$, where $t = 1$.

T Grapher Explorations

The formula

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}},$$

derived in Exercise 5, expresses the curvature $\kappa(x)$ of a twice-differentiable plane curve $y = f(x)$ as a function of x . Find the curvature function of each of the curves in Exercises 23–26. Then graph $f(x)$ together with $\kappa(x)$ over the given interval. You will find some surprises.

23. $y = x^2$, $-2 \leq x \leq 2$ 24. $y = x^4/4$, $-2 \leq x \leq 2$
 25. $y = \sin x$, $0 \leq x \leq 2\pi$ 26. $y = e^x$, $-1 \leq x \leq 2$

COMPUTER EXPLORATIONS

Circles of Curvature

In Exercises 27–34 you will use a CAS to explore the osculating circle at a point P on a plane curve where $\kappa \neq 0$. Use a CAS to perform the following steps:

- Plot the plane curve given in parametric or function form over the specified interval to see what it looks like.
- Calculate the curvature κ of the curve at the given value t_0 using the appropriate formula from Exercise 5 or 6. Use the parametrization $x = t$ and $y = f(t)$ if the curve is given as a function $y = f(x)$.

- Find the unit normal vector $\mathbf{N}$ at t_0 . Notice that the signs of the components of $\mathbf{N}$ depend on whether the unit tangent vector $\mathbf{T}$ is turning clockwise or counterclockwise at $t = t_0$. (See Exercise 7.)
- If $\mathbf{C} = a\mathbf{i} + b\mathbf{j}$ is the vector from the origin to the center (a, b) of the osculating circle, find the center $\mathbf{C}$ from the vector equation

$$\mathbf{C} = \mathbf{r}(t_0) + \frac{1}{\kappa(t_0)} \mathbf{N}(t_0).$$

The point $P(x_0, y_0)$ on the curve is given by the position vector $\mathbf{r}(t_0)$.

- Plot implicitly the equation $(x - a)^2 + (y - b)^2 = 1/\kappa^2$ of the osculating circle. Then plot the curve and osculating circle together. You may need to experiment with the size of the viewing window, but be sure it is square.
27. $\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (5 \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$, $t_0 = \pi/4$
 28. $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}$, $0 \leq t \leq 2\pi$, $t_0 = \pi/4$
 29. $\mathbf{r}(t) = t^2\mathbf{i} + (t^3 - 3t)\mathbf{j}$, $-4 \leq t \leq 4$, $t_0 = 3/5$
 30. $\mathbf{r}(t) = (t^3 - 2t^2 - t)\mathbf{i} + \frac{3t}{\sqrt{1 + t^2}}\mathbf{j}$, $-2 \leq t \leq 5$, $t_0 = 1$
 31. $\mathbf{r}(t) = (2t - \sin t)\mathbf{i} + (2 - 2 \cos t)\mathbf{j}$, $0 \leq t \leq 3\pi$, $t_0 = 3\pi/2$
 32. $\mathbf{r}(t) = (e^{-t} \cos t)\mathbf{i} + (e^{-t} \sin t)\mathbf{j}$, $0 \leq t \leq 6\pi$, $t_0 = \pi/4$
 33. $y = x^2 - x$, $-2 \leq x \leq 5$, $x_0 = 1$
 34. $y = x(1 - x)^{2/5}$, $-1 \leq x \leq 2$, $x_0 = 1/2$

13.5

Torsion and the Unit Binormal Vector **B**

Project



Project

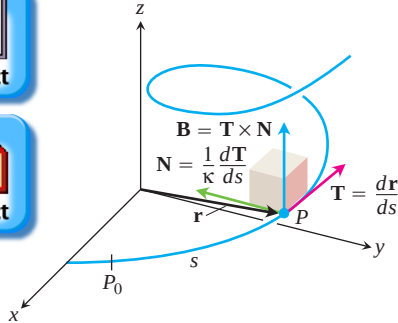


FIGURE 13.25 The **TNB** frame of mutually orthogonal unit vectors traveling along a curve in space.

If you are traveling along a space curve, the Cartesian **i**, **j**, and **k** coordinate system for representing the vectors describing your motion are not truly relevant to you. What is meaningful instead are the vectors representative of your forward direction (the unit tangent vector **T**), the direction in which your path is turning (the unit normal vector **N**), and the tendency of your motion to “twist” out of the plane created by these vectors in the direction perpendicular to this plane (defined by the *unit binormal vector* **B** = **T** × **N**). Expressing the acceleration vector along the curve as a linear combination of this **TNB** frame of mutually orthogonal unit vectors traveling with the motion (Figure 13.25) is particularly revealing of the nature of the path and motion along it.

Torsion

The **binormal vector** of a curve in space is **B** = **T** × **N**, a unit vector orthogonal to both **T** and **N** (Figure 13.26). Together **T**, **N**, and **B** define a moving right-handed vector frame that plays a significant role in calculating the paths of particles moving through space. It is

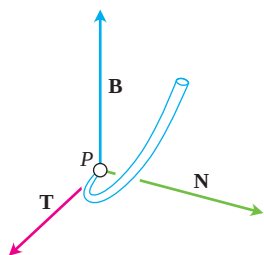


FIGURE 13.26 The vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ (in that order) make a right-handed frame of mutually orthogonal unit vectors in space.

called the **Frenet** (“fre-nay”) **frame** (after Jean-Frédéric Frenet, 1816–1900), or the **TNB frame**.

How does $d\mathbf{B}/ds$ behave in relation to $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$? From the rule for differentiating a cross product, we have

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds}.$$

Since $\mathbf{N}$ is the direction of $d\mathbf{T}/ds$, $(d\mathbf{T}/ds) \times \mathbf{N} = \mathbf{0}$ and

$$\frac{d\mathbf{B}}{ds} = \mathbf{0} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds}.$$

From this we see that $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$ since a cross product is orthogonal to its factors.

Since $d\mathbf{B}/ds$ is also orthogonal to $\mathbf{B}$ (the latter has constant length), it follows that $d\mathbf{B}/ds$ is orthogonal to the plane of $\mathbf{B}$ and $\mathbf{T}$. In other words, $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$, so $d\mathbf{B}/ds$ is a scalar multiple of $\mathbf{N}$. In symbols,

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

The negative sign in this equation is traditional. The scalar τ is called the *torsion* along the curve. Notice that

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\tau\mathbf{N} \cdot \mathbf{N} = -\tau(1) = -\tau,$$

so that

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$



DEFINITION Torsion

Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The **torsion** function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}. \quad (1)$$

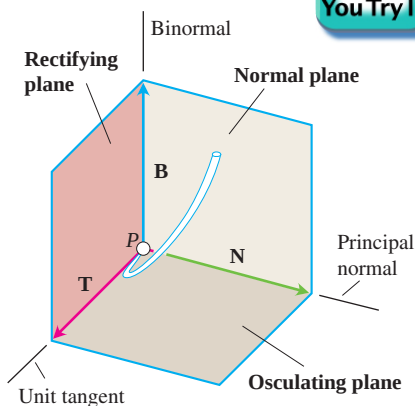


FIGURE 13.27 The names of the three planes determined by $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

Unlike the curvature κ , which is never negative, the torsion τ may be positive, negative, or zero.

The three planes determined by $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are named and shown in Figure 13.27. The curvature $\kappa = |d\mathbf{T}/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path. Similarly, the torsion $\tau = -(d\mathbf{B}/ds) \cdot \mathbf{N}$ is the rate at which the osculating plane turns about $\mathbf{T}$ as P moves along the curve. Torsion measures how the curve twists.

If we think of the curve as the path of a moving body, then $|d\mathbf{T}/ds|$ tells how much the path turns to the left or right as the object moves along; it is called the *curvature* of the object’s path. The number $-(d\mathbf{B}/ds) \cdot \mathbf{N}$ tells how much a body’s path rotates or

twists out of its plane of motion as the object moves along; it is called the *torsion* of the body's path. Look at Figure 13.28. If P is a train climbing up a curved track, the rate at which the headlight turns from side to side per unit distance is the curvature of the track. The rate at which the engine tends to twist out of the plane formed by $\mathbf{T}$ and $\mathbf{N}$ is the torsion.

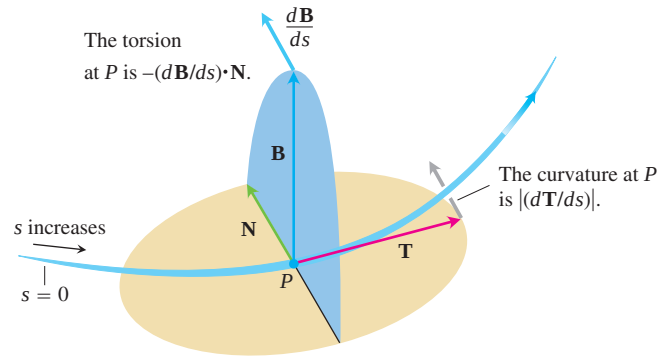


FIGURE 13.28 Every moving body travels with a **TNB** frame that characterizes the geometry of its path of motion.

Tangential and Normal Components of Acceleration

When a body is accelerated by gravity, brakes, a combination of rocket motors, or whatever, we usually want to know how much of the acceleration acts in the direction of motion, in the tangential direction $\mathbf{T}$. We can calculate this using the Chain Rule to rewrite $\mathbf{v}$ as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds} \frac{ds}{dt} = \mathbf{T} \frac{ds}{dt}$$

and differentiating both ends of this string of equalities to get

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left(\mathbf{T} \frac{ds}{dt} \right) = \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \frac{d\mathbf{T}}{dt} \\ &= \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \left(\frac{d\mathbf{T}}{ds} \frac{ds}{dt} \right) = \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \left(\kappa \mathbf{N} \frac{ds}{dt} \right) \quad \frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \\ &= \frac{d^2s}{dt^2} \mathbf{T} + \kappa \left(\frac{ds}{dt} \right)^2 \mathbf{N}. \end{aligned}$$

DEFINITION Tangential and Normal Components of Acceleration

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}, \quad (2)$$

where

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt} |\mathbf{v}| \quad \text{and} \quad a_N = \kappa \left(\frac{ds}{dt} \right)^2 = \kappa |\mathbf{v}|^2 \quad (3)$$

are the **tangential** and **normal** scalar components of acceleration.

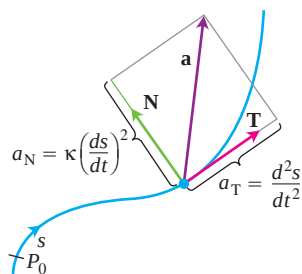


FIGURE 13.29 The tangential and normal components of acceleration. The acceleration $\mathbf{a}$ always lies in the plane of $\mathbf{T}$ and $\mathbf{N}$, orthogonal to $\mathbf{B}$.

Notice that the binormal vector $\mathbf{B}$ does not appear in Equation (2). No matter how the path of the moving body we are watching may appear to twist and turn in space, the acceleration $\mathbf{a}$ always lies in the plane of $\mathbf{T}$ and $\mathbf{N}$ orthogonal to $\mathbf{B}$. The equation also tells us exactly how much of the acceleration takes place tangent to the motion (d^2s/dt^2) and how much takes place normal to the motion [$\kappa(ds/dt)^2$] (Figure 13.29).

What information can we glean from Equations (3)? By definition, acceleration $\mathbf{a}$ is the rate of change of velocity $\mathbf{v}$, and in general, both the length and direction of $\mathbf{v}$ change as a body moves along its path. The tangential component of acceleration a_T measures the rate of change of the *length* of $\mathbf{v}$ (that is, the change in the speed). The normal component of acceleration a_N measures the rate of change of the *direction* of $\mathbf{v}$.

Notice that the normal scalar component of the acceleration is the curvature times the *square* of the speed. This explains why you have to hold on when your car makes a sharp (large κ), high-speed (large $|\mathbf{v}|$) turn. If you double the speed of your car, you will experience four times the normal component of acceleration for the same curvature.

If a body moves in a circle at a constant speed, d^2s/dt^2 is zero and all the acceleration points along $\mathbf{N}$ toward the circle's center. If the body is speeding up or slowing down, $\mathbf{a}$ has a nonzero tangential component (Figure 13.30).

To calculate a_N , we usually use the formula $a_N = \sqrt{|\mathbf{a}|^2 - a_T^2}$, which comes from solving the equation $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_T^2 + a_N^2$ for a_N . With this formula, we can find a_N without having to calculate κ first.

Formula for Calculating the Normal Component of Acceleration

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} \quad (4)$$

EXAMPLE 1 Finding the Acceleration Scalar Components a_T , a_N

Without finding $\mathbf{T}$ and $\mathbf{N}$, write the acceleration of the motion

$$\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad t > 0$$

in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$. (The path of the motion is the involute of the circle in Figure 13.31.)

Solution We use the first of Equations (3) to find a_T :

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-\sin t + \sin t + t \cos t)\mathbf{i} + (\cos t - \cos t + t \sin t)\mathbf{j}$$

$$= (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{t^2 \cos^2 t + t^2 \sin^2 t} = \sqrt{t^2} = |t| = t \quad t > 0$$

$$a_T = \frac{d}{dt} |\mathbf{v}| = \frac{d}{dt} (t) = 1. \quad \text{Equation (3)}$$

Knowing a_T , we use Equation (4) to find a_N :

$$\mathbf{a} = (\cos t - t \sin t)\mathbf{i} + (\sin t + t \cos t)\mathbf{j}$$

$$|\mathbf{a}|^2 = t^2 + 1 \quad \text{After some algebra}$$

$$\begin{aligned} a_N &= \sqrt{|\mathbf{a}|^2 - a_T^2} \\ &= \sqrt{(t^2 + 1) - (1)} = \sqrt{t^2} = t. \end{aligned}$$

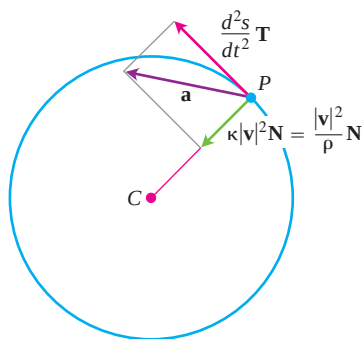


FIGURE 13.30 The tangential and normal components of the acceleration of a body that is speeding up as it moves counterclockwise around a circle of radius ρ .



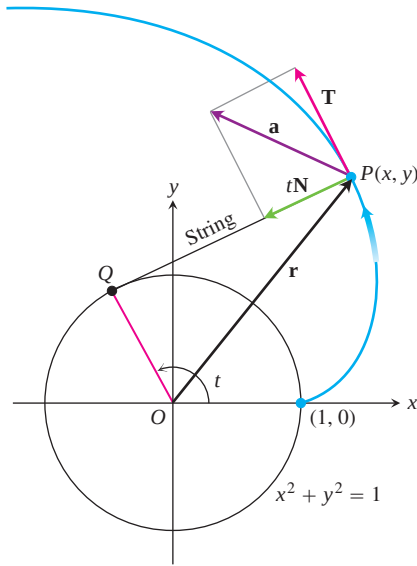


FIGURE 13.31 The tangential and normal components of the acceleration of the motion $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}$, for $t > 0$. If a string wound around a fixed circle is unwound while held taut in the plane of the circle, its end P traces an involute of the circle (Example 1).

We then use Equation (2) to find $\mathbf{a}$:

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} = (1)\mathbf{T} + (t)\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$

Formulas for Computing Curvature and Torsion

We now give some easy-to-use formulas for computing the curvature and torsion of a smooth curve. From Equation (2), we have

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \left(\frac{ds}{dt} \mathbf{T} \right) \times \left[\frac{d^2s}{dt^2} \mathbf{T} + \kappa \left(\frac{ds}{dt} \right)^2 \mathbf{N} \right] & \mathbf{v} = d\mathbf{r}/dt = (ds/dt)\mathbf{T} \\ &= \left(\frac{ds}{dt} \frac{d^2s}{dt^2} \right) (\mathbf{T} \times \mathbf{T}) + \kappa \left(\frac{ds}{dt} \right)^3 (\mathbf{T} \times \mathbf{N}) \\ &= \kappa \left(\frac{ds}{dt} \right)^3 \mathbf{B}. & \begin{aligned} \mathbf{T} \times \mathbf{T} &= \mathbf{0} \quad \text{and} \\ \mathbf{T} \times \mathbf{N} &= \mathbf{B} \end{aligned} \end{aligned}$$

It follows that

$$|\mathbf{v} \times \mathbf{a}| = \kappa \left| \frac{ds}{dt} \right|^3 |\mathbf{B}| = \kappa |\mathbf{v}|^3. \quad \frac{ds}{dt} = |\mathbf{v}| \quad \text{and} \quad |\mathbf{B}| = 1$$

Solving for κ gives the following formula.

Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \quad (5)$$

Equation (5) calculates the curvature, a geometric property of the curve, from the velocity and acceleration of any vector representation of the curve in which $|\mathbf{v}|$ is different from zero. Take a moment to think about how remarkable this really is: From any formula for motion along a curve, no matter how variable the motion may be (as long as $\mathbf{v}$ is never zero), we can calculate a physical property of the curve that seems to have nothing to do with the way the curve is traversed.

The most widely used formula for torsion, derived in more advanced texts, is

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \quad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}). \quad (6)$$

Newton's Dot Notation for Derivatives

The dots in Equation (6) denote differentiation with respect to t , one derivative for each dot. Thus, $\dot{x}$ ("x dot") means dx/dt , $\ddot{x}$ ("x double dot") means d^2x/dt^2 , and $\dddot{x}$ ("x triple dot") means d^3x/dt^3 . Similarly, $\dot{y} = dy/dt$, and so on.

This formula calculates the torsion directly from the derivatives of the component functions $x = f(t)$, $y = g(t)$, $z = h(t)$ that make up $\mathbf{r}$. The determinant's first row comes from $\mathbf{v}$, the second row comes from $\mathbf{a}$, and the third row comes from $\dot{\mathbf{a}} = d\mathbf{a}/dt$.



EXAMPLE 2 Finding Curvature and Torsion

Use Equations (5) and (6) to find κ and τ for the helix

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k}, \quad a, b \geq 0, \quad a^2 + b^2 \neq 0.$$

Solution We calculate the curvature with Equation (5):

$$\mathbf{v} = -(a \sin t)\mathbf{i} + (a \cos t)\mathbf{j} + b\mathbf{k}$$

$$\mathbf{a} = -(a \cos t)\mathbf{i} - (a \sin t)\mathbf{j}$$

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \end{vmatrix} \\ &= (ab \sin t)\mathbf{i} - (ab \cos t)\mathbf{j} + a^2\mathbf{k} \\ \kappa &= \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{a^2b^2 + a^4}}{(a^2 + b^2)^{3/2}} = \frac{a\sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}. \end{aligned} \quad (7)$$

Notice that Equation (7) agrees with the result in Example 5 in Section 13.4, where we calculated the curvature directly from its definition.

To evaluate Equation (6) for the torsion, we find the entries in the determinant by differentiating $\mathbf{r}$ with respect to t . We already have $\mathbf{v}$ and $\mathbf{a}$, and

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}}{dt} = (a \sin t)\mathbf{i} - (a \cos t)\mathbf{j}.$$

Hence,

$$\begin{aligned} \tau &= \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{\ddot{x}} & \ddot{\ddot{y}} & \ddot{\ddot{z}} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{\begin{vmatrix} -a \sin t & a \cos t & b \\ -a \cos t & -a \sin t & 0 \\ a \sin t & -a \cos t & 0 \end{vmatrix}}{(a\sqrt{a^2 + b^2})^2} \\ &= \frac{b(a^2 \cos^2 t + a^2 \sin^2 t)}{a^2(a^2 + b^2)} \\ &= \frac{b}{a^2 + b^2}. \end{aligned} \quad \begin{array}{l} \text{Value of } |\mathbf{v} \times \mathbf{a}| \\ \text{from Equation (7)} \end{array}$$

From this last equation we see that the torsion of a helix about a circular cylinder is constant. In fact, constant curvature and constant torsion characterize the helix among all curves in space. ■

Formulas for Curves in Space

Unit tangent vector: $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Curvature: $\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Torsion: $\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$

Tangential and normal scalar components of acceleration: $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$

$$a_T = \frac{d}{dt} |\mathbf{v}|$$

$$a_N = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

EXERCISES 13.5

Finding Torsion and the Binormal Vector

For Exercises 1–8 you found $\mathbf{T}$, $\mathbf{N}$, and κ in Section 13.4 (Exercises 9–16). Find now $\mathbf{B}$ and τ for these space curves.

1. $\mathbf{r}(t) = (3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4t\mathbf{k}$
2. $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} + 3t\mathbf{k}$
3. $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + 2t\mathbf{k}$
4. $\mathbf{r}(t) = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k}$
5. $\mathbf{r}(t) = (t^3/3)\mathbf{i} + (t^2/2)\mathbf{j}, \quad t > 0$
6. $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, \quad 0 < t < \pi/2$
7. $\mathbf{r}(t) = t\mathbf{i} + (a \cosh(t/a))\mathbf{j}, \quad a > 0$
8. $\mathbf{r}(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k}$

Tangential and Normal Components of Acceleration

In Exercises 9 and 10, write $\mathbf{a}$ in the form $a_T\mathbf{T} + a_N\mathbf{N}$ without finding $\mathbf{T}$ and $\mathbf{N}$.

9. $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k}$
10. $\mathbf{r}(t) = (1 + 3t)\mathbf{i} + (t - 2)\mathbf{j} - 3t\mathbf{k}$

In Exercises 11–14, write $\mathbf{a}$ in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ at the given value of t without finding $\mathbf{T}$ and $\mathbf{N}$.

11. $\mathbf{r}(t) = (t + 1)\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, \quad t = 1$
12. $\mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + t^2\mathbf{k}, \quad t = 0$
13. $\mathbf{r}(t) = t^2\mathbf{i} + (t + (1/3)t^3)\mathbf{j} + (t - (1/3)t^3)\mathbf{k}, \quad t = 0$
14. $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}, \quad t = 0$

In Exercises 15 and 16, find $\mathbf{r}$, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the given value of t . Then find equations for the osculating, normal, and rectifying planes at that value of t .

15. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - t\mathbf{k}, \quad t = \pi/4$
16. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad t = 0$

Physical Applications

17. The speedometer on your car reads a steady 35 mph. Could you be accelerating? Explain.
18. Can anything be said about the acceleration of a particle that is moving at a constant speed? Give reasons for your answer.
19. Can anything be said about the speed of a particle whose acceleration is always orthogonal to its velocity? Give reasons for your answer.



20. An object of mass m travels along the parabola $y = x^2$ with a constant speed of 10 units/sec. What is the force on the object due to its acceleration at $(0, 0)$? at $(2^{1/2}, 2)$? Write your answers in terms of $\mathbf{i}$ and $\mathbf{j}$. (Remember Newton's law, $\mathbf{F} = m\mathbf{a}$.)
21. The following is a quotation from an article in *The American Mathematical Monthly*, titled "Curvature in the Eighties" by Robert Osserman (October 1990, page 731):

Curvature also plays a key role in physics. The magnitude of a force required to move an object at constant speed along a curved path is, according to Newton's laws, a constant multiple of the curvature of the trajectories.

Explain mathematically why the second sentence of the quotation is true.

22. Show that a moving particle will move in a straight line if the normal component of its acceleration is zero.
23. **A sometime shortcut to curvature** If you already know $|a_N|$ and $|\mathbf{v}|$, then the formula $a_N = \kappa|\mathbf{v}|^2$ gives a convenient way to find the curvature. Use it to find the curvature and radius of curvature of the curve

$$\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad t > 0.$$

(Take a_N and $|\mathbf{v}|$ from Example 1.)

24. Show that κ and τ are both zero for the line

$$\mathbf{r}(t) = (x_0 + At)\mathbf{i} + (y_0 + Bt)\mathbf{j} + (z_0 + Ct)\mathbf{k}.$$

Theory and Examples

25. What can be said about the torsion of a smooth plane curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$? Give reasons for your answer.
26. **The torsion of a helix** In Example 2, we found the torsion of the helix

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k}, \quad a, b \geq 0$$

to be $\tau = b/(a^2 + b^2)$. What is the largest value τ can have for a given value of a ? Give reasons for your answer.

27. **Differentiable curves with zero torsion lie in planes** That a sufficiently differentiable curve with zero torsion lies in a plane is a special case of the fact that a particle whose velocity remains perpendicular to a fixed vector $\mathbf{C}$ moves in a plane perpendicular to $\mathbf{C}$. This, in turn, can be viewed as the solution of the following problem in calculus.

Suppose $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is twice differentiable for all t in an interval $[a, b]$, that $\mathbf{r} = 0$ when $t = a$, and that $\mathbf{v} \cdot \mathbf{k} = 0$ for all t in $[a, b]$. Then $h(t) = 0$ for all t in $[a, b]$.

Solve this problem. (*Hint:* Start with $\mathbf{a} = d^2\mathbf{r}/dt^2$ and apply the initial conditions in reverse order.)

28. **A formula that calculates τ from $\mathbf{B}$ and $\mathbf{v}$** If we start with the definition $\tau = -(\mathbf{dB}/ds) \cdot \mathbf{N}$ and apply the Chain Rule to rewrite $d\mathbf{B}/ds$ as

$$\frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}}{dt} \frac{dt}{ds} = \frac{d\mathbf{B}}{dt} \frac{1}{|\mathbf{v}|},$$

we arrive at the formula

$$\tau = -\frac{1}{|\mathbf{v}|} \left(\frac{d\mathbf{B}}{dt} \cdot \mathbf{N} \right).$$

The advantage of this formula over Equation (6) is that it is easier to derive and state. The disadvantage is that it can take a lot of work to evaluate without a computer. Use the new formula to find the torsion of the helix in Example 2.

COMPUTER EXPLORATIONS

Curvature, Torsion, and the TNB Frame

Rounding the answers to four decimal places, use a CAS to find $\mathbf{v}$, $\mathbf{a}$, speed, $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ , τ , and the tangential and normal components of acceleration for the curves in Exercises 29–32 at the given values of t .

29. $\mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + t\mathbf{k}, \quad t = \sqrt{3}$

30. $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}, \quad t = \ln 2$

31. $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j} + \sqrt{-t}\mathbf{k}, \quad t = -3\pi$

32. $\mathbf{r}(t) = (3t - t^2)\mathbf{i} + (3t^2)\mathbf{j} + (3t + t^3)\mathbf{k}, \quad t = 1$

13.6

Planetary Motion and Satellites

In this section, we derive Kepler's laws of planetary motion from Newton's laws of motion and gravitation and discuss the orbits of Earth satellites. The derivation of Kepler's laws from Newton's is one of the triumphs of calculus. It draws on almost everything we have studied so far, including the algebra and geometry of vectors in space, the calculus of vector functions, the solutions of differential equations and initial value problems, and the polar coordinate description of conic sections.

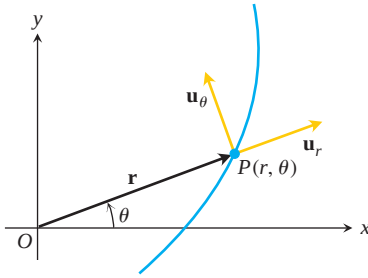


FIGURE 13.32 The length of $\mathbf{r}$ is the positive polar coordinate r of the point P . Thus, $\mathbf{u}_r$, which is $\mathbf{r}/|\mathbf{r}|$, is also $\mathbf{r}/r$. Equations (1) express $\mathbf{u}_r$ and $\mathbf{u}_\theta$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

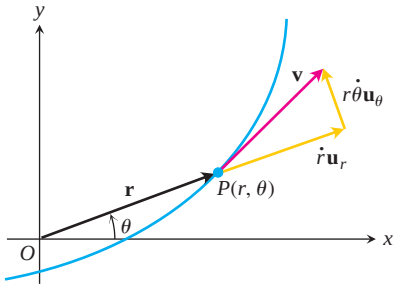


FIGURE 13.33 In polar coordinates, the velocity vector is

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta$$

Notice that $|\mathbf{r}| \neq r$ if $z \neq 0$.

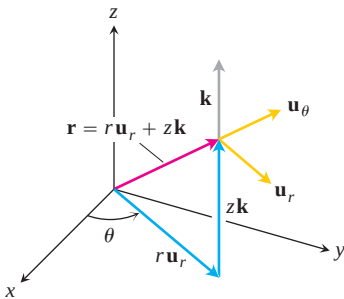


FIGURE 13.34 Position vector and basic unit vectors in cylindrical coordinates.

Motion in Polar and Cylindrical Coordinates

When a particle moves along a curve in the polar coordinate plane, we express its position, velocity, and acceleration in terms of the moving unit vectors

$$\mathbf{u}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}, \quad \mathbf{u}_\theta = -(\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j}, \quad (1)$$

shown in Figure 13.32. The vector $\mathbf{u}_r$ points along the position vector $\vec{OP}$, so $\mathbf{r} = r\mathbf{u}_r$. The vector $\mathbf{u}_\theta$, orthogonal to $\mathbf{u}_r$, points in the direction of increasing θ .

We find from Equations (1) that

$$\begin{aligned} \frac{d\mathbf{u}_r}{d\theta} &= -(\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j} = \mathbf{u}_\theta \\ \frac{d\mathbf{u}_\theta}{d\theta} &= -(\cos \theta)\mathbf{i} - (\sin \theta)\mathbf{j} = -\mathbf{u}_r. \end{aligned} \quad (2)$$

When we differentiate $\mathbf{u}_r$ and $\mathbf{u}_\theta$ with respect to t to find how they change with time, the Chain Rule gives

$$\dot{\mathbf{u}}_r = \frac{d\mathbf{u}_r}{d\theta} \dot{\theta} = \dot{\theta}\mathbf{u}_\theta, \quad \dot{\mathbf{u}}_\theta = \frac{d\mathbf{u}_\theta}{d\theta} \dot{\theta} = -\dot{\theta}\mathbf{u}_r. \quad (3)$$

Hence,

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{u}_r) = \dot{r}\mathbf{u}_r + r\dot{\mathbf{u}}_r = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta. \quad (4)$$

See Figure 13.33. As in the previous section, we use Newton's dot notation for time derivatives to keep the formulas as simple as we can: $\dot{\mathbf{u}}_r$ means $d\mathbf{u}_r/dt$, $\dot{\theta}$ means $d\theta/dt$, and so on.

The acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r) + (\dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta). \quad (5)$$

When Equations (3) are used to evaluate $\dot{\mathbf{u}}_r$ and $\dot{\mathbf{u}}_\theta$ and the components are separated, the equation for acceleration becomes

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta. \quad (6)$$

To extend these equations of motion to space, we add $z\mathbf{k}$ to the right-hand side of the equation $\mathbf{r} = r\mathbf{u}_r$. Then, in these *cylindrical coordinates*,

$$\begin{aligned} \mathbf{r} &= r\mathbf{u}_r + z\mathbf{k} \\ \mathbf{v} &= \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k} \\ \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}. \end{aligned} \quad (7)$$

The vectors $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{k}$ make a right-handed frame (Figure 13.34) in which

$$\mathbf{u}_r \times \mathbf{u}_\theta = \mathbf{k}, \quad \mathbf{u}_\theta \times \mathbf{k} = \mathbf{u}_r, \quad \mathbf{k} \times \mathbf{u}_r = \mathbf{u}_\theta. \quad (8)$$

Planets Move in Planes

Newton's law of gravitation says that if $\mathbf{r}$ is the radius vector from the center of a sun of mass M to the center of a planet of mass m , then the force $\mathbf{F}$ of the gravitational attraction between the planet and sun is

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|} \quad (9)$$

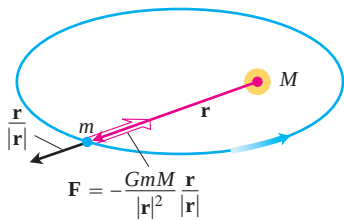


FIGURE 13.35 The force of gravity is directed along the line joining the centers of mass.

(Figure 13.35). The number G is the **universal gravitational constant**. If we measure mass in kilograms, force in newtons, and distance in meters, G is about $6.6726 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$.

Combining Equation (9) with Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}$, for the force acting on the planet gives

$$m\ddot{\mathbf{r}} = -\frac{GmM}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|},$$

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|}. \quad (10)$$

The planet is accelerated toward the sun's center at all times.

Equation (10) says that $\ddot{\mathbf{r}}$ is a scalar multiple of $\mathbf{r}$, so that

$$\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}. \quad (11)$$

A routine calculation shows $\mathbf{r} \times \ddot{\mathbf{r}}$ to be the derivative of $\mathbf{r} \times \dot{\mathbf{r}}$:

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \underbrace{\dot{\mathbf{r}} \times \dot{\mathbf{r}}}_{\mathbf{0}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \ddot{\mathbf{r}}. \quad (12)$$

Hence Equation (11) is equivalent to

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{0}, \quad (13)$$

which integrates to

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{C} \quad (14)$$

for some constant vector $\mathbf{C}$.

Equation (14) tells us that $\mathbf{r}$ and $\dot{\mathbf{r}}$ always lie in a plane perpendicular to $\mathbf{C}$. Hence, the planet moves in a fixed plane through the center of its sun (Figure 13.36).

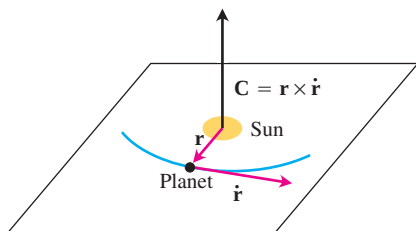


FIGURE 13.36 A planet that obeys Newton's laws of gravitation and motion travels in the plane through the sun's center of mass perpendicular to $\mathbf{C} = \mathbf{r} \times \dot{\mathbf{r}}$.

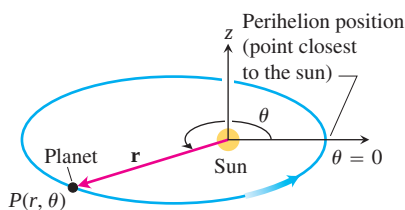


FIGURE 13.37 The coordinate system for planetary motion. The motion is counterclockwise when viewed from above, as it is here, and $\dot{\theta} > 0$.

Coordinates and Initial Conditions

We now introduce coordinates in a way that places the origin at the sun's center of mass and makes the plane of the planet's motion the polar coordinate plane. This makes $\mathbf{r}$ the planet's polar coordinate position vector and makes $|\mathbf{r}|$ equal to r and $\mathbf{r}/|\mathbf{r}|$ equal to $\mathbf{u}_r$. We also position the z -axis in a way that makes $\mathbf{k}$ the direction of $\mathbf{C}$. Thus, $\mathbf{k}$ has the same right-hand relation to $\mathbf{r} \times \dot{\mathbf{r}}$ that $\mathbf{C}$ does, and the planet's motion is counterclockwise when viewed from the positive z -axis. This makes θ increase with t , so that $\dot{\theta} > 0$ for all t . Finally, we rotate the polar coordinate plane about the z -axis, if necessary, to make the initial ray coincide with the direction $\mathbf{r}$ has when the planet is closest to the sun. This runs the ray through the planet's **perihelion** position (Figure 13.37).

If we measure time so that $t = 0$ at perihelion, we have the following initial conditions for the planet's motion.

1. $r = r_0$, the minimum radius, when $t = 0$
2. $\dot{r} = 0$ when $t = 0$ (because r has a minimum value then)
3. $\theta = 0$ when $t = 0$
4. $|\mathbf{v}| = v_0$ when $t = 0$

Since

$$\begin{aligned}
 v_0 &= |\mathbf{v}|_{t=0} \\
 &= |\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta|_{t=0} && \text{Equation (4)} \\
 &= |r\dot{\theta}\mathbf{u}_\theta|_{t=0} && \dot{r} = 0 \text{ when } t = 0 \\
 &= (|r\dot{\theta}||\mathbf{u}_\theta|)_{t=0} \\
 &= |r\dot{\theta}|_{t=0} && |\mathbf{u}_\theta| = 1 \\
 &= (r\dot{\theta})_{t=0}, && r \text{ and } \dot{\theta} \text{ both positive}
 \end{aligned}$$

we also know that

$$5. \quad r\dot{\theta} = v_0 \text{ when } t = 0.$$

HISTORICAL BIOGRAPHY

Johannes Kepler
(1571–1630)



Kepler's First Law (The Conic Section Law)

Kepler's first law says that a planet's path is a conic section with the sun at one focus. The eccentricity of the conic is

$$e = \frac{r_0 v_0^2}{GM} - 1 \quad (15)$$

and the polar equation is

$$r = \frac{(1 + e)r_0}{1 + e \cos \theta}. \quad (16)$$

The derivation uses Kepler's second law, so we will state and prove the second law before proving the first law.

Kepler's Second Law (The Equal Area Law)

Kepler's second law says that the radius vector from the sun to a planet (the vector $\mathbf{r}$ in our model) sweeps out equal areas in equal times (Figure 13.38). To derive the law, we use Equation (4) to evaluate the cross product $\mathbf{C} = \mathbf{r} \times \dot{\mathbf{r}}$ from Equation (14):

$$\begin{aligned}
 \mathbf{C} &= \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{v} \\
 &= r\mathbf{u}_r \times (\dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta) && \text{Equation (4)} \\
 &= r\dot{r}(\underbrace{\mathbf{u}_r \times \mathbf{u}_r}_0) + r(r\dot{\theta})(\underbrace{\mathbf{u}_r \times \mathbf{u}_\theta}_{\mathbf{k}}) && (17) \\
 &= r(r\dot{\theta})\mathbf{k}.
 \end{aligned}$$

Setting t equal to zero shows that

$$\mathbf{C} = [r(r\dot{\theta})]_{t=0} \mathbf{k} = r_0 v_0 \mathbf{k}. \quad (18)$$

Substituting this value for $\mathbf{C}$ in Equation (17) gives

$$r_0 v_0 \mathbf{k} = r^2 \dot{\theta} \mathbf{k}, \quad \text{or} \quad r^2 \dot{\theta} = r_0 v_0. \quad (19)$$

This is where the area comes in. The area differential in polar coordinates is

$$dA = \frac{1}{2} r^2 d\theta$$

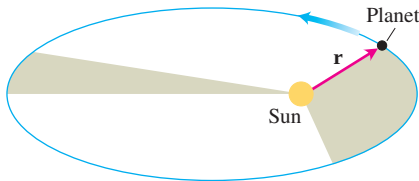


FIGURE 13.38 The line joining a planet to its sun sweeps over equal areas in equal times.

(Section 10.7). Accordingly, dA/dt has the constant value

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} r_0 v_0. \quad (20)$$

So dA/dt is constant, giving Kepler's second law.

For Earth, r_0 is about 150,000,000 km, v_0 is about 30 km/sec, and dA/dt is about 2,250,000,000 km²/sec. Every time your heart beats, Earth advances 30 km along its orbit, and the radius joining Earth to the sun sweeps out 2,250,000,000 km² of area.

Proof of Kepler's First Law

To prove that a planet moves along a conic section with one focus at its sun, we need to express the planet's radius r as a function of θ . This requires a long sequence of calculations and some substitutions that are not altogether obvious.

We begin with the equation that comes from equating the coefficients of $\mathbf{u}_r = \mathbf{r}/|\mathbf{r}|$ in Equations (6) and (10):

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}. \quad (21)$$

We eliminate $\dot{\theta}$ temporarily by replacing it with $r_0 v_0 / r^2$ from Equation (19) and rearrange the resulting equation to get

$$\ddot{r} = \frac{r_0^2 v_0^2}{r^3} - \frac{GM}{r^2}. \quad (22)$$

We change this into a first-order equation by a change of variable. With

$$p = \frac{dr}{dt}, \quad \frac{d^2 r}{dt^2} = \frac{dp}{dt} = \frac{dp}{dr} \frac{dr}{dt} = p \frac{dp}{dr}, \quad \text{Chain Rule}$$

Equation (22) becomes

$$p \frac{dp}{dr} = \frac{r_0^2 v_0^2}{r^3} - \frac{GM}{r^2}. \quad (23)$$

Multiplying through by 2 and integrating with respect to r gives

$$p^2 = (\dot{r})^2 = -\frac{r_0^2 v_0^2}{r^2} + \frac{2GM}{r} + C_1. \quad (24)$$

The initial conditions that $r = r_0$ and $\dot{r} = 0$ when $t = 0$ determine the value of C_1 to be

$$C_1 = v_0^2 - \frac{2GM}{r_0}.$$

Accordingly, Equation (24), after a suitable rearrangement, becomes

$$\dot{r}^2 = v_0^2 \left(1 - \frac{r_0^2}{r^2} \right) + 2GM \left(\frac{1}{r} - \frac{1}{r_0} \right). \quad (25)$$

The effect of going from Equation (21) to Equation (25) has been to replace a second-order differential equation in r by a first-order differential equation in r . Our goal is still to express r in terms of θ , so we now bring θ back into the picture. To accomplish this, we

divide both sides of Equation (25) by the squares of the corresponding sides of the equation $r^2\dot{\theta} = r_0v_0$ (Equation 19) and use the fact that $\dot{r}/\dot{\theta} = (dr/dt)/(d\theta/dt) = dr/d\theta$ to get

$$\begin{aligned}\frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 &= \frac{1}{r_0^2} - \frac{1}{r^2} + \frac{2GM}{r_0^2 v_0^2} \left(\frac{1}{r} - \frac{1}{r_0} \right) \\ &= \frac{1}{r_0^2} - \frac{1}{r^2} + 2h \left(\frac{1}{r} - \frac{1}{r_0} \right), \quad h = \frac{GM}{r_0^2 v_0^2}\end{aligned}\quad (26)$$

To simplify further, we substitute

$$u = \frac{1}{r}, \quad u_0 = \frac{1}{r_0}, \quad \frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta}, \quad \left(\frac{du}{d\theta} \right)^2 = \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2,$$

obtaining

$$\left(\frac{du}{d\theta} \right)^2 = u_0^2 - u^2 + 2hu - 2hu_0 = (u_0 - h)^2 - (u - h)^2, \quad (27)$$

$$\frac{du}{d\theta} = \pm \sqrt{(u_0 - h)^2 - (u - h)^2}. \quad (28)$$

Which sign do we take? We know that $\dot{\theta} = r_0v_0/r^2$ is positive. Also, r starts from a minimum value at $t = 0$, so it cannot immediately decrease, and $\dot{r} \geq 0$, at least for early positive values of t . Therefore,

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} \geq 0 \quad \text{and} \quad \frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta} \leq 0.$$

The correct sign for Equation (28) is the negative sign. With this determined, we rearrange Equation (28) and integrate both sides with respect to θ :

$$\begin{aligned}\frac{-1}{\sqrt{(u_0 - h)^2 - (u - h)^2}} \frac{du}{d\theta} &= 1 \\ \cos^{-1} \left(\frac{u - h}{u_0 - h} \right) &= \theta + C_2.\end{aligned}\quad (29)$$

The constant C_2 is zero because $u = u_0$ when $\theta = 0$ and $\cos^{-1}(1) = 0$. Therefore,

$$\frac{u - h}{u_0 - h} = \cos \theta$$

and

$$\frac{1}{r} = u = h + (u_0 - h) \cos \theta. \quad (30)$$

A few more algebraic maneuvers produce the final equation

$$r = \frac{(1 + e)r_0}{1 + e \cos \theta}, \quad (31)$$

where

$$e = \frac{1}{r_0 h} - 1 = \frac{r_0 v_0^2}{GM} - 1. \quad (32)$$

Together, Equations (31) and (32) say that the path of the planet is a conic section with one focus at the sun and with eccentricity $(r_0 v_0^2 / GM) - 1$. This is the modern formulation of Kepler's first law.

Kepler's Third Law (The Time–Distance Law)



The time T it takes a planet to go around its sun once is the planet's **orbital period**. Kepler's third law says that T and the orbit's semimajor axis a are related by the equation

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}. \quad (33)$$

Since the right-hand side of this equation is constant within a given solar system, the ratio of T^2 to a^3 is the same for every planet in the system.

Kepler's third law is the starting point for working out the size of our solar system. It allows the semimajor axis of each planetary orbit to be expressed in astronomical units, Earth's semimajor axis being one unit. The distance between any two planets at any time can then be predicted in astronomical units and all that remains is to find one of these distances in kilometers. This can be done by bouncing radar waves off Venus, for example. The astronomical unit is now known, after a series of such measurements, to be 149,597,870 km.

We derive Kepler's third law by combining two formulas for the area enclosed by the planet's elliptical orbit:

Formula 1: $\text{Area} = \pi ab$

The geometry formula in which a is the semimajor axis and b is the semiminor axis

Formula 2: $\text{Area} = \int_0^T dA$

$$= \int_0^T \frac{1}{2} r_0 v_0 dt \quad \text{Equation (20)}$$

$$= \frac{1}{2} T r_0 v_0.$$

Equating these gives

$$T = \frac{2\pi ab}{r_0 v_0} = \frac{2\pi a^2}{r_0 v_0} \sqrt{1 - e^2}. \quad \text{For any ellipse, } b = a\sqrt{1 - e^2} \quad (34)$$

It remains only to express a and e in terms of r_0 , v_0 , G , and M . Equation (32) does this for e . For a , we observe that setting θ equal to π in Equation (31) gives

$$r_{\max} = r_0 \frac{1 + e}{1 - e}.$$

Hence,

$$2a = r_0 + r_{\max} = \frac{2r_0}{1 - e} = \frac{2r_0 GM}{2GM - r_0 v_0^2}. \quad (35)$$

Squaring both sides of Equation (34) and substituting the results of Equations (32) and (35) now produces Kepler's third law (Exercise 15).

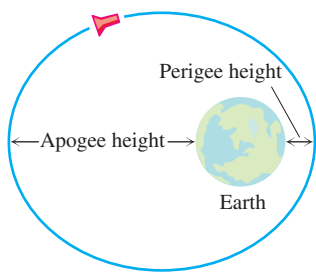


FIGURE 13.39 The orbit of an Earth satellite: $2a$ = diameter of Earth + perigee height + apogee height.

Orbit Data

Although Kepler discovered his laws empirically and stated them only for the six planets known at the time, the modern derivations of Kepler’s laws show that they apply to any body driven by a force that obeys an inverse square law like Equation (9). They apply to Halley’s comet and the asteroid Icarus. They apply to the moon’s orbit about Earth, and they applied to the orbit of the spacecraft *Apollo 8* about the moon.

Tables 13.1 through 13.3 give additional data for planetary orbits and for the orbits of seven of Earth’s artificial satellites (Figure 13.39). *Vanguard 1* sent back data that revealed differences between the levels of Earth’s oceans and provided the first determination of the precise locations of some of the more isolated Pacific islands. The data also verified that the gravitation of the sun and moon would affect the orbits of Earth’s satellites and that solar radiation could exert enough pressure to deform an orbit.

TABLE 13.1 Values of a , e , and T for the major planets

Planet	Semimajor axis a^*	Eccentricity e	Period T
Mercury	57.95	0.2056	87.967 days
Venus	108.11	0.0068	224.701 days
Earth	149.57	0.0167	365.256 days
Mars	227.84	0.0934	1.8808 years
Jupiter	778.14	0.0484	11.8613 years
Saturn	1427.0	0.0543	29.4568 years
Uranus	2870.3	0.0460	84.0081 years
Neptune	4499.9	0.0082	164.784 years
Pluto	5909	0.2481	248.35 years

*Millions of kilometers.

TABLE 13.2 Data on Earth’s satellites

Name	Launch date	Time or expected time aloft	Mass at launch (kg)	Period (min)	Perigee height (km)	Apogee height (km)	Semimajor axis a (km)	Eccentricity
<i>Sputnik 1</i>	Oct. 1957	57.6 days	83.6	96.2	215	939	6955	0.052
<i>Vanguard 1</i>	Mar. 1958	300 years	1.47	138.5	649	4340	8872	0.208
<i>Syncom 3</i>	Aug. 1964	$>10^6$ years	39	1436.2	35,718	35,903	42,189	0.002
<i>Skylab 4</i>	Nov. 1973	84.06 days	13,980	93.11	422	437	6808	0.001
<i>Tiros II</i>	Oct. 1978	500 years	734	102.12	850	866	7236	0.001
<i>GOES 4</i>	Sept. 1980	$>10^6$ years	627	1436.2	35,776	35,800	42,166	0.0003
<i>Intelsat 5</i>	Dec. 1980	$>10^6$ years	1928	1417.67	35,143	35,707	41,803	0.007



TABLE 13.3 Numerical data

Universal gravitational constant:	$G = 6.6726 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2}$
Sun’s mass:	$1.99 \times 10^{30} \text{ kg}$
Earth’s mass:	$5.975 \times 10^{24} \text{ kg}$
Equatorial radius of Earth:	6378.533 km
Polar radius of Earth:	6356.912 km
Earth’s rotational period:	1436.1 min
Earth’s orbital period:	1 year = 365.256 days

Syncom 3 is one of a series of U.S. Department of Defense telecommunications satellites. *Tiros II* (for “television infrared observation satellite”) is one of a series of weather satellites. *GOES 4* (for “geostationary operational environmental satellite”) is one of a series of satellites designed to gather information about Earth’s atmosphere. Its orbital period, 1436.2 min, is nearly the same as Earth’s rotational period of 1436.1 min, and its orbit is nearly circular ($e = 0.0003$). *Intelsat 5* is a heavy-capacity commercial telecommunications satellite.

EXERCISES 13.6

Reminder: When a calculation involves the gravitational constant G , express force in newtons, distance in meters, mass in kilograms, and time in seconds.



- 1. Period of *Skylab 4*** Since the orbit of *Skylab 4* had a semimajor axis of $a = 6808$ km, Kepler's third law with M equal to Earth's mass should give the period. Calculate it. Compare your result with the value in Table 13.2.
- 2. Earth's velocity at perihelion** Earth's distance from the sun at perihelion is approximately 149,577,000 km, and the eccentricity of Earth's orbit about the sun is 0.0167. Find the velocity v_0 of Earth in its orbit at perihelion. (Use Equation (15).)
- 3. Semimajor axis of *Proton 1*** In July 1965, the USSR launched *Proton 1*, weighing 12,200 kg (at launch), with a perigee height of 183 km, an apogee height of 589 km, and a period of 92.25 min. Using the relevant data for the mass of Earth and the gravitational constant G , find the semimajor axis a of the orbit from Equation (3). Compare your answer with the number you get by adding the perigee and apogee heights to the diameter of the Earth.
- 4. Semimajor axis of *Viking 1*** The *Viking 1* orbiter, which surveyed Mars from August 1975 to June 1976, had a period of 1639 min. Use this and the mass of Mars, 6.418×10^{23} kg, to find the semimajor axis of the *Viking 1* orbit.

- 5. Average diameter of Mars** (Continuation of Exercise 4.) The *Viking 1* orbiter was 1499 km from the surface of Mars at its closest point and 35,800 km from the surface at its farthest point. Use this information together with the value you obtained in Exercise 4 to estimate the average diameter of Mars.
- 6. Period of *Viking 2*** The *Viking 2* orbiter, which surveyed Mars from September 1975 to August 1976, moved in an ellipse whose semimajor axis was 22,030 km. What was the orbital period? (Express your answer in minutes.)
- 7. Geosynchronous orbits** Several satellites in Earth's equatorial plane have nearly circular orbits whose periods are the same as Earth's rotational period. Such orbits are *geosynchronous* or *geostationary* because they hold the satellite over the same spot on the Earth's surface.
 - a.** Approximately what is the semimajor axis of a geosynchronous orbit? Give reasons for your answer.
 - b.** About how high is a geosynchronous orbit above Earth's surface?
 - c.** Which of the satellites in Table 13.2 have (nearly) geosynchronous orbits?
- 8.** The mass of Mars is 6.418×10^{23} kg. If a satellite revolving about Mars is to hold a stationary orbit (have the same period as



the period of Mars's rotation, which is 1477.4 min), what must the semimajor axis of its orbit be? Give reasons for your answer.



- 9. Distance from Earth to the moon** The period of the moon's rotation about Earth is 2.36055×10^6 sec. About how far away is the moon?
- 10. Finding satellite speed** A satellite moves around Earth in a circular orbit. Express the satellite's speed as a function of the orbit's radius.
- 11. Orbital period** If T is measured in seconds and a in meters, what is the value of T^2/a^3 for planets in our solar system? For satellites orbiting Earth? For satellites orbiting the moon? (The moon's mass is 7.354×10^{22} kg.)
- 12. Type of orbit** For what values of v_0 in Equation (15) is the orbit in Equation (16) a circle? An ellipse? A parabola? A hyperbola?
- 13. Circular orbits** Show that a planet in a circular orbit moves with a constant speed. (*Hint:* This is a consequence of one of Kepler's laws.)
- 14.** Suppose that $\mathbf{r}$ is the position vector of a particle moving along a plane curve and dA/dt is the rate at which the vector sweeps out area. Without introducing coordinates, and assuming the necessary derivatives exist, give a geometric argument based on increments and limits for the validity of the equation

$$\frac{dA}{dt} = \frac{1}{2} |\mathbf{r} \times \dot{\mathbf{r}}|.$$

- 15. Kepler's third law** Complete the derivation of Kepler's third law (the part following Equation (34)).

In Exercises 16 and 17, two planets, planet A and planet B , are orbiting their sun in circular orbits with A being the inner planet and B being farther away from the sun. Suppose the positions of A and B at time t are

$$\mathbf{r}_A(t) = 2 \cos(2\pi t)\mathbf{i} + 2 \sin(2\pi t)\mathbf{j}$$

and

$$\mathbf{r}_B(t) = 3 \cos(\pi t)\mathbf{i} + 3 \sin(\pi t)\mathbf{j},$$

respectively, where the sun is assumed to be located at the origin and distance is measured in astronomical units. (Notice that planet A moves faster than planet B .)

The people on planet A regard their planet, not the sun, as the center of their planetary system (their solar system).

- 16.** Using planet A as the origin of a new coordinate system, give parametric equations for the location of planet B at time t . Write your answer in terms of $\cos(\pi t)$ and $\sin(\pi t)$.

- T 17.** Using planet A as the origin, graph the path of planet B .

This exercise illustrates the difficulty that people before Kepler's time, with an earth-centered (planet A) view of our solar system, had in understanding the motions of the planets (i.e., planet B = Mars). See D. G. Saari's article in the *American Mathematical Monthly*, Vol. 97 (Feb. 1990), pp. 105–119.

- 18.** Kepler discovered that the path of Earth around the sun is an ellipse with the sun at one of the foci. Let $\mathbf{r}(t)$ be the position vector from the center of the sun to the center of Earth at time t . Let $\mathbf{w}$ be the vector from Earth's South Pole to North Pole. It is known that $\mathbf{w}$ is constant and not orthogonal to the plane of the ellipse (Earth's axis is tilted). In terms of $\mathbf{r}(t)$ and $\mathbf{w}$, give the mathematical meaning of (i) perihelion, (ii) aphelion, (iii) equinox, (iv) summer solstice, (v) winter solstice.

Chapter 13

Questions to Guide Your Review

1. State the rules for differentiating and integrating vector functions. Give examples.
2. How do you define and calculate the velocity, speed, direction of motion, and acceleration of a body moving along a sufficiently differentiable space curve? Give an example.
3. What is special about the derivatives of vector functions of constant length? Give an example.
4. What are the vector and parametric equations for ideal projectile motion? How do you find a projectile's maximum height, flight time, and range? Give examples.
5. How do you define and calculate the length of a segment of a smooth space curve? Give an example. What mathematical assumptions are involved in the definition?
6. How do you measure distance along a smooth curve in space from a preselected base point? Give an example.
7. What is a differentiable curve's unit tangent vector? Give an example.
8. Define curvature, circle of curvature (osculating circle), center of curvature, and radius of curvature for twice-differentiable curves in the plane. Give examples. What curves have zero curvature? Constant curvature?
9. What is a plane curve's principal normal vector? When is it defined? Which way does it point? Give an example.
10. How do you define $\mathbf{N}$ and κ for curves in space? How are these quantities related? Give examples.

11. What is a curve's binormal vector? Give an example. How is this vector related to the curve's torsion? Give an example.
12. What formulas are available for writing a moving body's acceleration as a sum of its tangential and normal components? Give an example. Why might one want to write the acceleration this way? What if the body moves at a constant speed? At a constant speed around a circle?
13. State Kepler's laws. To what phenomena do they apply?

Chapter 13 Practice Exercises

Motion in a Cartesian Plane

In Exercises 1 and 2, graph the curves and sketch their velocity and acceleration vectors at the given values of t . Then write $\mathbf{a}$ in the form $\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$ without finding $\mathbf{T}$ and $\mathbf{N}$, and find the value of κ at the given values of t .

1. $\mathbf{r}(t) = (4 \cos t)\mathbf{i} + (\sqrt{2} \sin t)\mathbf{j}$, $t = 0$ and $\pi/4$

2. $\mathbf{r}(t) = (\sqrt{3} \sec t)\mathbf{i} + (\sqrt{3} \tan t)\mathbf{j}$, $t = 0$

3. The position of a particle in the plane at time t is

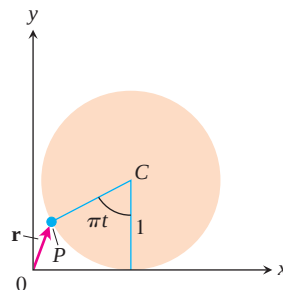
$$\mathbf{r} = \frac{1}{\sqrt{1+t^2}}\mathbf{i} + \frac{t}{\sqrt{1+t^2}}\mathbf{j}.$$

Find the particle's highest speed.

4. Suppose $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}$. Show that the angle between $\mathbf{r}$ and $\mathbf{a}$ never changes. What is the angle?
5. **Finding curvature** At point P , the velocity and acceleration of a particle moving in the plane are $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{a} = 5\mathbf{i} + 15\mathbf{j}$. Find the curvature of the particle's path at P .
6. Find the point on the curve $y = e^x$ where the curvature is greatest.
7. A particle moves around the unit circle in the xy -plane. Its position at time t is $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are differentiable functions of t . Find dy/dt if $\mathbf{v} \cdot \mathbf{i} = y$. Is the motion clockwise, or counterclockwise?
8. You send a message through a pneumatic tube that follows the curve $9y = x^3$ (distance in meters). At the point $(3, 3)$, $\mathbf{v} \cdot \mathbf{i} = 4$ and $\mathbf{a} \cdot \mathbf{i} = -2$. Find the values of $\mathbf{v} \cdot \mathbf{j}$ and $\mathbf{a} \cdot \mathbf{j}$ at $(3, 3)$.
9. **Characterizing circular motion** A particle moves in the plane so that its velocity and position vectors are always orthogonal. Show that the particle moves in a circle centered at the origin.
10. **Speed along a cycloid** A circular wheel with radius 1 ft and center C rolls to the right along the x -axis at a half-turn per second. (See the accompanying figure.) At time t seconds, the position vector of the point P on the wheel's circumference is

$$\mathbf{r} = (\pi t - \sin \pi t)\mathbf{i} + (1 - \cos \pi t)\mathbf{j}.$$

- a. Sketch the curve traced by P during the interval $0 \leq t \leq 3$.
- b. Find $\mathbf{v}$ and $\mathbf{a}$ at $t = 0, 1, 2$, and 3 and add these vectors to your sketch.
- c. At any given time, what is the forward speed of the topmost point of the wheel? Of C ?



Projectile Motion and Motion in a Plane

11. **Shot put** A shot leaves the thrower's hand 6.5 ft above the ground at a 45° angle at 44 ft/sec. Where is it 3 sec later?
12. **Javelin** A javelin leaves the thrower's hand 7 ft above the ground at a 45° angle at 80 ft/sec. How high does it go?
13. A golf ball is hit with an initial speed v_0 at an angle α to the horizontal from a point that lies at the foot of a straight-sided hill that is inclined at an angle ϕ to the horizontal, where

$$0 < \phi < \alpha < \frac{\pi}{2}.$$

Show that the ball lands at a distance

$$\frac{2v_0^2 \cos \alpha}{g \cos^2 \phi} \sin(\alpha - \phi),$$

measured up the face of the hill. Hence, show that the greatest range that can be achieved for a given v_0 occurs when $\alpha = (\phi/2) + (\pi/4)$, i.e., when the initial velocity vector bisects the angle between the vertical and the hill.

T 14. The Dictator The Civil War mortar Dictator weighed so much (17,120 lb) that it had to be mounted on a railroad car. It had a 13-in. bore and used a 20-lb powder charge to fire a 200-lb shell. The mortar was made by Mr. Charles Knapp in his ironworks in Pittsburgh, Pennsylvania, and was used by the Union army in 1864 in the siege of Petersburg, Virginia. How far did it shoot? Here we have a difference of opinion. The ordnance manual claimed 4325 yd, while field officers claimed 4752 yd. Assuming a 45° firing angle, what muzzle speeds are involved here?

T 15. The World's record for popping a champagne cork

- Until 1988, the world's record for popping a champagne cork was 109 ft. 6 in., once held by Captain Michael Hill of the British Royal Artillery (of course). Assuming Cpt. Hill held the bottle neck at ground level at a 45° angle, and the cork behaved like an ideal projectile, how fast was the cork going as it left the bottle?
- A new world record of 177 ft. 9 in. was set on June 5, 1988, by Prof. Emeritus Heinrich of Rensselaer Polytechnic Institute, firing from 4 ft. above ground level at the Woodbury Vineyards Winery, New York. Assuming an ideal trajectory, what was the cork's initial speed?

T 16. Javelin In Potsdam in 1988, Petra Felke of (then) East Germany set a women's world record by throwing a javelin 262 ft 5 in.

- Assuming that Felke launched the javelin at a 40° angle to the horizontal 6.5 ft above the ground, what was the javelin's initial speed?
- How high did the javelin go?

17. Synchronous curves By eliminating α from the ideal projectile equations

$$x = (v_0 \cos \alpha)t, \quad y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2,$$

show that $x^2 + (y + gt^2/2)^2 = v_0^2 t^2$. This shows that projectiles launched simultaneously from the origin at the same initial speed will, at any given instant, all lie on the circle of radius $v_0 t$ centered at $(0, -gt^2/2)$, regardless of their launch angle. These circles are the *synchronous curves* of the launching.

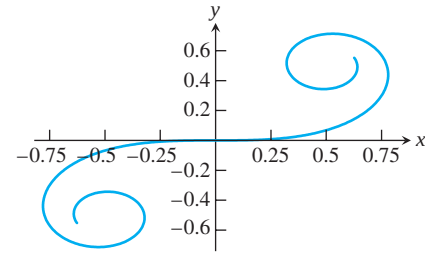
18. Radius of curvature Show that the radius of curvature of a twice-differentiable plane curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is given by the formula

$$\rho = \frac{\dot{x}^2 + \dot{y}^2}{\sqrt{\dot{x}^2 + \dot{y}^2 - \dot{s}^2}}, \quad \text{where} \quad \dot{s} = \frac{d}{dt} \sqrt{\dot{x}^2 + \dot{y}^2}.$$

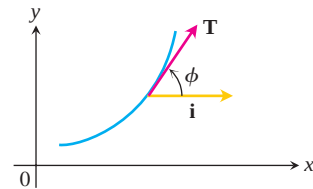
19. Curvature Express the curvature of the curve

$$\mathbf{r}(t) = \left(\int_0^t \cos \left(\frac{1}{2} \pi \theta^2 \right) d\theta \right) \mathbf{i} + \left(\int_0^t \sin \left(\frac{1}{2} \pi \theta^2 \right) d\theta \right) \mathbf{j}$$

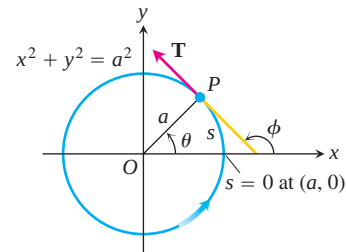
as a function of the directed distance s measured along the curve from the origin. (See the accompanying figure.)



20. An alternative definition of curvature in the plane An alternative definition gives the curvature of a sufficiently differentiable plane curve to be $|d\phi/ds|$, where ϕ is the angle between $\mathbf{T}$ and $\mathbf{i}$ (Figure 13.40a). Figure 13.40b shows the distance s measured counterclockwise around the circle $x^2 + y^2 = a^2$ from the point $(a, 0)$ to a point P , along with the angle ϕ at P . Calculate the circle's curvature using the alternative definition. (Hint: $\phi = \theta + \pi/2$.)



(a)



(b)

FIGURE 13.40 Figures for Exercise 20.

Motion in Space

Find the lengths of the curves in Exercises 21 and 22.

21. $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + t^2\mathbf{k}, \quad 0 \leq t \leq \pi/4$

22. $\mathbf{r}(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + 2t^{3/2}\mathbf{k}, \quad 0 \leq t \leq 3$

In Exercises 23–26, find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ , and τ at the given value of t .

23. $\mathbf{r}(t) = \frac{4}{9}(1+t)^{3/2}\mathbf{i} + \frac{4}{9}(1-t)^{3/2}\mathbf{j} + \frac{1}{3}t\mathbf{k}, \quad t = 0$

24. $\mathbf{r}(t) = (e^t \sin 2t)\mathbf{i} + (e^t \cos 2t)\mathbf{j} + 2e^t\mathbf{k}, \quad t = 0$

25. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}e^{2t}\mathbf{j}, \quad t = \ln 2$

26. $\mathbf{r}(t) = (3 \cosh 2t)\mathbf{i} + (3 \sinh 2t)\mathbf{j} + 6t\mathbf{k}, \quad t = \ln 2$

In Exercises 27 and 28, write $\mathbf{a}$ in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ at $t = 0$ without finding $\mathbf{T}$ and $\mathbf{N}$.

27. $\mathbf{r}(t) = (2 + 3t + 3t^2)\mathbf{i} + (4t + 4t^2)\mathbf{j} - (6 \cos t)\mathbf{k}$

28. $\mathbf{r}(t) = (2 + t)\mathbf{i} + (t + 2t^2)\mathbf{j} + (1 + t^2)\mathbf{k}$

29. Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, κ , and τ as functions of t if $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\sqrt{2} \cos t)\mathbf{j} + (\sin t)\mathbf{k}$.

30. At what times in the interval $0 \leq t \leq \pi$ are the velocity and acceleration vectors of the motion $\mathbf{r}(t) = \mathbf{i} + (5 \cos t)\mathbf{j} + (3 \sin t)\mathbf{k}$ orthogonal?

31. The position of a particle moving in space at time $t \geq 0$ is

$$\mathbf{r}(t) = 2\mathbf{i} + \left(4 \sin \frac{t}{2}\right)\mathbf{j} + \left(3 - \frac{t}{\pi}\right)\mathbf{k}.$$

Find the first time $\mathbf{r}$ is orthogonal to the vector $\mathbf{i} - \mathbf{j}$.

32. Find equations for the osculating, normal, and rectifying planes of the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point $(1, 1, 1)$.

33. Find parametric equations for the line that is tangent to the curve $\mathbf{r}(t) = e^t\mathbf{i} + (\sin t)\mathbf{j} + \ln(1 - t)\mathbf{k}$ at $t = 0$.

34. Find parametric equations for the line tangent to the helix $\mathbf{r}(t) = \left(\frac{1}{2} \cos t\right)\mathbf{i} + \left(\frac{1}{2} \sin t\right)\mathbf{j} + t\mathbf{k}$ at the point where $t = \pi/4$.

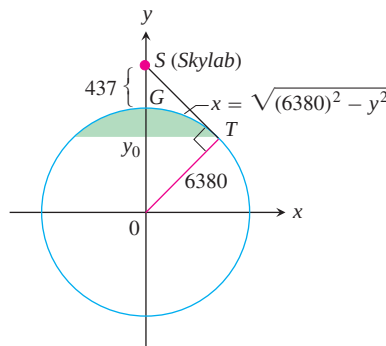
35. The view from Skylab 4 What percentage of Earth's surface area could the astronauts see when *Skylab 4* was at its apogee height, 437 km above the surface? To find out, model the visible surface as the surface generated by revolving the circular arc GT , shown here, about the y -axis. Then carry out these steps:

1. Use similar triangles in the figure to show that $y_0/6380 = 6380/(6380 + 437)$. Solve for y_0 .

2. To four significant digits, calculate the visible area as

$$VA = \int_{y_0}^{6380} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

3. Express the result as a percentage of Earth's surface area.



Chapter 13

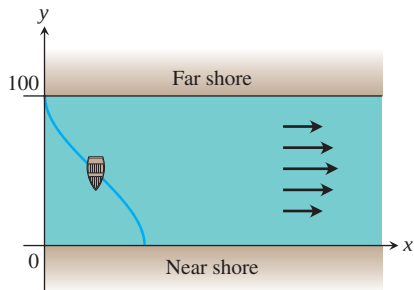
Additional and Advanced Exercises

Applications

1. A straight river is 100 m wide. A rowboat leaves the far shore at time $t = 0$. The person in the boat rows at a rate of 20 m/min, always toward the near shore. The velocity of the river at (x, y) is

$$\mathbf{v} = \left(-\frac{1}{250}(y - 50)^2 + 10 \right) \mathbf{i} \text{ m/min}, \quad 0 < y < 100.$$

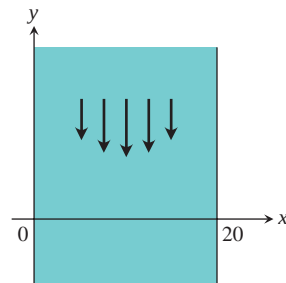
- a. Given that $\mathbf{r}(0) = 0\mathbf{i} + 100\mathbf{j}$, what is the position of the boat at time t ?
- b. How far downstream will the boat land on the near shore?



2. A straight river is 20 m wide. The velocity of the river at (x, y) is

$$\mathbf{v} = -\frac{3x(20 - x)}{100} \mathbf{j} \text{ m/min}, \quad 0 \leq x \leq 20.$$

A boat leaves the shore at $(0, 0)$ and travels through the water with a constant velocity. It arrives at the opposite shore at $(20, 0)$. The speed of the boat is always $\sqrt{20}$ m/min.



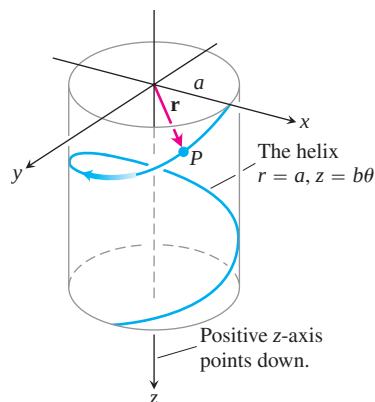
- a. Find the velocity of the boat.
- b. Find the location of the boat at time t .
- c. Sketch the path of the boat.

3. A frictionless particle P , starting from rest at time $t = 0$ at the point $(a, 0, 0)$, slides down the helix

$$\mathbf{r}(\theta) = (a \cos \theta)\mathbf{i} + (a \sin \theta)\mathbf{j} + b\theta\mathbf{k} \quad (a, b > 0)$$

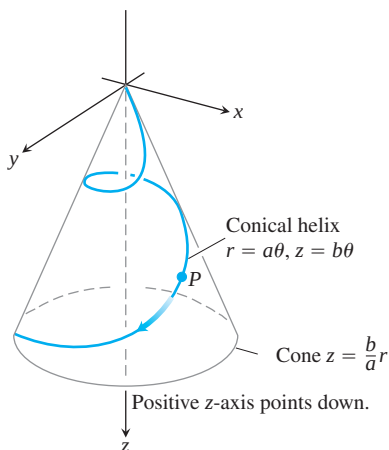
under the influence of gravity, as in the accompanying figure. The θ in this equation is the cylindrical coordinate θ and the helix is the curve $r = a, z = b\theta, \theta \geq 0$, in cylindrical coordinates. We assume θ to be a differentiable function of t for the motion. The law of conservation of energy tells us that the particle's speed after it has fallen straight down a distance z is $\sqrt{2gz}$, where g is the constant acceleration of gravity.

- Find the angular velocity $d\theta/dt$ when $\theta = 2\pi$.
- Express the particle's θ - and z -coordinates as functions of t .
- Express the tangential and normal components of the velocity $d\mathbf{r}/dt$ and acceleration $d^2\mathbf{r}/dt^2$ as functions of t . Does the acceleration have any nonzero component in the direction of the binormal vector $\mathbf{B}$?



4. Suppose the curve in Exercise 3 is replaced by the conical helix $r = a\theta, z = b\theta$ shown in the accompanying figure.

- Express the angular velocity $d\theta/dt$ as a function of θ .
- Express the distance the particle travels along the helix as a function of θ .



Polar Coordinate Systems and Motion in Space

5. Deduce from the orbit equation

$$r = \frac{(1 + e)r_0}{1 + e \cos \theta}$$

that a planet is closest to its sun when $\theta = 0$ and show that $r = r_0$ at that time.

- T** 6. **A Kepler equation** The problem of locating a planet in its orbit at a given time and date eventually leads to solving “Kepler” equations of the form

$$f(x) = x - 1 - \frac{1}{2} \sin x = 0.$$

- Show that this particular equation has a solution between $x = 0$ and $x = 2$.
 - With your computer or calculator in radian mode, use Newton's method to find the solution to as many places as you can.
7. In Section 13.6, we found the velocity of a particle moving in the plane to be

$$\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta.$$

- Express $\dot{x}$ and $\dot{y}$ in terms of $\dot{r}$ and $r\dot{\theta}$ by evaluating the dot products $\mathbf{v} \cdot \mathbf{i}$ and $\mathbf{v} \cdot \mathbf{j}$.
 - Express $\dot{r}$ and $r\dot{\theta}$ in terms of $\dot{x}$ and $\dot{y}$ by evaluating the dot products $\mathbf{v} \cdot \mathbf{u}_r$ and $\mathbf{v} \cdot \mathbf{u}_\theta$.
8. Express the curvature of a twice-differentiable curve $r = f(\theta)$ in the polar coordinate plane in terms of f and its derivatives.
9. A slender rod through the origin of the polar coordinate plane rotates (in the plane) about the origin at the rate of 3 rad/min. A beetle starting from the point $(2, 0)$ crawls along the rod toward the origin at the rate of 1 in./min.
- Find the beetle's acceleration and velocity in polar form when it is halfway to (1 in. from) the origin.
- T** b. To the nearest tenth of an inch, what will be the length of the path the beetle has traveled by the time it reaches the origin?
10. **Conservation of angular momentum** Let $\mathbf{r}(t)$ denote the position in space of a moving object at time t . Suppose the force acting on the object at time t is

$$\mathbf{F}(t) = -\frac{c}{|\mathbf{r}(t)|^3} \mathbf{r}(t),$$

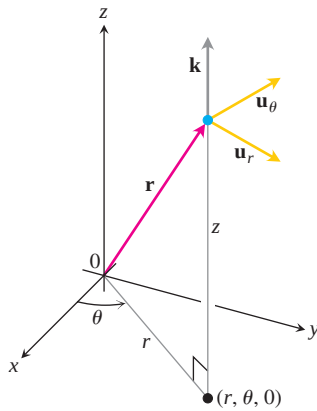
where c is a constant. In physics the **angular momentum** of an object at time t is defined to be $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t)$, where m is the mass of the object and $\mathbf{v}(t)$ is the velocity. Prove that angular momentum is a conserved quantity; i.e., prove that $\mathbf{L}(t)$ is a constant vector, independent of time. Remember Newton's law $\mathbf{F} = m\mathbf{a}$. (This is a calculus problem, not a physics problem.)

Cylindrical Coordinate Systems

11. Unit vectors for position and motion in cylindrical coordinates When the position of a particle moving in space is given in cylindrical coordinates, the unit vectors we use to describe its position and motion are

$$\mathbf{u}_r = (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j}, \quad \mathbf{u}_\theta = -(\sin \theta)\mathbf{i} + (\cos \theta)\mathbf{j},$$

and $\mathbf{k}$ (see accompanying figure). The particle's position vector is then $\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$, where r is the positive polar distance coordinate of the particle's position.



- Show that $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{k}$, in this order, form a right-handed frame of unit vectors.
- Show that

$$\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta \quad \text{and} \quad \frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r.$$

- Assuming that the necessary derivatives with respect to t exist, express $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{a} = \ddot{\mathbf{r}}$ in terms of $\mathbf{u}_r$, $\mathbf{u}_\theta$, $\mathbf{k}$, $\dot{r}$, and $\dot{\theta}$. (The dots indicate derivatives with respect to t : $\dot{\mathbf{r}}$ means $d\mathbf{r}/dt$, $\ddot{\mathbf{r}}$ means $d^2\mathbf{r}/dt^2$, and so on.) Section 13.6 derives these formulas and shows how the vectors mentioned here are used in describing planetary motion.

12. Arc length in cylindrical coordinates

- Show that when you express $ds^2 = dx^2 + dy^2 + dz^2$ in terms of cylindrical coordinates, you get $ds^2 = dr^2 + r^2 d\theta^2 + dz^2$.
- Interpret this result geometrically in terms of the edges and a diagonal of a box. Sketch the box.
- Use the result in part (a) to find the length of the curve $r = e^\theta$, $z = e^\theta$, $0 \leq \theta \leq \theta \ln 8$.

Chapter 13 Technology Application Projects

Mathematica/Maple Module

Radar Tracking of a Moving Object

Visualize position, velocity, and acceleration vectors to analyze motion.

Mathematica/Maple Module

Parametric and Polar Equations with a Figure Skater

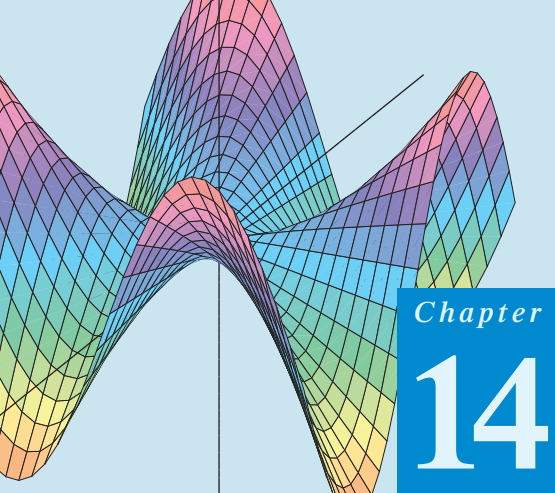
Visualize position, velocity, and acceleration vectors to analyze motion.

Mathematica/Maple Module

Moving in Three Dimensions

Compute distance traveled, speed, curvature, and torsion for motion along a space curve. Visualize and compute the tangential, normal, and binormal vectors associated with motion along a space curve.





Chapter

14

PARTIAL DERIVATIVES



OVERVIEW In studying a real-world phenomenon, a quantity being investigated usually depends on two or more independent variables. So we need to extend the basic ideas of the calculus of functions of a single variable to functions of several variables. Although the calculus rules remain essentially the same, the calculus is even richer. The derivatives of functions of several variables are more varied and more interesting because of the different ways in which the variables can interact. Their integrals lead to a greater variety of applications. The studies of probability, statistics, fluid dynamics, and electricity, to mention only a few, all lead in natural ways to functions of more than one variable.

14.1

Functions of Several Variables



Many functions depend on more than one independent variable. The function $V = \pi r^2 h$ calculates the volume of a right circular cylinder from its radius and height. The function $f(x, y) = x^2 + y^2$ calculates the height of the paraboloid $z = x^2 + y^2$ above the point $P(x, y)$ from the two coordinates of P . The temperature T of a point on Earth's surface depends on its latitude x and longitude y , expressed by writing $T = f(x, y)$. In this section, we define functions of more than one independent variable and discuss ways to graph them.

Real-valued functions of several independent real variables are defined much the way you would imagine from the single-variable case. The domains are sets of ordered pairs (triples, quadruples, n -tuples) of real numbers, and the ranges are sets of real numbers of the kind we have worked with all along.

DEFINITIONS Function of n Independent Variables

Suppose D is a set of n -tuples of real numbers $(x_1, x_2, \dots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D . The set D is the function's **domain**. The set of w -values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f , and f is said to be a function of the n **independent variables** x_1 to x_n . We also call the x_j 's the function's **input variables** and call w the function's **output variable**.

If f is a function of two independent variables, we usually call the independent variables x and y and picture the domain of f as a region in the xy -plane. If f is a function of three independent variables, we call the variables x , y , and z and picture the domain as a region in space.

In applications, we tend to use letters that remind us of what the variables stand for. To say that the volume of a right circular cylinder is a function of its radius and height, we might write $V = f(r, h)$. To be more specific, we might replace the notation $f(r, h)$ by the formula that calculates the value of V from the values of r and h , and write $V = \pi r^2 h$. In either case, r and h would be the independent variables and V the dependent variable of the function.

As usual, we evaluate functions defined by formulas by substituting the values of the independent variables in the formula and calculating the corresponding value of the dependent variable.

EXAMPLE 1 Evaluating a Function

The value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 0, 4)$ is

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.$$

From Section 12.1, we recognize f as the distance function from the origin to the point (x, y, z) in Cartesian space coordinates.

Domains and Ranges

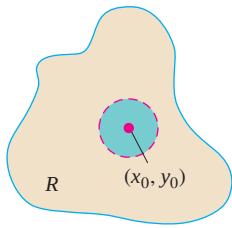
In defining a function of more than one variable, we follow the usual practice of excluding inputs that lead to complex numbers or division by zero. If $f(x, y) = \sqrt{y - x^2}$, y cannot be less than x^2 . If $f(x, y) = 1/(xy)$, xy cannot be zero. The domain of a function is assumed to be the largest set for which the defining rule generates real numbers, unless the domain is otherwise specified explicitly. The range consists of the set of output values for the dependent variable.

EXAMPLE 2(a) Functions of Two Variables

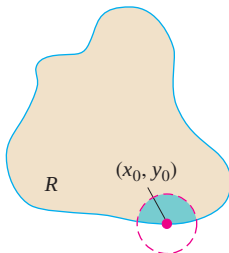
Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Entire plane	$[-1, 1]$

(b) Functions of Three Variables

Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \frac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$



(a) Interior point



(b) Boundary point

FIGURE 14.1 Interior points and boundary points of a plane region R . An interior point is necessarily a point of R . A boundary point of R need not belong to R .

Functions of Two Variables

Regions in the plane can have interior points and boundary points just like intervals on the real line. Closed intervals $[a, b]$ include their boundary points, open intervals (a, b) don't include their boundary points, and intervals such as $[a, b)$ are neither open nor closed.

DEFINITIONS Interior and Boundary Points, Open, Closed

A point (x_0, y_0) in a region (set) R in the xy -plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.1). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R . (The boundary point itself need not belong to R .)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.2).

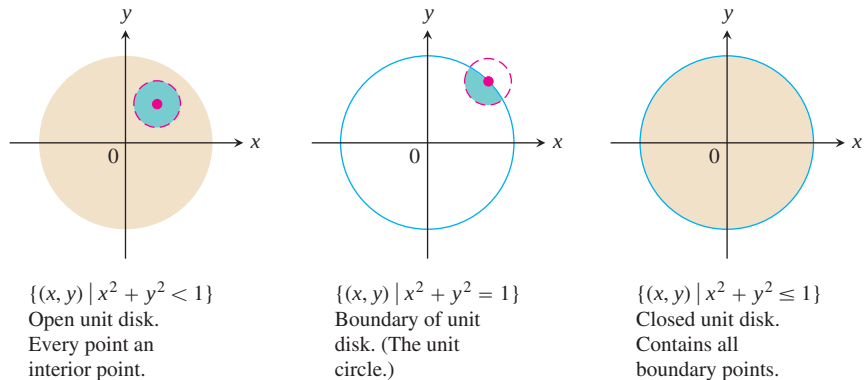


FIGURE 14.2 Interior points and boundary points of the unit disk in the plane.

As with intervals of real numbers, some regions in the plane are neither open nor closed. If you start with the open disk in Figure 14.2 and add to it some of but not all its boundary points, the resulting set is neither open nor closed. The boundary points that *are* there keep the set from being open. The absence of the remaining boundary points keeps the set from being closed.

DEFINITIONS Bounded and Unbounded Regions in the Plane

A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

Examples of *bounded* sets in the plane include line segments, triangles, interiors of triangles, rectangles, circles, and disks. Examples of *unbounded* sets in the plane include lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

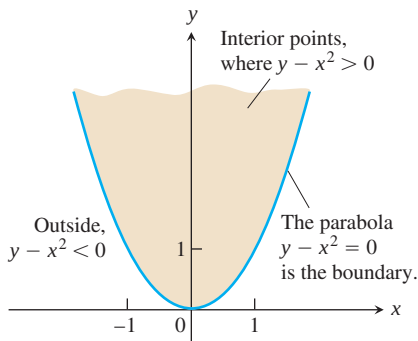


FIGURE 14.3 The domain of $f(x, y) = \sqrt{y - x^2}$ consists of the shaded region and its bounding parabola $y = x^2$ (Example 3).

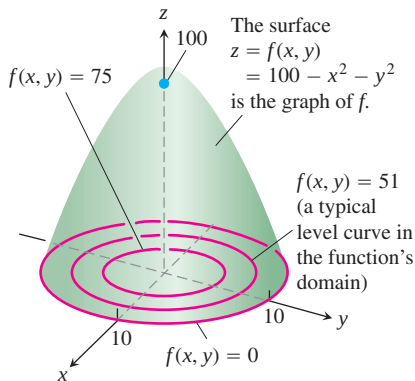


FIGURE 14.4 The graph and selected level curves of the function $f(x, y) = 100 - x^2 - y^2$ (Example 4).

EXAMPLE 3 Describing the Domain of a Function of Two Variables

Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

Solution Since f is defined only where $y - x^2 \geq 0$, the domain is the closed, unbounded region shown in Figure 14.3. The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior. ■

Graphs, Level Curves, and Contours of Functions of Two Variables

There are two standard ways to picture the values of a function $f(x, y)$. One is to draw and label curves in the domain on which f has a constant value. The other is to sketch the surface $z = f(x, y)$ in space.

DEFINITIONS Level Curve, Graph, Surface

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f . The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f , is called the **graph** of f . The graph of f is also called the **surface** $z = f(x, y)$.

EXAMPLE 4 Graphing a Function of Two Variables

Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

Solution The domain of f is the entire xy -plane, and the range of f is the set of real numbers less than or equal to 100. The graph is the paraboloid $z = 100 - x^2 - y^2$, a portion of which is shown in Figure 14.4.

The level curve $f(x, y) = 0$ is the set of points in the xy -plane at which

$$f(x, y) = 100 - x^2 - y^2 = 0, \quad \text{or} \quad x^2 + y^2 = 100,$$

which is the circle of radius 10 centered at the origin. Similarly, the level curves $f(x, y) = 51$ and $f(x, y) = 75$ (Figure 14.4) are the circles

$$f(x, y) = 100 - x^2 - y^2 = 51, \quad \text{or} \quad x^2 + y^2 = 49$$

$$f(x, y) = 100 - x^2 - y^2 = 75, \quad \text{or} \quad x^2 + y^2 = 25.$$

The level curve $f(x, y) = 100$ consists of the origin alone. (It is still a level curve.) ■

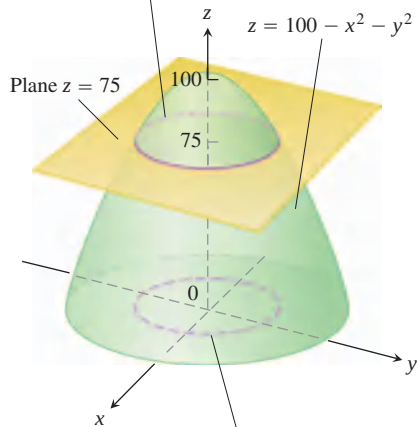
The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$. It is called the **contour curve** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of f . Figure 14.5 shows the contour curve $f(x, y) = 75$ on the surface $z = 100 - x^2 - y^2$ defined by the function $f(x, y) = 100 - x^2 - y^2$. The contour curve lies directly above the circle $x^2 + y^2 = 25$, which is the level curve $f(x, y) = 75$ in the function's domain.

Not everyone makes this distinction, however, and you may wish to call both kinds of curves by a single name and rely on context to convey which one you have in mind. On most maps, for example, the curves that represent constant elevation (height above sea level) are called contours, not level curves (Figure 14.6).





The contour curve $f(x, y) = 100 - x^2 - y^2 = 75$ is the circle $x^2 + y^2 = 25$ in the plane $z = 75$.



The level curve $f(x, y) = 100 - x^2 - y^2 = 75$ is the circle $x^2 + y^2 = 25$ in the xy -plane.

FIGURE 14.5 A plane $z = c$ parallel to the xy -plane intersecting a surface $z = f(x, y)$ produces a contour curve.

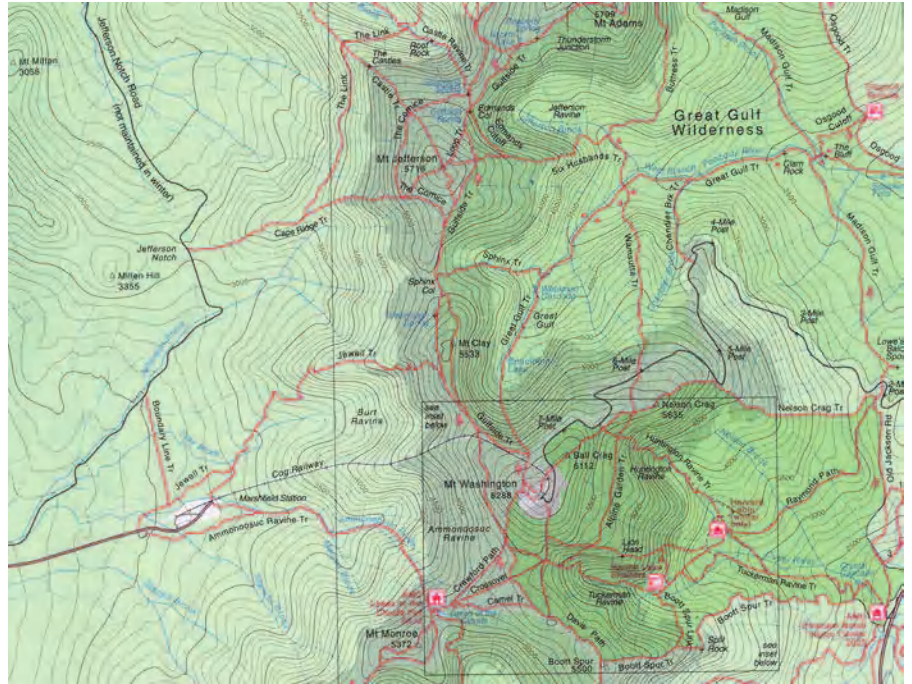


FIGURE 14.6 Contours on Mt. Washington in New Hampshire. (Reproduced by permission from the Appalachian Mountain Club.)

Functions of Three Variables

In the plane, the points where a function of two independent variables has a constant value $f(x, y) = c$ make a curve in the function's domain. In space, the points where a function of three independent variables has a constant value $f(x, y, z) = c$ make a surface in the function's domain.

DEFINITION Level Surface

The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f .

Since the graphs of functions of three variables consist of points $(x, y, z, f(x, y, z))$ lying in a four-dimensional space, we cannot sketch them effectively in our three-dimensional frame of reference. We can see how the function behaves, however, by looking at its three-dimensional level surfaces.



EXAMPLE 5 Describing Level Surfaces of a Function of Three Variables

Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

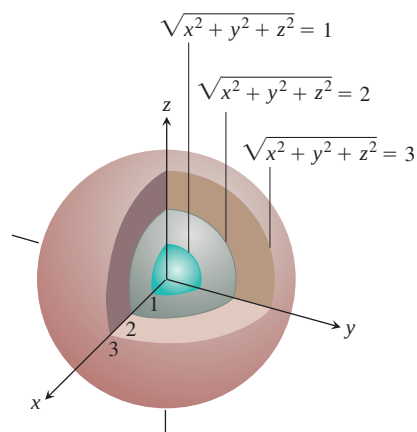


FIGURE 14.7 The level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ are concentric spheres (Example 5).

Solution The value of f is the distance from the origin to the point (x, y, z) . Each level surface $\sqrt{x^2 + y^2 + z^2} = c$, $c > 0$, is a sphere of radius c centered at the origin. Figure 14.7 shows a cutaway view of three of these spheres. The level surface $\sqrt{x^2 + y^2 + z^2} = 0$ consists of the origin alone.

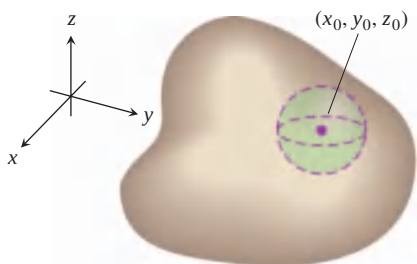
We are not graphing the function here; we are looking at level surfaces in the function's domain. The level surfaces show how the function's values change as we move through its domain. If we remain on a sphere of radius c centered at the origin, the function maintains a constant value, namely c . If we move from one sphere to another, the function's value changes. It increases if we move away from the origin and decreases if we move toward the origin. The way the values change depends on the direction we take. The dependence of change on direction is important. We return to it in Section 14.5. ■

The definitions of interior, boundary, open, closed, bounded, and unbounded for regions in space are similar to those for regions in the plane. To accommodate the extra dimension, we use solid balls of positive radius instead of disks.

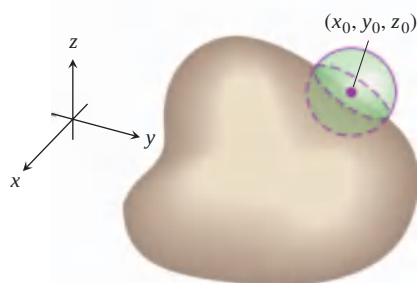
DEFINITIONS Interior and Boundary Points for Space Regions

A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if it is the center of a solid ball that lies entirely in R (Figure 14.8a). A point (x_0, y_0, z_0) is a **boundary point** of R if every sphere centered at (x_0, y_0, z_0) encloses points that lie outside of R as well as points that lie inside R (Figure 14.8b). The **interior** of R is the set of interior points of R . The **boundary** of R is the set of boundary points of R .

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.



(a) Interior point



(b) Boundary point

FIGURE 14.8 Interior points and boundary points of a region in space.

Examples of *open* sets in space include the interior of a sphere, the open half-space $z > 0$, the first octant (where x , y , and z are all positive), and space itself.

Examples of *closed* sets in space include lines, planes, the closed half-space $z \geq 0$, the first octant together with its bounding planes, and space itself (since it has no boundary points).

A solid sphere with part of its boundary removed or a solid cube with a missing face, edge, or corner point would be *neither open nor closed*.

Functions of more than three independent variables are also important. For example, the temperature on a surface in space may depend not only on the location of the point $P(x, y, z)$ on the surface, but also on time t when it is visited, so we would write $T = f(x, y, z, t)$.

Computer Graphing

Three-dimensional graphing programs for computers and calculators make it possible to graph functions of two variables with only a few keystrokes. We can often get information more quickly from a graph than from a formula.

EXAMPLE 6 Modeling Temperature Beneath Earth's Surface

The temperature beneath the Earth's surface is a function of the depth x beneath the surface and the time t of the year. If we measure x in feet and t as the number of days elapsed from the expected date of the yearly highest surface temperature, we can model the variation in temperature with the function

$$w = \cos(1.7 \times 10^{-2}t - 0.2x)e^{-0.2x}.$$

(The temperature at 0 ft is scaled to vary from $+1$ to -1 , so that the variation at x feet can be interpreted as a fraction of the variation at the surface.)

Figure 14.9 shows a computer-generated graph of the function. At a depth of 15 ft, the variation (change in vertical amplitude in the figure) is about 5% of the surface variation. At 30 ft, there is almost no variation during the year.

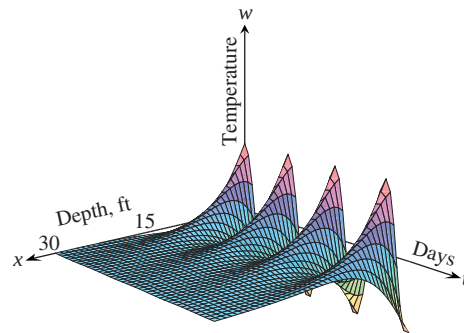


FIGURE 14.9 This computer-generated graph of

$$w = \cos(1.7 \times 10^{-2}t - 0.2x)e^{-0.2x}$$

shows the seasonal variation of the temperature belowground as a fraction of surface temperature. At $x = 15$ ft, the variation is only 5% of the variation at the surface. At $x = 30$ ft, the variation is less than 0.25% of the surface variation (Example 6). (Adapted from art provided by Norton Starr.)

The graph also shows that the temperature 15 ft below the surface is about half a year out of phase with the surface temperature. When the temperature is lowest on the surface (late January, say), it is at its highest 15 ft below. Fifteen feet below the ground, the seasons are reversed. ■

Figure 14.10 shows computer-generated graphs of a number of functions of two variables together with their level curves.

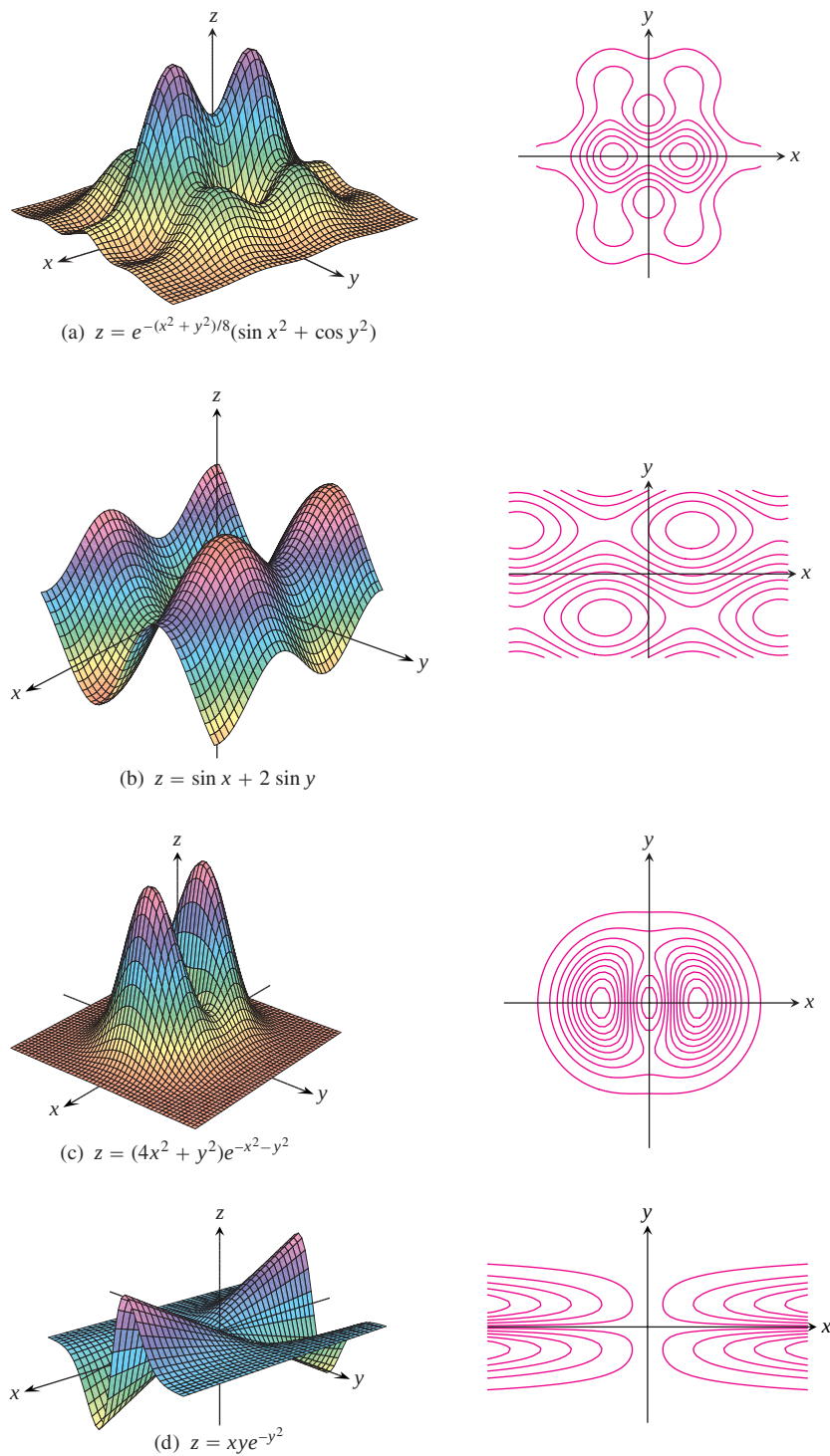


FIGURE 14.10 Computer-generated graphs and level surfaces of typical functions of two variables.

EXERCISES 14.1

Domain, Range, and Level Curves

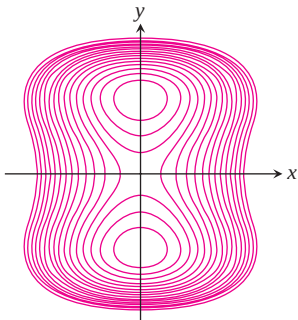
In Exercises 1–12, **(a)** find the function's domain, **(b)** find the function's range, **(c)** describe the function's level curves, **(d)** find the boundary of the function's domain, **(e)** determine if the domain is an open region, a closed region, or neither, and **(f)** decide if the domain is bounded or unbounded.

1. $f(x, y) = y - x$
2. $f(x, y) = \sqrt{y - x}$
3. $f(x, y) = 4x^2 + 9y^2$
4. $f(x, y) = x^2 - y^2$
5. $f(x, y) = xy$
6. $f(x, y) = y/x^2$
7. $f(x, y) = \frac{1}{\sqrt{16 - x^2 - y^2}}$
8. $f(x, y) = \sqrt{9 - x^2 - y^2}$
9. $f(x, y) = \ln(x^2 + y^2)$
10. $f(x, y) = e^{-(x^2 + y^2)}$
11. $f(x, y) = \sin^{-1}(y - x)$
12. $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$

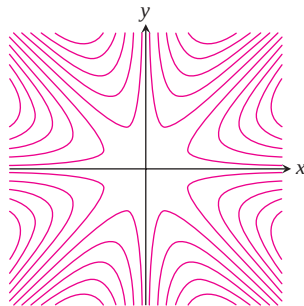
Identifying Surfaces and Level Curves

Exercises 13–18 show level curves for the functions graphed in (a)–(f). Match each set of curves with the appropriate function.

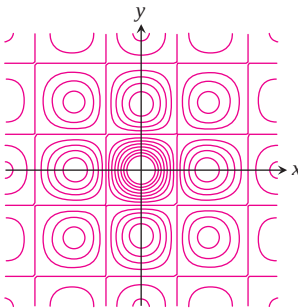
13.



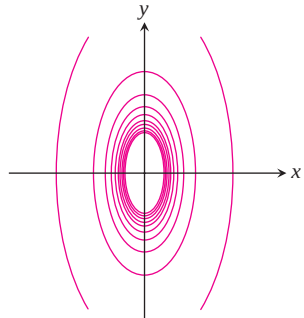
14.



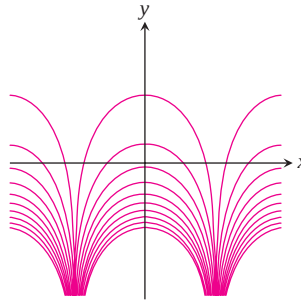
15.



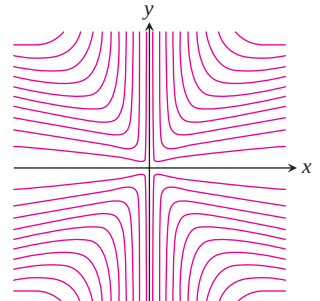
16.



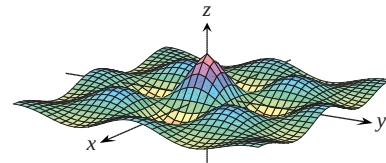
17.



18.

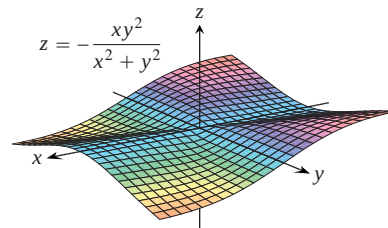


a.

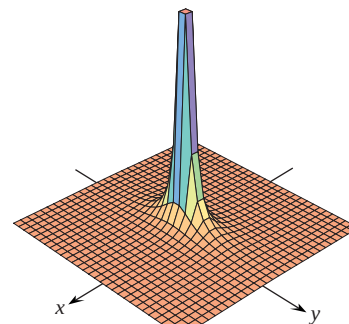


$$z = (\cos x)(\cos y) e^{-\sqrt{x^2 + y^2}/4}$$

b.



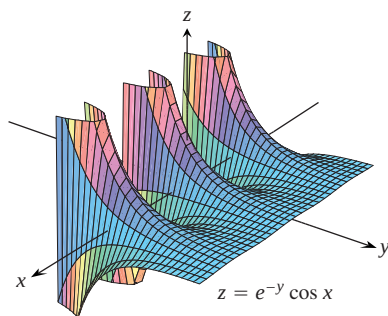
c.



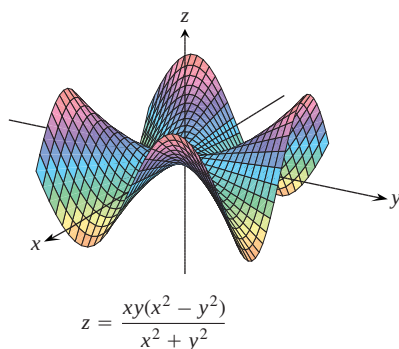
$$z = \frac{1}{4x^2 + y^2}$$



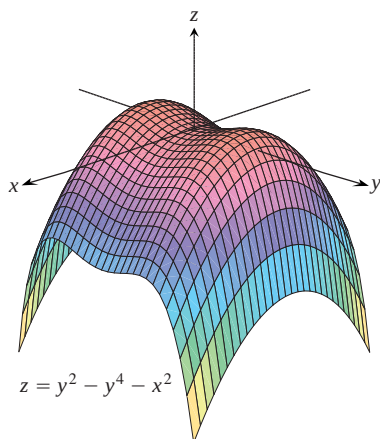
d.



e.



f.



Identifying Functions of Two Variables

Display the values of the functions in Exercises 19–28 in two ways: **(a)** by sketching the surface $z = f(x, y)$ and **(b)** by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.



19. $f(x, y) = y^2$ 20. $f(x, y) = 4 - y^2$
 21. $f(x, y) = x^2 + y^2$ 22. $f(x, y) = \sqrt{x^2 + y^2}$
 23. $f(x, y) = -(x^2 + y^2)$ 24. $f(x, y) = 4 - x^2 - y^2$

25. $f(x, y) = 4x^2 + y^2$ 26. $f(x, y) = 4x^2 + y^2 + 1$
 27. $f(x, y) = 1 - |y|$ 28. $f(x, y) = 1 - |x| - |y|$



Finding a Level Curve

In Exercises 29–32, find an equation for the level curve of the function $f(x, y)$ that passes through the given point.

29. $f(x, y) = 16 - x^2 - y^2$, $(2\sqrt{2}, \sqrt{2})$
 30. $f(x, y) = \sqrt{x^2 - 1}$, $(1, 0)$
 31. $f(x, y) = \int_x^y \frac{dt}{1 + t^2}$, $(-\sqrt{2}, \sqrt{2})$
 32. $f(x, y) = \sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n$, $(1, 2)$

Sketching Level Surfaces

In Exercises 33–40, sketch a typical level surface for the function.

33. $f(x, y, z) = x^2 + y^2 + z^2$ 34. $f(x, y, z) = \ln(x^2 + y^2 + z^2)$
 35. $f(x, y, z) = x + z$ 36. $f(x, y, z) = z$
 37. $f(x, y, z) = x^2 + y^2$ 38. $f(x, y, z) = y^2 + z^2$
 39. $f(x, y, z) = z - x^2 - y^2$
 40. $f(x, y, z) = (x^2/25) + (y^2/16) + (z^2/9)$

Finding a Level Surface

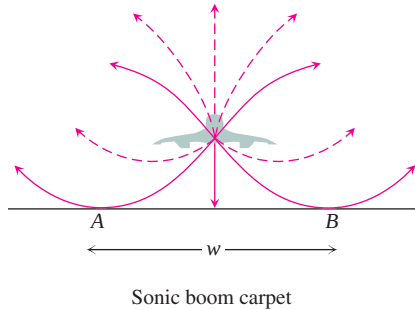
In Exercises 41–44, find an equation for the level surface of the function through the given point.

41. $f(x, y, z) = \sqrt{x - y} - \ln z$, $(3, -1, 1)$
 42. $f(x, y, z) = \ln(x^2 + y + z^2)$, $(-1, 2, 1)$
 43. $g(x, y, z) = \sum_{n=0}^{\infty} \frac{(x + y)^n}{n! z^n}$, $(\ln 2, \ln 4, 3)$
 44. $g(x, y, z) = \int_x^y \frac{d\theta}{\sqrt{1 - \theta^2}} + \int_{\sqrt{2}}^z \frac{dt}{t\sqrt{t^2 - 1}}$, $(0, 1/2, 2)$

Theory and Examples

45. **The maximum value of a function on a line in space** Does the function $f(x, y, z) = xyz$ have a maximum value on the line $x = 20 - t$, $y = t$, $z = 20$? If so, what is it? Give reasons for your answer. (*Hint:* Along the line, $w = f(x, y, z)$ is a differentiable function of t .)
46. **The minimum value of a function on a line in space** Does the function $f(x, y, z) = xy - z$ have a minimum value on the line $x = t - 1$, $y = t - 2$, $z = t + 7$? If so, what is it? Give reasons for your answer. (*Hint:* Along the line, $w = f(x, y, z)$ is a differentiable function of t .)
47. **The Concorde's sonic booms** Sound waves from the *Concorde* bend as the temperature changes above and below the altitude at which the plane flies. The sonic boom carpet is the region on the

ground that receives shock waves directly from the plane, not reflected from the atmosphere or diffracted along the ground. The carpet is determined by the grazing rays striking the ground from the point directly under the plane. (See accompanying figure.)



The width w of the region in which people on the ground hear the *Concorde's* sonic boom directly, not reflected from a layer in the atmosphere, is a function of

T = air temperature at ground level (in degrees Kelvin)

h = the *Concorde's* altitude (in kilometers)

d = the vertical temperature gradient (temperature drop in degrees Kelvin per kilometer).

The formula for w is

$$w = 4 \left(\frac{Th}{d} \right)^{1/2}.$$

The Washington-bound *Concorde* approached the United States from Europe on a course that took it south of Nantucket Island at an altitude of 16.8 km. If the surface temperature is 290 K and the vertical temperature gradient is 5 K/km, how many kilometers south of Nantucket did the plane have to be flown to keep its sonic boom carpet away from the island? (From "Concorde Sonic Booms as an Atmospheric Probe" by N. K. Balachandra, W. L. Donn, and D. H. Rind, *Science*, Vol. 197 (July 1, 1977), pp. 47–49.)

48. As you know, the graph of a real-valued function of a single real variable is a set in a two-coordinate space. The graph of a real-valued function of two independent real variables is a set in a three-coordinate space. The graph of a real-valued function of three independent real variables is a set in a four-coordinate space. How would you define the graph of a real-valued function $f(x_1, x_2, x_3, x_4)$ of four independent real variables? How would you define the graph of a real-valued function $f(x_1, x_2, x_3, \dots, x_n)$ of n independent real variables?

COMPUTER EXPLORATIONS

Explicit Surfaces

Use a CAS to perform the following steps for each of the functions in Exercises 49–52.

- Plot the surface over the given rectangle.
- Plot several level curves in the rectangle.
- Plot the level curve of f through the given point.

49. $f(x, y) = x \sin \frac{y}{2} + y \sin 2x$, $0 \leq x \leq 5\pi$ $0 \leq y \leq 5\pi$,
 $P(3\pi, 3\pi)$

50. $f(x, y) = (\sin x)(\cos y)e^{\sqrt{x^2+y^2}/8}$, $0 \leq x \leq 5\pi$,
 $0 \leq y \leq 5\pi$, $P(4\pi, 4\pi)$

51. $f(x, y) = \sin(x + 2 \cos y)$, $-2\pi \leq x \leq 2\pi$,
 $-2\pi \leq y \leq 2\pi$, $P(\pi, \pi)$

52. $f(x, y) = e^{(x^{0.1}-y)} \sin(x^2 + y^2)$, $0 \leq x \leq 2\pi$,
 $-2\pi \leq y \leq \pi$, $P(\pi, -\pi)$

Implicit Surfaces

Use a CAS to plot the level surfaces in Exercises 53–56.

53. $4 \ln(x^2 + y^2 + z^2) = 1$ 54. $x^2 + z^2 = 1$

55. $x + y^2 - 3z^2 = 1$

56. $\sin\left(\frac{x}{2}\right) - (\cos y)\sqrt{x^2 + z^2} = 2$

Parametrized Surfaces

Just as you describe curves in the plane parametrically with a pair of equations $x = f(t)$, $y = g(t)$ defined on some parameter interval I , you can sometimes describe surfaces in space with a triple of equations $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$ defined on some parameter rectangle $a \leq u \leq b$, $c \leq v \leq d$. Many computer algebra systems permit you to plot such surfaces in *parametric mode*. (Parametrized surfaces are discussed in detail in Section 16.6.) Use a CAS to plot the surfaces in Exercises 57–60. Also plot several level curves in the xy -plane.

57. $x = u \cos v$, $y = u \sin v$, $z = u$, $0 \leq u \leq 2$,
 $0 \leq v \leq 2\pi$

58. $x = u \cos v$, $y = u \sin v$, $z = v$, $0 \leq u \leq 2$,
 $0 \leq v \leq 2\pi$

59. $x = (2 + \cos u) \cos v$, $y = (2 + \cos u) \sin v$, $z = \sin u$,
 $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$

60. $x = 2 \cos u \cos v$, $y = 2 \cos u \sin v$, $z = 2 \sin u$,
 $0 \leq u \leq 2\pi$, $0 \leq v \leq \pi$

14.2

Limits and Continuity in Higher Dimensions

This section treats limits and continuity for multivariable functions. The definition of the limit of a function of two or three variables is similar to the definition of the limit of a function of a single variable but with a crucial difference, as we now see.

Limits

If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0) , we say that f approaches the limit L as (x, y) approaches (x_0, y_0) . This is similar to the informal definition for the limit of a function of a single variable. Notice, however, that if (x_0, y_0) lies in the interior of f 's domain, (x, y) can approach (x_0, y_0) from any direction. The direction of approach can be an issue, as in some of the examples that follow.

DEFINITION Limit of a Function of Two Variables

We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0) , and write

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f ,

$$|f(x, y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

The definition of limit says that the distance between $f(x, y)$ and L becomes arbitrarily small whenever the distance from (x, y) to (x_0, y_0) is made sufficiently small (but not 0).

The definition of limit applies to boundary points (x_0, y_0) as well as interior points of the domain of f . The only requirement is that the point (x, y) remain in the domain at all times. It can be shown, as for functions of a single variable, that

$$\begin{aligned} \lim_{(x, y) \rightarrow (x_0, y_0)} x &= x_0 \\ \lim_{(x, y) \rightarrow (x_0, y_0)} y &= y_0 \\ \lim_{(x, y) \rightarrow (x_0, y_0)} k &= k \quad (\text{any number } k). \end{aligned}$$

For example, in the first limit statement above, $f(x, y) = x$ and $L = x_0$. Using the definition of limit, suppose that $\epsilon > 0$ is chosen. If we let δ equal this ϵ , we see that

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta = \epsilon$$

implies

$$\begin{aligned} 0 < \sqrt{(x - x_0)^2} &< \epsilon \\ |x - x_0| &< \epsilon & \sqrt{a^2} = |a| \\ |f(x, y) - x_0| &< \epsilon & x = f(x, y) \end{aligned}$$

That is,

$$|f(x, y) - x_0| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

So

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = \lim_{(x, y) \rightarrow (x_0, y_0)} x = x_0.$$

It can also be shown that the limit of the sum of two functions is the sum of their limits (when they both exist), with similar results for the limits of the differences, products, constant multiples, quotients, and powers.

THEOREM 1 Properties of Limits of Functions of Two Variables

The following rules hold if L , M , and k are real numbers and

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x, y) \rightarrow (x_0, y_0)} g(x, y) = M.$$

1. *Sum Rule:* $\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$
2. *Difference Rule:* $\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$
3. *Product Rule:* $\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{(x, y) \rightarrow (x_0, y_0)} (kf(x, y)) = kL \quad (\text{any number } k)$
5. *Quotient Rule:* $\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, and $s \neq 0$, then

$$\lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y))^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

While we won't prove Theorem 1 here, we give an informal discussion of why it's true. If (x, y) is sufficiently close to (x_0, y_0) , then $f(x, y)$ is close to L and $g(x, y)$ is close to M (from the informal interpretation of limits). It is then reasonable that $f(x, y) + g(x, y)$ is close to $L + M$; $f(x, y) - g(x, y)$ is close to $L - M$; $f(x, y)g(x, y)$ is close to LM ; $kf(x, y)$ is close to kL ; and that $f(x, y)/g(x, y)$ is close to L/M if $M \neq 0$.

When we apply Theorem 1 to polynomials and rational functions, we obtain the useful result that the limits of these functions as $(x, y) \rightarrow (x_0, y_0)$ can be calculated by evaluating the functions at (x_0, y_0) . The only requirement is that the rational functions be defined at (x_0, y_0) .



EXAMPLE 1 Calculating Limits

$$(a) \quad \lim_{(x, y) \rightarrow (0, 1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

$$(b) \quad \lim_{(x, y) \rightarrow (3, -4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$



EXAMPLE 2 Calculating Limits

Find

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \rightarrow (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} && \text{Algebra} \\ &= \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) && \text{Cancel the nonzero factor } (x - y). \\ &= 0(\sqrt{0} + \sqrt{0}) = 0 \end{aligned}$$

We can cancel the factor $(x - y)$ because the path $y = x$ (along which $x - y = 0$) is *not* in the domain of the function

$$\frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

EXAMPLE 3 Applying the Limit Definition

Find $\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^2}$ if it exists.

Solution We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches $(0, 0)$, the value of the limit must be 0. To see if this is true, we apply the definition of limit.

Let $\epsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2 + y^2} < \epsilon \quad \text{whenever} \quad 0 < \sqrt{x^2 + y^2} < \delta.$$

Since $y^2 \leq x^2 + y^2$ we have that

$$\frac{4|x|y^2}{x^2 + y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2 + y^2}.$$

So if we choose $\delta = \epsilon/4$ and let $0 < \sqrt{x^2 + y^2} < \delta$, we get

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| \leq 4\sqrt{x^2 + y^2} < 4\delta = 4\left(\frac{\epsilon}{4}\right) = \epsilon.$$

It follows from the definition that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4xy^2}{x^2 + y^2} = 0.$$

Continuity

As with functions of a single variable, continuity is defined in terms of limits.

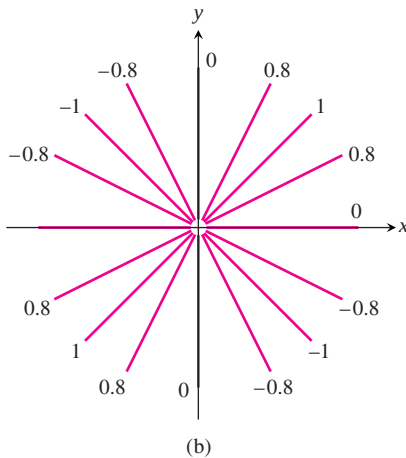
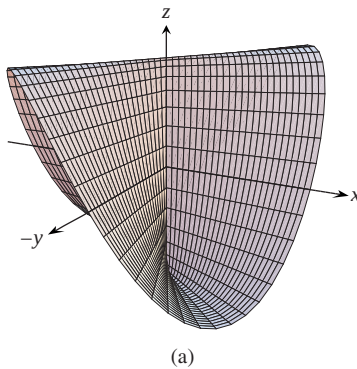


FIGURE 14.11 (a) The graph of

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

The function is continuous at every point except the origin. (b) The level curves of f (Example 4).

DEFINITION Continuous Function of Two Variables

A function $f(x, y)$ is **continuous at the point** (x_0, y_0) if

1. f is defined at (x_0, y_0) ,
2. $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

As with the definition of limit, the definition of continuity applies at boundary points as well as interior points of the domain of f . The only requirement is that the point (x, y) remain in the domain at all times.

As you may have guessed, one of the consequences of Theorem 1 is that algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined. This means that sums, differences, products, constant multiples, quotients, and powers of continuous functions are continuous where defined. In particular, polynomials and rational functions of two variables are continuous at every point at which they are defined.

EXAMPLE 4 A Function with a Single Point of Discontinuity

Show that

$$f(x, y) = \begin{cases} \frac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.11).

Solution The function f is continuous at any point $(x, y) \neq (0, 0)$ because its values are then given by a rational function of x and y .

At $(0, 0)$, the value of f is defined, but f , we claim, has no limit as $(x, y) \rightarrow (0, 0)$. The reason is that different paths of approach to the origin can lead to different results, as we now see.



For every value of m , the function f has a constant value on the “punctured” line $y = mx, x \neq 0$, because

$$f(x, y) \Big|_{y=mx} = \frac{2xy}{x^2 + y^2} \Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

Therefore, f has this number as its limit as (x, y) approaches $(0, 0)$ along the line:

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } y=mx}} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \left[f(x, y) \Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

This limit changes with m . There is therefore no single number we may call the limit of f as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous. ■

Example 4 illustrates an important point about limits of functions of two variables (or even more variables, for that matter). For a limit to exist at a point, the limit must be the same along every approach path. This result is analogous to the single-variable case where both the left- and right-sided limits had to have the same value; therefore, for functions of two or more variables, if we ever find paths with different limits, we know the function has no limit at the point they approach.

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths as (x, y) approaches (x_0, y_0) , then $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$ does not exist.

EXAMPLE 5 Applying the Two-Path Test

Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.12) has no limit as (x, y) approaches $(0, 0)$.

Solution The limit cannot be found by direct substitution, which gives the form $0/0$. We examine the values of f along curves that end at $(0, 0)$. Along the curve $y = kx^2, x \neq 0$, the function has the constant value

$$f(x, y) \Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2} \Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } y=kx^2}} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \left[f(x, y) \Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches $(0, 0)$ along the x -axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches $(0, 0)$.

The language here may seem contradictory. You might well ask, “What do you mean f has no limit as (x, y) approaches the origin—it has lots of limits.” But that is

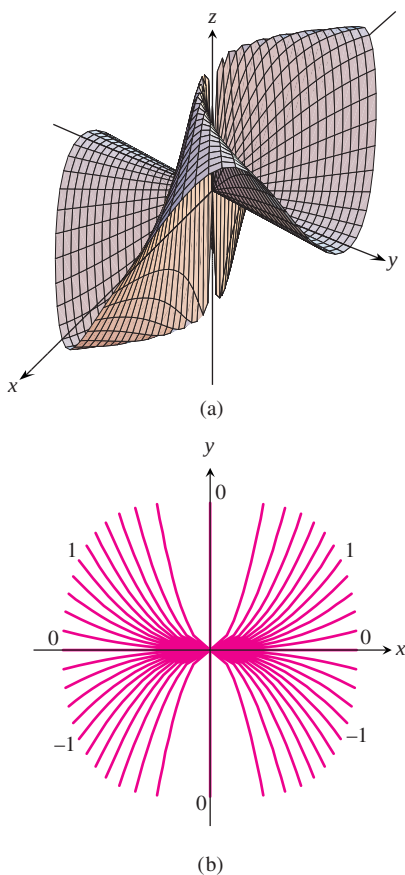


FIGURE 14.12 (a) The graph of $f(x, y) = 2x^2y/(x^4 + y^2)$. As the graph suggests and the level-curve values in part (b) confirm, $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist (Example 5).

the point. There is no *single* path-independent limit, and therefore, by the definition, $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist. ■

Compositions of continuous functions are also continuous. The proof, omitted here, is similar to that for functions of a single variable (Theorem 10 in Section 2.6).

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0) .

For example, the composite functions

$$e^{x-y}, \quad \cos \frac{xy}{x^2 + 1}, \quad \ln(1 + x^2 y^2)$$

are continuous at every point (x, y) .

As with functions of a single variable, the general rule is that composites of continuous functions are continuous. The only requirement is that each function be continuous where it is applied.

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln(x + y + z) \quad \text{and} \quad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P \rightarrow (1,0,-1)} \frac{e^{x+z}}{z^2 + \cos \sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z) , may be found by direct substitution.

Extreme Values of Continuous Functions on Closed, Bounded Sets

We have seen that a function of a single variable that is continuous throughout a closed, bounded interval $[a, b]$ takes on an absolute maximum value and an absolute minimum value at least once in $[a, b]$. The same is true of a function $z = f(x, y)$ that is continuous on a closed, bounded set R in the plane (like a line segment, a disk, or a filled-in triangle). The function takes on an absolute maximum value at some point in R and an absolute minimum value at some point in R .

Theorems similar to these and other theorems of this section hold for functions of three or more variables. A continuous function $w = f(x, y, z)$, for example, must take on absolute maximum and minimum values on any closed, bounded set (solid ball or cube, spherical shell, rectangular solid) on which it is defined.

We learn how to find these extreme values in Section 14.7, but first we need to study derivatives in higher dimensions. That is the topic of the next section.

EXERCISES 14.2

Limits with Two Variables

Find the limits in Exercises 1–12.

1. $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$ 2. $\lim_{(x,y) \rightarrow (0,4)} \frac{x}{\sqrt{y}}$
3. $\lim_{(x,y) \rightarrow (3,4)} \sqrt{x^2 + y^2 - 1}$ 4. $\lim_{(x,y) \rightarrow (2,-3)} \left(\frac{1}{x} + \frac{1}{y} \right)^2$
5. $\lim_{(x,y) \rightarrow (0,\pi/4)} \sec x \tan y$ 6. $\lim_{(x,y) \rightarrow (0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$
7. $\lim_{(x,y) \rightarrow (0, \ln 2)} e^{x-y}$ 8. $\lim_{(x,y) \rightarrow (1,1)} \ln |1 + x^2 y^2|$
9. $\lim_{(x,y) \rightarrow (0,0)} \frac{e^y \sin x}{x}$ 10. $\lim_{(x,y) \rightarrow (1,1)} \cos \sqrt[3]{|xy| - 1}$
11. $\lim_{(x,y) \rightarrow (1,0)} \frac{x \sin y}{x^2 + 1}$ 12. $\lim_{(x,y) \rightarrow (\pi/2, 0)} \frac{\cos y + 1}{y - \sin x}$

Limits of Quotients

Find the limits in Exercises 13–20 by rewriting the fractions first.

13. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - 2xy + y^2}{x - y}$ 14. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq y}} \frac{x^2 - y^2}{x - y}$
15. $\lim_{\substack{(x,y) \rightarrow (1,1) \\ x \neq 1}} \frac{xy - y - 2x + 2}{x - 1}$
16. $\lim_{\substack{(x,y) \rightarrow (2,-4) \\ y \neq -4, x \neq x^2}} \frac{y + 4}{x^2 y - xy + 4x^2 - 4x}$
17. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ x \neq y}} \frac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$
18. $\lim_{\substack{(x,y) \rightarrow (2,2) \\ x+y \neq 4}} \frac{x + y - 4}{\sqrt{x} + y - 2}$ 19. $\lim_{\substack{(x,y) \rightarrow (2,0) \\ 2x-y \neq 4}} \frac{\sqrt{2x-y} - 2}{2x - y - 4}$
20. $\lim_{\substack{(x,y) \rightarrow (4,3) \\ x \neq y+1}} \frac{\sqrt{x} - \sqrt{y+1}}{x - y - 1}$

Limits with Three Variables

Find the limits in Exercises 21–26.

21. $\lim_{P \rightarrow (1,3,4)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$ 22. $\lim_{P \rightarrow (1,-1,-1)} \frac{2xy + yz}{x^2 + z^2}$
23. $\lim_{P \rightarrow (3,3,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$
24. $\lim_{P \rightarrow (-1/4, \pi/2, 2)} \tan^{-1} xyz$ 25. $\lim_{P \rightarrow (\pi, 0, 3)} ze^{-2y} \cos 2x$
26. $\lim_{P \rightarrow (0, -2, 0)} \ln \sqrt{x^2 + y^2 + z^2}$

Continuity in the Plane

At what points (x, y) in the plane are the functions in Exercises 27–30 continuous?

27. a. $f(x, y) = \sin(x + y)$ b. $f(x, y) = \ln(x^2 + y^2)$
28. a. $f(x, y) = \frac{x + y}{x - y}$ b. $f(x, y) = \frac{y}{x^2 + 1}$
29. a. $g(x, y) = \sin \frac{1}{xy}$ b. $g(x, y) = \frac{x + y}{2 + \cos x}$
30. a. $g(x, y) = \frac{x^2 + y^2}{x^2 - 3x + 2}$ b. $g(x, y) = \frac{1}{x^2 - y}$

Continuity in Space

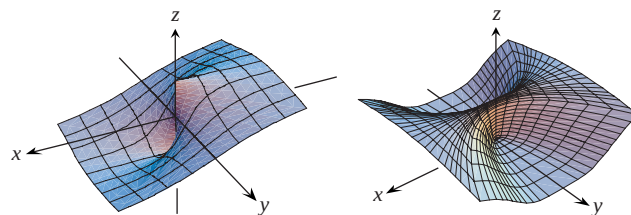
At what points (x, y, z) in space are the functions in Exercises 31–34 continuous?

31. a. $f(x, y, z) = x^2 + y^2 - 2z^2$
b. $f(x, y, z) = \sqrt{x^2 + y^2 - 1}$
32. a. $f(x, y, z) = \ln xyz$ b. $f(x, y, z) = e^{x+y} \cos z$
33. a. $h(x, y, z) = xy \sin \frac{1}{z}$ b. $h(x, y, z) = \frac{1}{x^2 + z^2 - 1}$
34. a. $h(x, y, z) = \frac{1}{|y| + |z|}$ b. $h(x, y, z) = \frac{1}{|xy| + |z|}$

No Limit at a Point

By considering different paths of approach, show that the functions in Exercises 35–42 have no limit as $(x, y) \rightarrow (0, 0)$.

35. $f(x, y) = -\frac{x}{\sqrt{x^2 + y^2}}$ 36. $f(x, y) = \frac{x^4}{x^4 + y^2}$



37. $f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$ 38. $f(x, y) = \frac{xy}{|xy|}$
39. $g(x, y) = \frac{x - y}{x + y}$ 40. $g(x, y) = \frac{x + y}{x - y}$
41. $h(x, y) = \frac{x^2 + y}{y}$ 42. $h(x, y) = \frac{x^2}{x^2 - y}$

Theory and Examples

43. If $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$, must f be defined at (x_0, y_0) ? Give reasons for your answer.

44. If $f(x_0, y_0) = 3$, what can you say about

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y)$$

if f is continuous at (x_0, y_0) ? If f is not continuous at (x_0, y_0) ? Give reasons for your answer.

The Sandwich Theorem for functions of two variables states that if $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \rightarrow (x_0, y_0)$, then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L.$$

Use this result to support your answers to the questions in Exercises 45–48.

45. Does knowing that

$$1 - \frac{x^2 y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\tan^{-1} xy}{xy}?$$

Give reasons for your answer.

46. Does knowing that

$$2|xy| - \frac{x^2 y^2}{6} < 4 - 4 \cos \sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} \frac{4 - 4 \cos \sqrt{|xy|}}{|xy|}?$$

Give reasons for your answer.

47. Does knowing that $|\sin(1/x)| \leq 1$ tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} y \sin \frac{1}{x}?$$

Give reasons for your answer.

48. Does knowing that $|\cos(1/y)| \leq 1$ tell you anything about

$$\lim_{(x,y) \rightarrow (0,0)} x \cos \frac{1}{y}?$$

Give reasons for your answer.

49. (Continuation of Example 4.)

a. Reread Example 4. Then substitute $m = \tan \theta$ into the formula

$$f(x, y) \Big|_{y=mx} = \frac{2m}{1+m^2}$$

and simplify the result to show how the value of f varies with the line's angle of inclination.

b. Use the formula you obtained in part (a) to show that the limit of f as $(x, y) \rightarrow (0, 0)$ along the line $y = mx$ varies from -1 to 1 depending on the angle of approach.

50. **Continuous extension** Define $f(0, 0)$ in a way that extends

$$f(x, y) = xy \frac{x^2 - y^2}{x^2 + y^2}$$

to be continuous at the origin.

Changing to Polar Coordinates

If you cannot make any headway with $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ in rectangular coordinates, try changing to polar coordinates. Substitute $x = r \cos \theta$, $y = r \sin \theta$, and investigate the limit of the resulting expression as $r \rightarrow 0$. In other words, try to decide whether there exists a number L satisfying the following criterion:

Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ ,

$$|r| < \delta \implies |f(r, \theta) - L| < \epsilon. \quad (1)$$

If such an L exists, then

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{r \rightarrow 0} f(r, \theta) = L.$$

For instance,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta}{r^2} = \lim_{r \rightarrow 0} r \cos^3 \theta = 0.$$

To verify the last of these equalities, we need to show that Equation (1) is satisfied with $f(r, \theta) = r \cos^3 \theta$ and $L = 0$. That is, we need to show that given any $\epsilon > 0$ there exists a $\delta > 0$ such that for all r and θ ,

$$|r| < \delta \implies |r \cos^3 \theta - 0| < \epsilon.$$

Since

$$|r \cos^3 \theta| = |r| |\cos^3 \theta| \leq |r| \cdot 1 = |r|,$$

the implication holds for all r and θ if we take $\delta = \epsilon$.

In contrast,

$$\frac{x^2}{x^2 + y^2} = \frac{r^2 \cos^2 \theta}{r^2} = \cos^2 \theta$$

takes on all values from 0 to 1 regardless of how small $|r|$ is, so that $\lim_{(x,y) \rightarrow (0,0)} x^2/(x^2 + y^2)$ does not exist.

In each of these instances, the existence or nonexistence of the limit as $r \rightarrow 0$ is fairly clear. Shifting to polar coordinates does not always help, however, and may even tempt us to false conclusions. For example, the limit may exist along every straight line (or ray) $\theta = \text{constant}$ and yet fail to exist in the broader sense. Example 4 illustrates this point. In polar coordinates, $f(x, y) = (2x^2 y)/(x^4 + y^2)$ becomes

$$f(r \cos \theta, r \sin \theta) = \frac{r \cos \theta \sin 2\theta}{r^2 \cos^4 \theta + \sin^2 \theta}$$

for $r \neq 0$. If we hold θ constant and let $r \rightarrow 0$, the limit is 0. On the path $y = x^2$, however, we have $r \sin \theta = r^2 \cos^2 \theta$ and

$$\begin{aligned} f(r \cos \theta, r \sin \theta) &= \frac{r \cos \theta \sin 2\theta}{r^2 \cos^4 \theta + (r \cos^2 \theta)^2} \\ &= \frac{2r \cos^2 \theta \sin \theta}{2r^2 \cos^4 \theta} = \frac{r \sin \theta}{r^2 \cos^2 \theta} = 1. \end{aligned}$$

In Exercises 51–56, find the limit of f as $(x, y) \rightarrow (0, 0)$ or show that the limit does not exist.

$$51. f(x, y) = \frac{x^3 - xy^2}{x^2 + y^2} \qquad 52. f(x, y) = \cos \left(\frac{x^3 - y^3}{x^2 + y^2} \right)$$

$$53. f(x, y) = \frac{y^2}{x^2 + y^2} \qquad 54. f(x, y) = \frac{2x}{x^2 + x + y^2}$$

$$55. f(x, y) = \tan^{-1} \left(\frac{|x| + |y|}{x^2 + y^2} \right)$$

$$56. f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

In Exercises 57 and 58, define $f(0, 0)$ in a way that extends f to be continuous at the origin.

$$57. f(x, y) = \ln \left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2} \right)$$

$$58. f(x, y) = \frac{3x^2y}{x^2 + y^2}$$

Using the δ - ϵ Definition

Each of Exercises 59–62 gives a function $f(x, y)$ and a positive number ϵ . In each exercise, show that there exists a $\delta > 0$ such that for all (x, y) ,

$$\sqrt{x^2 + y^2} < \delta \implies |f(x, y) - f(0, 0)| < \epsilon.$$

$$59. f(x, y) = x^2 + y^2, \quad \epsilon = 0.01$$

$$60. f(x, y) = y/(x^2 + 1), \quad \epsilon = 0.05$$

$$61. f(x, y) = (x + y)/(x^2 + 1), \quad \epsilon = 0.01$$

$$62. f(x, y) = (x + y)/(2 + \cos x), \quad \epsilon = 0.02$$

Each of Exercises 63–66 gives a function $f(x, y, z)$ and a positive number ϵ . In each exercise, show that there exists a $\delta > 0$ such that for all (x, y, z) ,

$$\sqrt{x^2 + y^2 + z^2} < \delta \implies |f(x, y, z) - f(0, 0, 0)| < \epsilon.$$

$$63. f(x, y, z) = x^2 + y^2 + z^2, \quad \epsilon = 0.015$$

$$64. f(x, y, z) = xyz, \quad \epsilon = 0.008$$

$$65. f(x, y, z) = \frac{x + y + z}{x^2 + y^2 + z^2 + 1}, \quad \epsilon = 0.015$$

$$66. f(x, y, z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \epsilon = 0.03$$

$$67. \text{ Show that } f(x, y, z) = x + y - z \text{ is continuous at every point } (x_0, y_0, z_0).$$

$$68. \text{ Show that } f(x, y, z) = x^2 + y^2 + z^2 \text{ is continuous at the origin.}$$

14.3

Partial Derivatives

The calculus of several variables is basically single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a “partial” derivative. This section shows how partial derivatives are defined and interpreted geometrically, and how to calculate them by applying the rules for differentiating functions of a single variable.

Partial Derivatives of a Function of Two Variables

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$ (Figure 14.13). This curve is the graph of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x ; the vertical coordinate is z . The y -value is held constant at y_0 , so y is not a variable.

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$. To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than the d previously used.

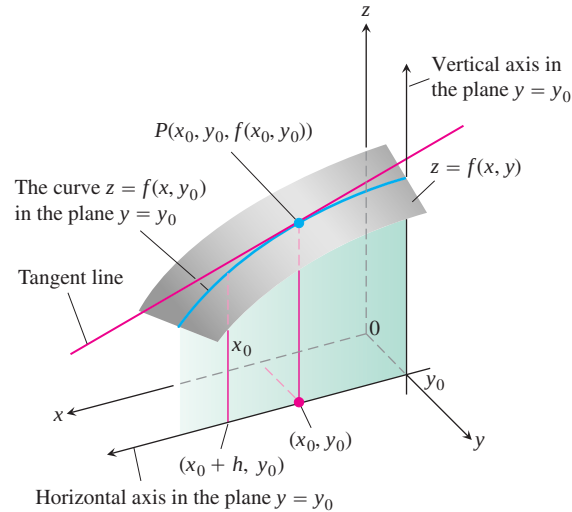


FIGURE 14.13 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy -plane.

DEFINITION Partial Derivative with Respect to x

The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

An equivalent expression for the partial derivative is

$$\left. \frac{d}{dx} f(x, y_0) \right|_{x=x_0}.$$

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0) . The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $\partial f / \partial x$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0 . This is the rate of change of f in the direction of $\mathbf{i}$ at (x_0, y_0) .

The notation for a partial derivative depends on what we want to emphasize:

$\frac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$ “Partial derivative of f with respect to x at (x_0, y_0) ” or “ f sub x at (x_0, y_0) .” Convenient for stressing the point (x_0, y_0) .

$\left. \frac{\partial z}{\partial x} \right|_{(x_0, y_0)}$ “Partial derivative of z with respect to x at (x_0, y_0) .” Common in science and engineering when you are dealing with variables and do not mention the function explicitly.

$f_x, \frac{\partial f}{\partial x}, z_x,$ or $\frac{\partial z}{\partial x}$ “Partial derivative of f (or z) with respect to x .” Convenient when you regard the partial derivative as a function in its own right.

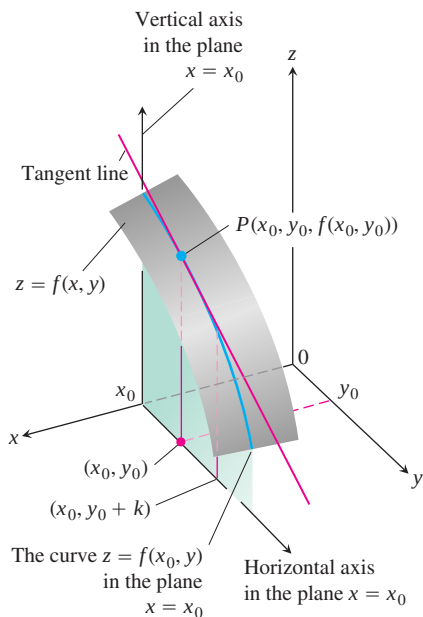


FIGURE 14.14 The intersection of the plane $x = x_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy -plane.

The definition of the partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is similar to the definition of the partial derivative of f with respect to x . We hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0, y)$ with respect to y at y_0 .

DEFINITION Partial Derivative with Respect to y

The **partial derivative of $f(x, y)$ with respect to y** at the point (x_0, y_0) is

$$\left. \frac{\partial f}{\partial y} \right|_{(x_0, y_0)} = \left. \frac{d}{dy} f(x_0, y) \right|_{y=y_0} = \lim_{h \rightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

The slope of the curve $z = f(x_0, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$ (Figure 14.14) is the partial derivative of f with respect to y at (x_0, y_0) . The tangent line to the curve at P is the line in the plane $x = x_0$ that passes through P with this slope. The partial derivative gives the rate of change of f with respect to y at (x_0, y_0) when x is held fixed at the value x_0 . This is the rate of change of f in the direction of $\mathbf{j}$ at (x_0, y_0) .

The partial derivative with respect to y is denoted the same way as the partial derivative with respect to x :

$$\frac{\partial f}{\partial y}(x_0, y_0), \quad f_y(x_0, y_0), \quad \frac{\partial f}{\partial y}, \quad f_y.$$

Notice that we now have two tangent lines associated with the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ (Figure 14.15). Is the plane they determine tangent to the surface at P ? We will see that it is, but we have to learn more about partial derivatives before we can find out why.

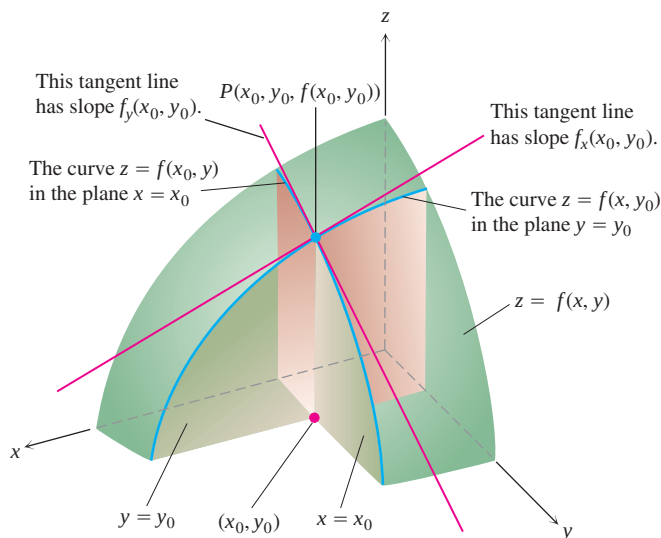


FIGURE 14.15 Figures 14.13 and 14.14 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.



Calculations

The definitions of $\partial f/\partial x$ and $\partial f/\partial y$ give us two different ways of differentiating f at a point: with respect to x in the usual way while treating y as a constant and with respect to y in the usual way while treating x as constant. As the following examples show, the values of these partial derivatives are usually different at a given point (x_0, y_0) .



EXAMPLE 1 Finding Partial Derivatives at a Point

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x :

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y :

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$. ■



EXAMPLE 2 Finding a Partial Derivative as a Function

Find $\partial f/\partial y$ if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y) \\ &= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy. \end{aligned}$$

■

USING TECHNOLOGY Partial Differentiation

A simple grapher can support your calculations even in multiple dimensions. If you specify the values of all but one independent variable, the grapher can calculate partial derivatives and can plot traces with respect to that remaining variable. Typically, a CAS can compute partial derivatives symbolically and numerically as easily as it can compute simple derivatives. Most systems use the same command to differentiate a function, regardless of the number of variables. (Simply specify the variable with which differentiation is to take place).



EXAMPLE 3 Partial Derivatives May Be Different Functions

Find f_x and f_y if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we get

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial x} (2y) - 2y \frac{\partial}{\partial x} (y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}. \end{aligned}$$

With x held constant, we get

$$\begin{aligned} f_y &= \frac{\partial}{\partial y} \left(\frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial y} (2y) - 2y \frac{\partial}{\partial y} (y + \cos x)}{(y + \cos x)^2} \\ &= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}. \end{aligned}$$

Implicit differentiation works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.



EXAMPLE 4 Implicit Partial Differentiation

Find $\partial z / \partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x , holding y constant and treating z as a differentiable function of x :

$$\begin{aligned} \frac{\partial}{\partial x} (yz) - \frac{\partial}{\partial x} \ln z &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} \\ y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} &= 1 + 0 \quad \text{With } y \text{ constant,} \\ \left(y - \frac{1}{z} \right) \frac{\partial z}{\partial x} &= 1 \quad \frac{\partial}{\partial x} (yz) = y \frac{\partial z}{\partial x}. \\ \frac{\partial z}{\partial x} &= \frac{z}{yz - 1}. \end{aligned}$$

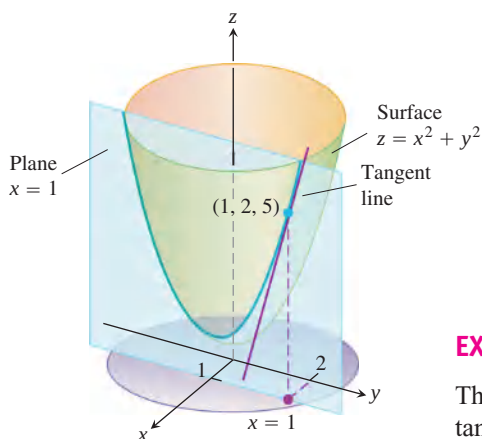


FIGURE 14.16 The tangent to the curve of intersection of the plane $x = 1$ and surface $z = x^2 + y^2$ at the point $(1, 2, 5)$ (Example 5).

EXAMPLE 5 Finding the Slope of a Surface in the y -Direction

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$ (Figure 14.16).

Solution The slope is the value of the partial derivative $\partial z / \partial y$ at $(1, 2)$:

$$\left. \frac{\partial z}{\partial y} \right|_{(1,2)} = \left. \frac{\partial}{\partial y} (x^2 + y^2) \right|_{(1,2)} = 2y \Big|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left. \frac{dz}{dy} \right|_{y=2} = \left. \frac{d}{dy} (1 + y^2) \right|_{y=2} = 2y \Big|_{y=2} = 4. \quad \blacksquare$$

Functions of More Than Two Variables

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.



EXAMPLE 6 A Function of Three Variables

If x , y , and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} [x \sin(y + 3z)] = x \frac{\partial}{\partial z} \sin(y + 3z) \\ &= x \cos(y + 3z) \frac{\partial}{\partial z} (y + 3z) = 3x \cos(y + 3z). \end{aligned} \quad \blacksquare$$

EXAMPLE 7 Electrical Resistors in Parallel

If resistors of R_1 , R_2 , and R_3 ohms are connected in parallel to make an R -ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

(Figure 14.17). Find the value of $\partial R / \partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.

Solution To find $\partial R / \partial R_2$, we treat R_1 and R_3 as constants and, using implicit differentiation, differentiate both sides of the equation with respect to R_2 :

$$\begin{aligned} \frac{\partial}{\partial R_2} \left(\frac{1}{R} \right) &= \frac{\partial}{\partial R_2} \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \\ -\frac{1}{R^2} \frac{\partial R}{\partial R_2} &= 0 - \frac{1}{R_2^2} + 0 \\ \frac{\partial R}{\partial R_2} &= \frac{R^2}{R_2^2} = \left(\frac{R}{R_2} \right)^2. \end{aligned}$$

When $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$,

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3 + 2 + 1}{90} = \frac{6}{90} = \frac{1}{15},$$

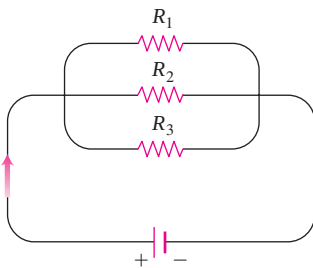


FIGURE 14.17 Resistors arranged this way are said to be connected in parallel (Example 7). Each resistor lets a portion of the current through. Their equivalent resistance R is calculated with the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

so $R = 15$ and

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Partial Derivatives and Continuity

A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without the function being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. If the partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at (x_0, y_0) , however, then f is continuous at (x_0, y_0) , as we see at the end of this section.

EXAMPLE 8 Partials Exist, But f Discontinuous

Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Figure 14.18).

- (a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.
- (b) Prove that f is not continuous at the origin.
- (c) Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

Solution

- (a) Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin), we have

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) \Big|_{y=x} = \lim_{(x, y) \rightarrow (0, 0)} 0 = 0.$$

- (b) Since $f(0, 0) = 1$, the limit in part (a) proves that f is not continuous at $(0, 0)$.
- (c) To find $\partial f/\partial x$ at $(0, 0)$, we hold y fixed at $y = 0$. Then $f(x, y) = 1$ for all x , and the graph of f is the line L_1 in Figure 14.18. The slope of this line at any x is $\partial f/\partial x = 0$. In particular, $\partial f/\partial x = 0$ at $(0, 0)$. Similarly, $\partial f/\partial y$ is the slope of line L_2 at any y , so $\partial f/\partial y = 0$ at $(0, 0)$.

Example 8 notwithstanding, it is still true in higher dimensions that *differentiability* at a point implies continuity. What Example 8 suggests is that we need a stronger requirement for differentiability in higher dimensions than the mere existence of the partial derivatives. We define differentiability for functions of two variables at the end of this section and revisit the connection to continuity.

Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\begin{array}{llll} \frac{\partial^2 f}{\partial x^2} & \text{"d squared f dx squared"} & \text{or} & f_{xx} \quad \text{"f sub xx"} \\ \frac{\partial^2 f}{\partial y^2} & \text{"d squared f dy squared"} & \text{or} & f_{yy} \quad \text{"f sub yy"} \end{array}$$

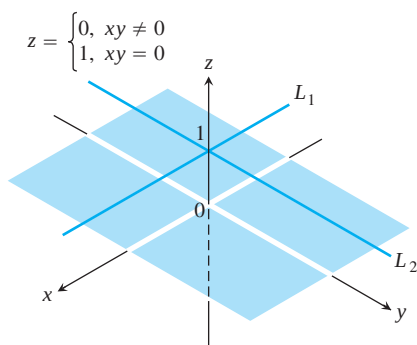


FIGURE 14.18 The graph of

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

consists of the lines L_1 and L_2 and the four open quadrants of the xy -plane. The function has partial derivatives at the origin but is not continuous there (Example 8).

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{“}d^2 \text{ squared } f dx dy\text{”} \quad \text{or} \quad f_{yx} \quad \text{“}f \text{ sub } yx\text{”}$$

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{“}d^2 \text{ squared } f dy dx\text{”} \quad \text{or} \quad f_{xy} \quad \text{“}f \text{ sub } xy\text{”}$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right),$$

and so on. Notice the order in which the derivatives are taken:

$$\frac{\partial^2 f}{\partial x \partial y} \quad \text{Differentiate first with respect to } y, \text{ then with respect to } x.$$

$$f_{yx} = (f_y)_x \quad \text{Means the same thing.}$$

HISTORICAL BIOGRAPHY

Pierre-Simon Laplace
(1749–1827)



EXAMPLE 9 Finding Second-Order Partial Derivatives

If $f(x, y) = x \cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} (x \cos y + ye^x) \\ &= \cos y + ye^x \end{aligned}$$

So

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = -\sin y + e^x \\ \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = ye^x. \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} (x \cos y + ye^x) \\ &= -x \sin y + e^x \end{aligned}$$

So

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = -\sin y + e^x \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = -x \cos y. \end{aligned}$$

The Mixed Derivative Theorem

You may have noticed that the “mixed” second-order partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}$$

in Example 9 were equal. This was not a coincidence. They must be equal whenever f , f_x , f_y , f_{xy} , and f_{yx} are continuous, as stated in the following theorem.

THEOREM 2 The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x , f_y , f_{xy} , and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b) , then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

HISTORICAL BIOGRAPHY

Alexis Clairaut
(1713–1765)

Theorem 2 is also known as Clairaut's Theorem, named after the French mathematician Alexis Clairaut who discovered it. A proof is given in Appendix 7. Theorem 2 says that to calculate a mixed second-order derivative, we may differentiate in either order, provided the continuity conditions are satisfied. This can work to our advantage.

EXAMPLE 10 Choosing the Order of Differentiation

Find $\partial^2 w / \partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w / \partial x \partial y$ tells us to differentiate first with respect to y and then with respect to x . If we postpone the differentiation with respect to y and differentiate first with respect to x , however, we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y \partial x} = 1.$$

If we differentiate first with respect to y , we obtain $\partial^2 w / \partial x \partial y = 1$ as well. ■

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx},$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

EXAMPLE 11 Calculating a Partial Derivative of Fourth-Order

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution We first differentiate with respect to the variable y , then x , then y again, and finally with respect to z :

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yxy} = -4z$$

$$f_{yxyz} = -4$$

Differentiability

The starting point for differentiability is not Fermat's difference quotient but rather the idea of increment. You may recall from our work with functions of a single variable in Section 3.8 that if $y = f(x)$ is differentiable at $x = x_0$, then the change in the value of f that results from changing x from x_0 to $x_0 + \Delta x$ is given by an equation of the form

$$\Delta y = f'(x_0)\Delta x + \epsilon\Delta x$$

in which $\epsilon \rightarrow 0$ as $\Delta x \rightarrow 0$. For functions of two variables, the analogous property becomes the definition of differentiability. The Increment Theorem (from advanced calculus) tells us when to expect the property to hold.

THEOREM 3 The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0) . Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

in which each of $\epsilon_1, \epsilon_2 \rightarrow 0$ as both $\Delta x, \Delta y \rightarrow 0$.

You can see where the epsilons come from in the proof in Appendix 7. You will also see that similar results hold for functions of more than two independent variables.

DEFINITION Differentiable Function

A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

in which each of $\epsilon_1, \epsilon_2 \rightarrow 0$ as both $\Delta x, \Delta y \rightarrow 0$. We call f **differentiable** if it is differentiable at every point in its domain.

In light of this definition, we have the immediate corollary of Theorem 3 that a function is differentiable if its first partial derivatives are *continuous*.

COROLLARY OF THEOREM 3 Continuity of Partial Derivatives Implies Differentiability

If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R , then f is differentiable at every point of R .

If $z = f(x, y)$ is differentiable, then the definition of differentiability assures that $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ approaches 0 as Δx and Δy approach 0. This tells us that a function of two variables is continuous at every point where it is differentiable.

THEOREM 4 Differentiability Implies Continuity

If a function $f(x, y)$ is differentiable at (x_0, y_0) , then f is continuous at (x_0, y_0) .

As we can see from Theorems 3 and 4, a function $f(x, y)$ must be continuous at a point (x_0, y_0) if f_x and f_y are continuous throughout an open region containing (x_0, y_0) . Remember, however, that it is still possible for a function of two variables to be discontinuous at a point where its first partial derivatives exist, as we saw in Example 8. Existence alone of the partial derivative at a point is not enough.

EXERCISES 14.3

Calculating First-Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

1. $f(x, y) = 2x^2 - 3y - 4$ 2. $f(x, y) = x^2 - xy + y^2$
 3. $f(x, y) = (x^2 - 1)(y + 2)$
 4. $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$
 5. $f(x, y) = (xy - 1)^2$ 6. $f(x, y) = (2x - 3y)^3$
 7. $f(x, y) = \sqrt{x^2 + y^2}$ 8. $f(x, y) = (x^3 + (y/2))^{2/3}$
 9. $f(x, y) = 1/(x + y)$ 10. $f(x, y) = x/(x^2 + y^2)$
 11. $f(x, y) = (x + y)/(xy - 1)$ 12. $f(x, y) = \tan^{-1}(y/x)$
 13. $f(x, y) = e^{(x+y+1)}$ 14. $f(x, y) = e^{-x} \sin(x + y)$
 15. $f(x, y) = \ln(x + y)$ 16. $f(x, y) = e^{xy} \ln y$
 17. $f(x, y) = \sin^2(x - 3y)$ 18. $f(x, y) = \cos^2(3x - y^2)$
 19. $f(x, y) = x^y$ 20. $f(x, y) = \log_y x$
21. $f(x, y) = \int_x^y g(t) dt$ (g continuous for all t)

$$22. f(x, y) = \sum_{n=0}^{\infty} (xy)^n \quad (|xy| < 1)$$

In Exercises 23–34, find f_x , f_y , and f_z .

23. $f(x, y, z) = 1 + xy^2 - 2z^2$ 24. $f(x, y, z) = xy + yz + xz$
 25. $f(x, y, z) = x - \sqrt{y^2 + z^2}$
 26. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

27. $f(x, y, z) = \sin^{-1}(xyz)$ 28. $f(x, y, z) = \sec^{-1}(x + yz)$
 29. $f(x, y, z) = \ln(x + 2y + 3z)$
 30. $f(x, y, z) = yz \ln(xy)$ 31. $f(x, y, z) = e^{-(x^2+y^2+z^2)}$
 32. $f(x, y, z) = e^{-xyz}$
 33. $f(x, y, z) = \tanh(x + 2y + 3z)$
 34. $f(x, y, z) = \sinh(xy - z^2)$

In Exercises 35–40, find the partial derivative of the function with respect to each variable.

35. $f(t, \alpha) = \cos(2\pi t - \alpha)$ 36. $g(u, v) = v^2 e^{(2u/v)}$
 37. $h(\rho, \phi, \theta) = \rho \sin \phi \cos \theta$ 38. $g(r, \theta, z) = r(1 - \cos \theta) - z$
 39. **Work done by the heart** (Section 3.8, Exercise 51)

$$W(P, V, \delta, v, g) = PV + \frac{V\delta v^2}{2g}$$

40. **Wilson lot size formula** (Section 4.5, Exercise 45)

$$A(c, h, k, m, q) = \frac{km}{q} + cm + \frac{hq}{2}$$

Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–46.

41. $f(x, y) = x + y + xy$ 42. $f(x, y) = \sin xy$





43. $g(x, y) = x^2y + \cos y + y \sin x$
 44. $h(x, y) = xe^y + y + 1$ 45. $r(x, y) = \ln(x + y)$
 46. $s(x, y) = \tan^{-1}(y/x)$

Mixed Partial Derivatives

In Exercises 47–50, verify that $w_{xy} = w_{yx}$.

47. $w = \ln(2x + 3y)$ 48. $w = e^x + x \ln y + y \ln x$
 49. $w = xy^2 + x^2y^3 + x^3y^4$ 50. $w = x \sin y + y \sin x + xy$
 51. Which order of differentiation will calculate f_{xy} faster: x first or y first? Try to answer without writing anything down.
 a. $f(x, y) = x \sin y + e^y$
 b. $f(x, y) = 1/x$
 c. $f(x, y) = y + (x/y)$
 d. $f(x, y) = y + x^2y + 4y^3 - \ln(y^2 + 1)$
 e. $f(x, y) = x^2 + 5xy + \sin x + 7e^x$
 f. $f(x, y) = x \ln xy$
 52. The fifth-order partial derivative $\partial^5 f / \partial x^2 \partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y ? Try to answer without writing anything down.
 a. $f(x, y) = y^2x^4e^x + 2$
 b. $f(x, y) = y^2 + y(\sin x - x^4)$
 c. $f(x, y) = x^2 + 5xy + \sin x + 7e^x$
 d. $f(x, y) = xe^{y^2/2}$

Using the Partial Derivative Definition

In Exercises 53 and 54, use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.



53. $f(x, y) = 1 - x + y - 3x^2y$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(1, 2)$
 54. $f(x, y) = 4 + 2x - 3y - xy^2$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(-2, 1)$
 55. **Three variables** Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f / \partial z$ at (x_0, y_0, z_0) . Use this definition to find $\partial f / \partial z$ at $(1, 2, 3)$ for $f(x, y, z) = x^2yz^2$.
 56. **Three variables** Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f / \partial y$ at (x_0, y_0, z_0) . Use this definition to find $\partial f / \partial y$ at $(-1, 0, 3)$ for $f(x, y, z) = -2xy^2 + yz^2$.

Differentiating Implicitly

57. Find the value of $\partial z / \partial x$ at the point $(1, 1, 1)$ if the equation

$$xy + z^3x - 2yz = 0$$

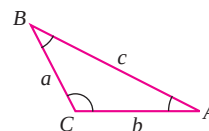
defines z as a function of the two independent variables x and y and the partial derivative exists.

58. Find the value of $\partial x / \partial z$ at the point $(1, -1, -3)$ if the equation

$$xz + y \ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

Exercises 59 and 60 are about the triangle shown here.



59. Express A implicitly as a function of a , b , and c and calculate $\partial A / \partial a$ and $\partial A / \partial b$.
 60. Express a implicitly as a function of A , b , and C and calculate $\partial a / \partial A$ and $\partial a / \partial C$.
 61. **Two dependent variables** Express v_x in terms of u and v if the equations $x = v \ln u$ and $y = u \ln v$ define u and v as functions of the independent variables x and y , and if v_x exists. (Hint: Differentiate both equations with respect to x and solve for v_x by eliminating u_x .)
 62. **Two dependent variables** Find $\partial x / \partial u$ and $\partial y / \partial u$ if the equations $u = x^2 - y^2$ and $v = x^2 - y$ define x and y as functions of the independent variables u and v , and the partial derivatives exist. (See the hint in Exercise 61.) Then let $s = x^2 + y^2$ and find $\partial s / \partial u$.

Laplace Equations

The **three-dimensional Laplace equation**

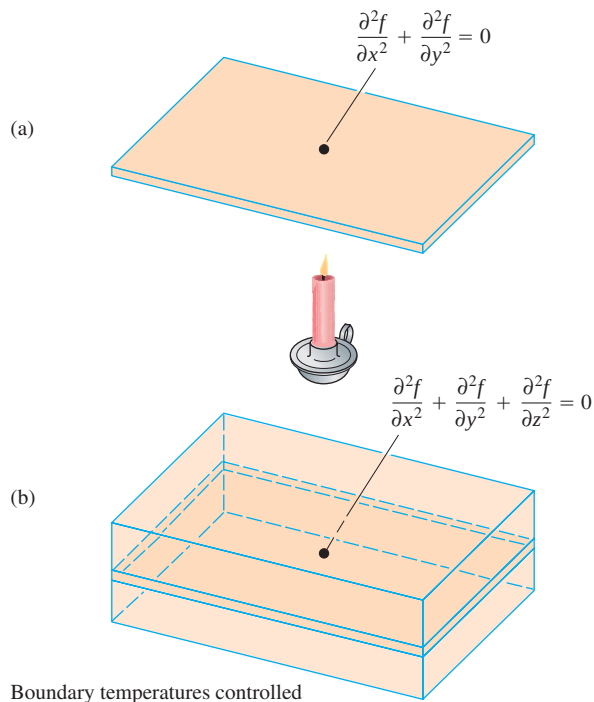
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is satisfied by steady-state temperature distributions $T = f(x, y, z)$ in space, by gravitational potentials, and by electrostatic potentials. The **two-dimensional Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

obtained by dropping the $\partial^2 f / \partial z^2$ term from the previous equation, describes potentials and steady-state temperature distributions in a plane (see the accompanying figure). The plane (a) may be treated as a thin slice of the solid (b) perpendicular to the z -axis.





Show that each function in Exercises 63–68 satisfies a Laplace equation.

63. $f(x, y, z) = x^2 + y^2 - 2z^2$
 64. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$
 65. $f(x, y) = e^{-2y} \cos 2x$
 66. $f(x, y) = \ln \sqrt{x^2 + y^2}$
 67. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
 68. $f(x, y, z) = e^{3x+4y} \cos 5z$

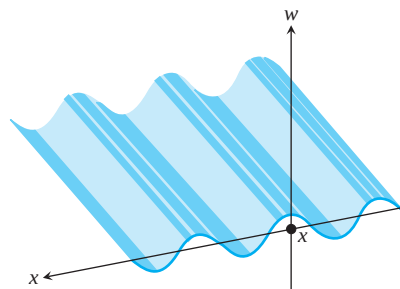
The Wave Equation

If we stand on an ocean shore and take a snapshot of the waves, the picture shows a regular pattern of peaks and valleys in an instant of time. We see periodic vertical motion in space, with respect to distance. If we stand in the water, we can feel the rise and fall of the

water as the waves go by. We see periodic vertical motion in time. In physics, this beautiful symmetry is expressed by the **one-dimensional wave equation**

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},$$

where w is the wave height, x is the distance variable, t is the time variable, and c is the velocity with which the waves are propagated.



In our example, x is the distance across the ocean's surface, but in other applications, x might be the distance along a vibrating string, distance through air (sound waves), or distance through space (light waves). The number c varies with the medium and type of wave.

Show that the functions in Exercises 69–75 are all solutions of the wave equation.

69. $w = \sin(x + ct)$ 70. $w = \cos(2x + 2ct)$
 71. $w = \sin(x + ct) + \cos(2x + 2ct)$
 72. $w = \ln(2x + 2ct)$ 73. $w = \tan(2x - 2ct)$
 74. $w = 5 \cos(3x + 3ct) + e^{x+ct}$
 75. $w = f(u)$, where f is a differentiable function of u , and $u = a(x + ct)$, where a is a constant

Continuous Partial Derivatives

76. Does a function $f(x, y)$ with continuous first partial derivatives throughout an open region R have to be continuous on R ? Give reasons for your answer.
 77. If a function $f(x, y)$ has continuous second partial derivatives throughout an open region R , must the first-order partial derivatives of f be continuous on R ? Give reasons for your answer.

14.4

The Chain Rule

The Chain Rule for functions of a single variable studied in Section 3.5 said that when $w = f(x)$ was a differentiable function of x and $x = g(t)$ was a differentiable function of t , w became a differentiable function of t and dw/dt could be calculated with the formula

$$\frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt}.$$

For functions of two or more variables the Chain Rule has several forms. The form depends on how many variables are involved but works like the Chain Rule in Section 3.5 once we account for the presence of additional variables.

Functions of Two Variables

The Chain Rule formula for a function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

THEOREM 5 Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y , and if $x = x(t)$, $y = y(t)$ are differentiable functions of t , then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

Proof The proof consists of showing that if x and y are differentiable at $t = t_0$, then w is differentiable at t_0 and

$$\left(\frac{dw}{dt} \right)_{t_0} = \left(\frac{\partial w}{\partial x} \right)_{P_0} \left(\frac{dx}{dt} \right)_{t_0} + \left(\frac{\partial w}{\partial y} \right)_{P_0} \left(\frac{dy}{dt} \right)_{t_0},$$

where $P_0 = (x(t_0), y(t_0))$. The subscripts indicate where each of the derivatives are to be evaluated.

Let Δx , Δy , and Δw be the increments that result from changing t from t_0 to $t_0 + \Delta t$. Since f is differentiable (see the definition in Section 14.3),

$$\Delta w = \left(\frac{\partial w}{\partial x} \right)_{P_0} \Delta x + \left(\frac{\partial w}{\partial y} \right)_{P_0} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1, \epsilon_2 \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$. To find dw/dt , we divide this equation through by Δt and let Δt approach zero. The division gives

$$\frac{\Delta w}{\Delta t} = \left(\frac{\partial w}{\partial x} \right)_{P_0} \frac{\Delta x}{\Delta t} + \left(\frac{\partial w}{\partial y} \right)_{P_0} \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}.$$

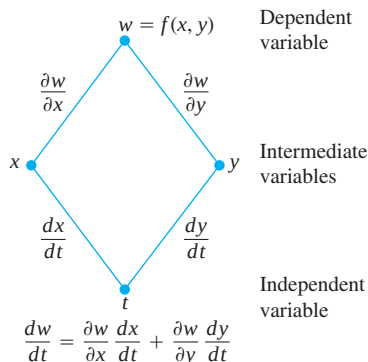
Letting Δt approach zero gives

$$\begin{aligned} \left(\frac{dw}{dt} \right)_{t_0} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta w}{\Delta t} \\ &= \left(\frac{\partial w}{\partial x} \right)_{P_0} \left(\frac{dx}{dt} \right)_{t_0} + \left(\frac{\partial w}{\partial y} \right)_{P_0} \left(\frac{dy}{dt} \right)_{t_0} + 0 \cdot \left(\frac{dx}{dt} \right)_{t_0} + 0 \cdot \left(\frac{dy}{dt} \right)_{t_0}. \quad \blacksquare \end{aligned}$$

The **tree diagram** in the margin provides a convenient way to remember the Chain Rule. From the diagram, you see that when $t = t_0$, the derivatives dx/dt and dy/dt are

To remember the Chain Rule picture the diagram below. To find dw/dt , start at w and read down each route to t , multiplying derivatives along the way. Then add the products.

Chain Rule



evaluated at t_0 . The value of t_0 then determines the value x_0 for the differentiable function x and the value y_0 for the differentiable function y . The partial derivatives $\partial w/\partial x$ and $\partial w/\partial y$ (which are themselves functions of x and y) are evaluated at the point $P_0(x_0, y_0)$ corresponding to t_0 . The “true” independent variable is t , whereas x and y are *intermediate variables* (controlled by t) and w is the dependent variable.

A more precise notation for the Chain Rule shows how the various derivatives in Theorem 5 are evaluated:

$$\frac{dw}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0, y_0) \cdot \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \frac{dy}{dt}(t_0).$$



EXAMPLE 1 Applying the Chain Rule

Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution We apply the Chain Rule to find dw/dt as follows:

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial(xy)}{\partial x} \cdot \frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y} \cdot \frac{d}{dt}(\sin t) \\ &= (y)(-\sin t) + (x)(\cos t) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) \\ &= -\sin^2 t + \cos^2 t \\ &= \cos 2t. \end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of t ,

$$w = xy = \cos t \sin t = \frac{1}{2} \sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt} \left(\frac{1}{2} \sin 2t \right) = \frac{1}{2} \cdot 2 \cos 2t = \cos 2t.$$

In either case, at the given value of t ,

$$\left(\frac{dw}{dt} \right)_{t=\pi/2} = \cos \left(2 \cdot \frac{\pi}{2} \right) = \cos \pi = -1. \quad \blacksquare$$

Functions of Three Variables

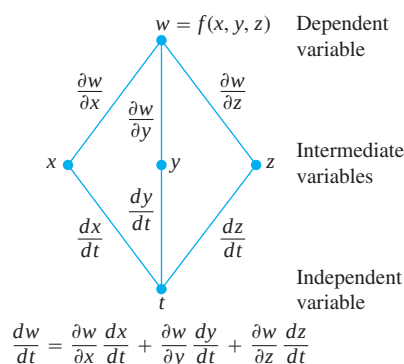
You can probably predict the Chain Rule for functions of three variables, as it only involves adding the expected third term to the two-variable formula.

THEOREM 6 Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and x , y , and z are differentiable functions of t , then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.$$

Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule

The proof is identical with the proof of Theorem 5 except that there are now three intermediate variables instead of two. The diagram we use for remembering the new equation is similar as well, with three routes from w to t .

EXAMPLE 2 Changes in a Function's Values Along a Helix

Find dw/dt if

$$w = xy + z, \quad x = \cos t, \quad y = \sin t, \quad z = t.$$

In this example the values of w are changing along the path of a helix (Section 13.1). What is the derivative's value at $t = 0$?

Solution

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\ &= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t. \end{aligned}$$

Substitute for
the intermediate
variables.

$$\left(\frac{dw}{dt} \right)_{t=0} = 1 + \cos(0) = 2.$$

Here is a physical interpretation of change along a curve. If $w = T(x, y, z)$ is the temperature at each point (x, y, z) along a curve C with parametric equations $x = x(t)$, $y = y(t)$, and $z = z(t)$, then the composite function $w = T(x(t), y(t), z(t))$ represents the temperature relative to t along the curve. The derivative dw/dt is then the instantaneous rate of change of temperature along the curve, as calculated in Theorem 6.

Functions Defined on Surfaces

If we are interested in the temperature $w = f(x, y, z)$ at points (x, y, z) on a globe in space, we might prefer to think of x , y , and z as functions of the variables r and s that give the points' longitudes and latitudes. If $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$, we could then express the temperature as a function of r and s with the composite function

$$w = f(g(r, s), h(r, s), k(r, s)).$$

Under the right conditions, w would have partial derivatives with respect to both r and s that could be calculated in the following way.





THEOREM 7 Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s , given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}.$$

The first of these equations can be derived from the Chain Rule in Theorem 6 by holding s fixed and treating r as t . The second can be derived in the same way, holding r fixed and treating s as t . The tree diagrams for both equations are shown in Figure 14.19.

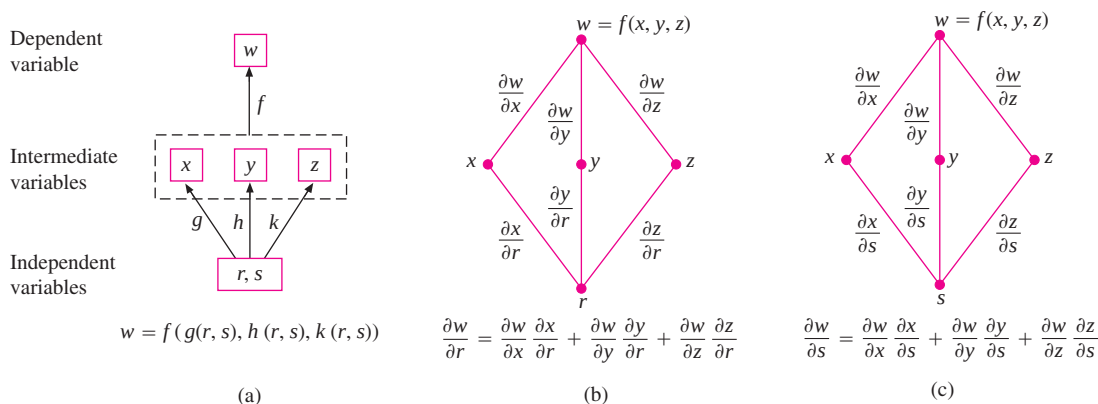


FIGURE 14.19 Composite function and tree diagrams for Theorem 7.



EXAMPLE 3 Partial Derivatives Using Theorem 7

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r.$$

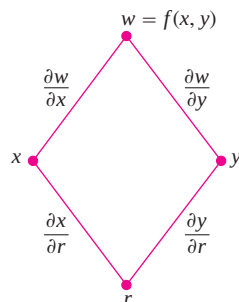
Solution

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r} \\ &= (1) \left(\frac{1}{s} \right) + (2)(2r) + (2z)(2) \\ &= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \end{aligned}$$

Substitute for intermediate variable z .

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} \\ &= (1) \left(-\frac{r}{s^2} \right) + (2) \left(\frac{1}{s} \right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2} \end{aligned}$$

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r}$$

FIGURE 14.20 Tree diagram for the equation

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r}$$

If f is a function of two variables instead of three, each equation in Theorem 7 becomes correspondingly one term shorter.

If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}.$$

Figure 14.20 shows the tree diagram for the first of these equations. The diagram for the second equation is similar; just replace r with s .

EXAMPLE 4 More Partial Derivatives

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \quad x = r - s, \quad y = r + s.$$

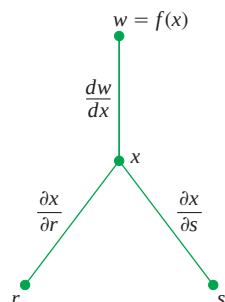
Solution

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial r} & \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} \\ &= (2x)(1) + (2y)(1) & &= (2x)(-1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) & &= -2(r - s) + 2(r + s) \\ &= 4r & &= 4s \end{aligned}$$

Substitute
for the
intermediate
variables.

If f is a function of x alone, our equations become even simpler.

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{dw}{dx} \frac{dx}{dr}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx} \frac{dx}{ds}$$

FIGURE 14.21 Tree diagram for differentiating f as a composite function of r and s with one intermediate variable.

If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx} \frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx} \frac{\partial x}{\partial s}.$$

In this case, we can use the ordinary (single-variable) derivative, dw/dx . The tree diagram is shown in Figure 14.21.

Implicit Differentiation Revisited

The two-variable Chain Rule in Theorem 5 leads to a formula that takes most of the work out of implicit differentiation. Suppose that

1. The function $F(x, y)$ is differentiable and
2. The equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x , say $y = h(x)$.

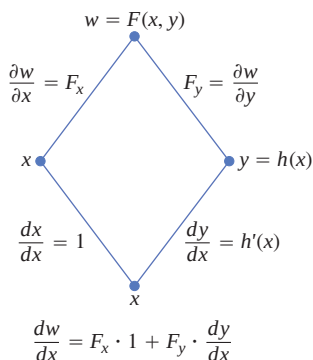


FIGURE 14.22 Tree diagram for differentiating $w = F(x, y)$ with respect to x . Setting $dw/dx = 0$ leads to a simple computational formula for implicit differentiation (Theorem 8).

Since $w = F(x, y) = 0$, the derivative dw/dx must be zero. Computing the derivative from the Chain Rule (tree diagram in Figure 14.22), we find

$$\begin{aligned} 0 &= \frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx} && \text{Theorem 5 with } t = x \text{ and } f = F \\ &= F_x \cdot 1 + F_y \cdot \frac{dy}{dx}. \end{aligned}$$

If $F_y = \partial w / \partial y \neq 0$, we can solve this equation for dy/dx to get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

This relationship gives a surprisingly simple shortcut to finding derivatives of implicitly defined functions, which we state here as a theorem.

THEOREM 8 A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x . Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$



EXAMPLE 5 Implicit Differentiation

Use Theorem 8 to find dy/dx if $y^2 - x^2 - \sin xy = 0$.

Solution Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\begin{aligned} \frac{dy}{dx} &= -\frac{F_x}{F_y} = -\frac{-2x - y \cos xy}{2y - x \cos xy} \\ &= \frac{2x + y \cos xy}{2y - x \cos xy}. \end{aligned}$$

This calculation is significantly shorter than the single-variable calculation with which we found dy/dx in Section 3.6, Example 3. ■

Functions of Many Variables

We have seen several different forms of the Chain Rule in this section, but you do not have to memorize them all if you can see them as special cases of the same general formula. When solving particular problems, it may help to draw the appropriate tree diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom. To find the derivative of the dependent variable with respect to the selected independent variable, start at the dependent variable and read down each route of the tree to the independent variable, calculating and multiplying the derivatives along each route. Then add the products you found for the different routes.

In general, suppose that $w = f(x, y, \dots, v)$ is a differentiable function of the variables $x, y, \dots, v$ (a finite set) and the $x, y, \dots, v$ are differentiable functions of $p, q, \dots, t$ (another finite set). Then w is a differentiable function of the variables p through t and the partial derivatives of w with respect to these variables are given by equations of the form

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial v} \frac{\partial v}{\partial p}.$$

The other equations are obtained by replacing p by $q, \dots, t$, one at a time.

One way to remember this equation is to think of the right-hand side as the dot product of two vectors with components

$$\underbrace{\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \dots, \frac{\partial w}{\partial v} \right)}_{\text{Derivatives of } w \text{ with respect to the intermediate variables}} \quad \text{and} \quad \underbrace{\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \dots, \frac{\partial v}{\partial p} \right)}_{\text{Derivatives of the intermediate variables with respect to the selected independent variable}}.$$

EXERCISES 14.4

Chain Rule: One Independent Variable

In Exercises 1–6, **(a)** express dw/dt as a function of t , both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t . Then **(b)** evaluate dw/dt at the given value of t .

1. $w = x^2 + y^2$, $x = \cos t$, $y = \sin t$; $t = \pi$
2. $w = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$; $t = 0$
3. $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = 1/t$; $t = 3$
4. $w = \ln(x^2 + y^2 + z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$; $t = 3$
5. $w = 2ye^x - \ln z$, $x = \ln(t^2 + 1)$, $y = \tan^{-1} t$, $z = e^t$; $t = 1$
6. $w = z - \sin xy$, $x = t$, $y = \ln t$, $z = e^{t-1}$; $t = 1$

Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v) .

7. $z = 4e^x \ln y$, $x = \ln(u \cos v)$, $y = u \sin v$;
 $(u, v) = (2, \pi/4)$
8. $z = \tan^{-1}(x/y)$, $x = u \cos v$, $y = u \sin v$;
 $(u, v) = (1.3, \pi/6)$

In Exercises 9 and 10, **(a)** express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of u and v both by using the Chain Rule and by expressing w directly in

terms of u and v before differentiating. Then **(b)** evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point (u, v) .

9. $w = xy + yz + xz$, $x = u + v$, $y = u - v$, $z = uv$;
 $(u, v) = (1/2, 1)$
10. $w = \ln(x^2 + y^2 + z^2)$, $x = ue^v \sin u$, $y = ue^v \cos u$,
 $z = ue^v$; $(u, v) = (-2, 0)$

In Exercises 11 and 12, **(a)** express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of x , y , and z both by using the Chain Rule and by expressing u directly in terms of x , y , and z before differentiating. Then **(b)** evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point (x, y, z) .

11. $u = \frac{p-q}{q-r}$, $p = x + y + z$, $q = x - y + z$,
 $r = x + y - z$; $(x, y, z) = (\sqrt{3}, 2, 1)$
12. $u = e^{qr} \sin^{-1} p$, $p = \sin x$, $q = z^2 \ln y$, $r = 1/z$;
 $(x, y, z) = (\pi/4, 1/2, -1/2)$

Using a Tree Diagram

In Exercises 13–24, draw a tree diagram and write a Chain Rule formula for each derivative.

13. $\frac{dz}{dt}$ for $z = f(x, y)$, $x = g(t)$, $y = h(t)$
14. $\frac{dz}{dt}$ for $z = f(u, v, w)$, $u = g(t)$, $v = h(t)$, $w = k(t)$
15. $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = h(x, y, z)$, $x = f(u, v)$, $y = g(u, v)$,
 $z = k(u, v)$





16. $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = f(r, s, t)$, $r = g(x, y)$, $s = h(x, y)$, $t = k(x, y)$
17. $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = g(x, y)$, $x = h(u, v)$, $y = k(u, v)$
18. $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = g(u, v)$, $u = h(x, y)$, $v = k(x, y)$
19. $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$ for $z = f(x, y)$, $x = g(t, s)$, $y = h(t, s)$
20. $\frac{\partial y}{\partial r}$ for $y = f(u)$, $u = g(r, s)$
21. $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ for $w = g(u)$, $u = h(s, t)$
22. $\frac{\partial w}{\partial p}$ for $w = f(x, y, z, v)$, $x = g(p, q)$, $y = h(p, q)$, $z = j(p, q)$, $v = k(p, q)$
23. $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ for $w = f(x, y)$, $x = g(r)$, $y = h(s)$
24. $\frac{\partial w}{\partial s}$ for $w = g(x, y)$, $x = h(r, s, t)$, $y = k(r, s, t)$

Implicit Differentiation

Assuming that the equations in Exercises 25–28 define y as a differentiable function of x , use Theorem 8 to find the value of dy/dx at the given point.



25. $x^3 - 2y^2 + xy = 0$, $(1, 1)$
26. $xy + y^2 - 3x - 3 = 0$, $(-1, 1)$
27. $x^2 + xy + y^2 - 7 = 0$, $(1, 2)$
28. $xe^y + \sin xy + y - \ln 2 = 0$, $(0, \ln 2)$

Three-Variable Implicit Differentiation

Theorem 8 can be generalized to functions of three variables and even more. The three-variable version goes like this: If the equation $F(x, y, z) = 0$ determines z as a differentiable function of x and y , then, at points where $F_z \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Use these equations to find the values of $\partial z/\partial x$ and $\partial z/\partial y$ at the points in Exercises 29–32.



29. $z^3 - xy + yz + y^3 - 2 = 0$, $(1, 1, 1)$
30. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$, $(2, 3, 6)$
31. $\sin(x + y) + \sin(y + z) + \sin(x + z) = 0$, (π, π, π)
32. $xe^y + ye^z + 2 \ln x - 2 - 3 \ln 2 = 0$, $(1, \ln 2, \ln 3)$

Finding Specified Partial Derivatives

33. Find $\partial w/\partial r$ when $r = 1, s = -1$ if $w = (x + y + z)^2$, $x = r - s, y = \cos(r + s), z = \sin(r + s)$.
34. Find $\partial w/\partial v$ when $u = -1, v = 2$ if $w = xy + \ln z$, $x = v^2/u, y = u + v, z = \cos u$.
35. Find $\partial w/\partial v$ when $u = 0, v = 0$ if $w = x^2 + (y/x)$, $x = u - 2v + 1, y = 2u + v - 2$.
36. Find $\partial z/\partial u$ when $u = 0, v = 1$ if $z = \sin xy + x \sin y$, $x = u^2 + v^2, y = uv$.
37. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = \ln 2, v = 1$ if $z = 5 \tan^{-1} x$ and $x = e^u + \ln v$.
38. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = 1$ and $v = -2$ if $z = \ln q$ and $q = \sqrt{v + 3} \tan^{-1} u$.

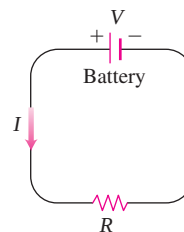


Theory and Examples

39. **Changing voltage in a circuit** The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I} \frac{dI}{dt} + \frac{\partial V}{\partial R} \frac{dR}{dt}$$

to find how the current is changing at the instant when $R = 600$ ohms, $I = 0.04$ amp, $dR/dt = 0.5$ ohm/sec, and $dV/dt = -0.01$ volt/sec.



40. **Changing dimensions in a box** The lengths a, b , and c of the edges of a rectangular box are changing with time. At the instant in question, $a = 1$ m, $b = 2$ m, $c = 3$ m, $da/dt = db/dt = 1$ m/sec, and $dc/dt = -3$ m/sec. At what rates are the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length or decreasing?
41. If $f(u, v, w)$ is differentiable and $u = x - y, v = y - z$, and $w = z - x$, show that
- $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$
42. **Polar coordinates** Suppose that we substitute polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$ in a differentiable function $w = f(x, y)$.

a. Show that

$$\frac{\partial w}{\partial r} = f_x \cos \theta + f_y \sin \theta$$

and

$$\frac{1}{r} \frac{\partial w}{\partial \theta} = -f_x \sin \theta + f_y \cos \theta.$$

b. Solve the equations in part (a) to express f_x and f_y in terms of $\partial w / \partial r$ and $\partial w / \partial \theta$.

c. Show that

$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial w}{\partial \theta} \right)^2.$$

43. Laplace equations Show that if $w = f(u, v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

44. Laplace equations Let $w = f(u) + g(v)$, where $u = x + iy$ and $v = x - iy$ and $i = \sqrt{-1}$. Show that w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$ if all the necessary functions are differentiable.

Changes in Functions Along Curves

45. Extreme values on a helix Suppose that the partial derivatives of a function $f(x, y, z)$ at points on the helix $x = \cos t$, $y = \sin t$, $z = t$ are

$$f_x = \cos t, \quad f_y = \sin t, \quad f_z = t^2 + t - 2.$$

At what points on the curve, if any, can f take on extreme values?

46. A space curve Let $w = x^2 e^{2y} \cos 3z$. Find the value of dw/dt at the point $(1, \ln 2, 0)$ on the curve $x = \cos t$, $y = \ln(t + 2)$, $z = t$.

47. Temperature on a circle Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$ and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \quad \frac{\partial T}{\partial y} = 8y - 4x.$$

a. Find where the maximum and minimum temperatures on the circle occur by examining the derivatives dT/dt and d^2T/dt^2 .

b. Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of T on the circle.

48. Temperature on an ellipse Let $T = g(x, y)$ be the temperature at the point (x, y) on the ellipse

$$x = 2\sqrt{2} \cos t, \quad y = \sqrt{2} \sin t, \quad 0 \leq t \leq 2\pi,$$

and suppose that

$$\frac{\partial T}{\partial x} = y, \quad \frac{\partial T}{\partial y} = x.$$

a. Locate the maximum and minimum temperatures on the ellipse by examining dT/dt and d^2T/dt^2 .

b. Suppose that $T = xy - 2$. Find the maximum and minimum values of T on the ellipse.

Differentiating Integrals

Under mild continuity restrictions, it is true that if

$$F(x) = \int_a^b g(t, x) dt,$$

then $F'(x) = \int_a^b g_x(t, x) dt$. Using this fact and the Chain Rule, we can find the derivative of

$$F(x) = \int_a^{f(x)} g(t, x) dt$$

by letting

$$G(u, x) = \int_a^u g(t, x) dt,$$

where $u = f(x)$. Find the derivatives of the functions in Exercises 49 and 50.

49. $F(x) = \int_0^{x^2} \sqrt{t^4 + x^3} dt$

50. $F(x) = \int_{x^2}^1 \sqrt{t^3 + x^2} dt$

14.5

Directional Derivatives and Gradient Vectors



If you look at the map (Figure 14.23) showing contours on the West Point Area along the Hudson River in New York, you will notice that the tributary streams flow perpendicular to the contours. The streams are following paths of steepest descent so the waters reach the Hudson as quickly as possible. Therefore, the instantaneous rate of change in a stream's

altitude above sea level has a particular direction. In this section, you see why this direction, called the “downhill” direction, is perpendicular to the contours.

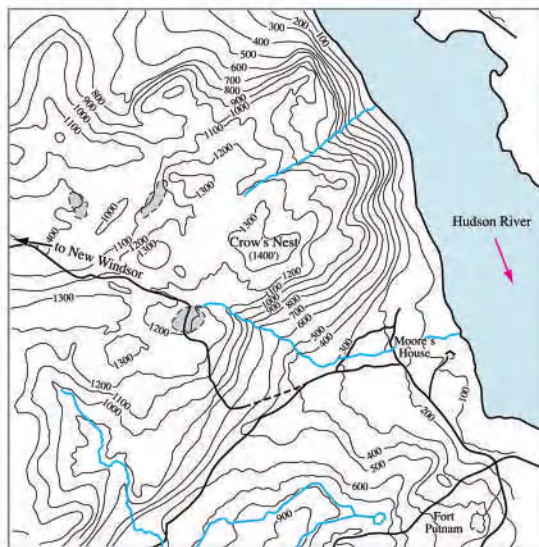


FIGURE 14.23 Contours of the West Point Area in New York show streams, which follow paths of steepest descent, running perpendicular to the contours.

Directional Derivatives in the Plane

We know from Section 14.4 that if $f(x, y)$ is differentiable, then the rate at which f changes with respect to t along a differentiable curve $x = g(t)$, $y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

At any point $P_0(x_0, y_0) = P_0(g(t_0), h(t_0))$, this equation gives the rate of change of f with respect to increasing t and therefore depends, among other things, on the direction of motion along the curve. If the curve is a straight line and t is the arc length parameter along the line measured from P_0 in the direction of a given unit vector $\mathbf{u}$, then df/dt is the rate of change of f with respect to distance in its domain in the direction of $\mathbf{u}$. By varying $\mathbf{u}$, we find the rates at which f changes with respect to distance as we move through P_0 in different directions. We now define this idea more precisely.

Suppose that the function $f(x, y)$ is defined throughout a region R in the xy -plane, that $P_0(x_0, y_0)$ is a point in R , and that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector. Then the equations

$$x = x_0 + su_1, \quad y = y_0 + su_2$$

parametrize the line through P_0 parallel to $\mathbf{u}$. If the parameter s measures arc length from P_0 in the direction of $\mathbf{u}$, we find the rate of change of f at P_0 in the direction of $\mathbf{u}$ by calculating df/ds at P_0 (Figure 14.24).

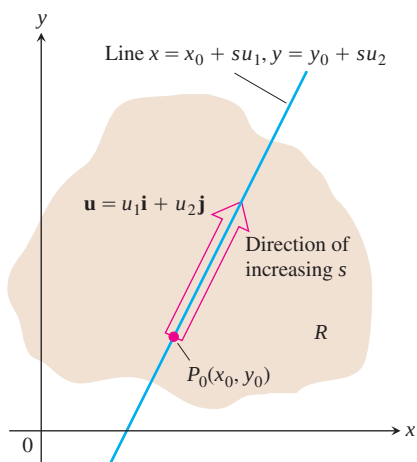


FIGURE 14.24 The rate of change of f in the direction of $\mathbf{u}$ at a point P_0 is the rate at which f changes along this line at P_0 .

DEFINITION Directional Derivative

The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \quad (1)$$

provided the limit exists.

The directional derivative is also denoted by

$$(D_{\mathbf{u}}f)_{P_0}.$$

“The derivative of f at P_0
in the direction of $\mathbf{u}$ ”

EXAMPLE 1 Finding a Directional Derivative Using the Definition

Find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$.

Solution

$$\begin{aligned} \left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} &= \lim_{s \rightarrow 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} && \text{Equation (1)} \\ &= \lim_{s \rightarrow 0} \frac{f\left(1 + s \cdot \frac{1}{\sqrt{2}}, 2 + s \cdot \frac{1}{\sqrt{2}}\right) - f(1, 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1 \cdot 2)}{s} \\ &= \lim_{s \rightarrow 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s} \\ &= \lim_{s \rightarrow 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s \rightarrow 0} \left(\frac{5}{\sqrt{2}} + s\right) = \left(\frac{5}{\sqrt{2}} + 0\right) = \frac{5}{\sqrt{2}}. \end{aligned}$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction $\mathbf{u} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$ is $5/\sqrt{2}$. ■

Interpretation of the Directional Derivative

The equation $z = f(x, y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the point $P(x_0, y_0, z_0)$ lies on S . The vertical plane that passes through P and $P_0(x_0, y_0)$ parallel to $\mathbf{u}$

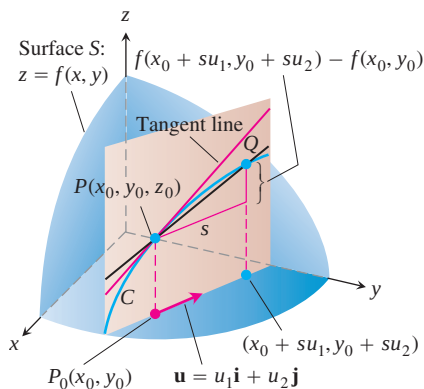


FIGURE 14.25 The slope of curve C at P_0 is $\lim_{Q \rightarrow P} \text{slope}(PQ)$; this is the directional derivative

$$\left(\frac{df}{ds} \right)_{\mathbf{u}, P_0} = (D_{\mathbf{u}}f)_{P_0}.$$

intersects S in a curve C (Figure 14.25). The rate of change of f in the direction of $\mathbf{u}$ is the slope of the tangent to C at P .

When $\mathbf{u} = \mathbf{i}$, the directional derivative at P_0 is $\partial f / \partial x$ evaluated at (x_0, y_0) . When $\mathbf{u} = \mathbf{j}$, the directional derivative at P_0 is $\partial f / \partial y$ evaluated at (x_0, y_0) . The directional derivative generalizes the two partial derivatives. We can now ask for the rate of change of f in any direction $\mathbf{u}$, not just the directions $\mathbf{i}$ and $\mathbf{j}$.

Here's a physical interpretation of the directional derivative. Suppose that $T = f(x, y)$ is the temperature at each point (x, y) over a region in the plane. Then $f(x_0, y_0)$ is the temperature at the point $P_0(x_0, y_0)$ and $(D_{\mathbf{u}}f)_{P_0}$ is the instantaneous rate of change of the temperature at P_0 stepping off in the direction $\mathbf{u}$.

Calculation and Gradients

We now develop an efficient formula to calculate the directional derivative for a differentiable function f . We begin with the line

$$x = x_0 + su_1, \quad y = y_0 + su_2, \quad (2)$$

through $P_0(x_0, y_0)$, parametrized with the arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$. Then

$$\begin{aligned} \left(\frac{df}{ds} \right)_{\mathbf{u}, P_0} &= \left(\frac{\partial f}{\partial x} \right)_{P_0} \frac{dx}{ds} + \left(\frac{\partial f}{\partial y} \right)_{P_0} \frac{dy}{ds} && \text{Chain Rule for differentiable } f \\ &= \left(\frac{\partial f}{\partial x} \right)_{P_0} \cdot u_1 + \left(\frac{\partial f}{\partial y} \right)_{P_0} \cdot u_2 && \text{From Equations (2), } dx/ds = u_1 \text{ and } dy/ds = u_2 \\ &= \underbrace{\left[\left(\frac{\partial f}{\partial x} \right)_{P_0} \mathbf{i} + \left(\frac{\partial f}{\partial y} \right)_{P_0} \mathbf{j} \right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1 \mathbf{i} + u_2 \mathbf{j} \right]}_{\text{Direction } \mathbf{u}}. && (3) \end{aligned}$$



DEFINITION Gradient Vector

The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0 .

The notation ∇f is read “grad f ” as well as “gradient of f ” and “del f .” The symbol ∇ by itself is read “del.” Another notation for the gradient is $\text{grad } f$, read the way it is written.

Equation (3) says that the derivative of a differentiable function f in the direction of $\mathbf{u}$ at P_0 is the dot product of $\mathbf{u}$ with the gradient of f at P_0 .

THEOREM 9 The Directional Derivative Is a Dot Product

If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \quad (4)$$

the dot product of the gradient f at P_0 and $\mathbf{u}$.

**EXAMPLE 2** Finding the Directional Derivative Using the Gradient

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution The direction of $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y \sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x \sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.26). The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$\begin{aligned} (D_{\mathbf{u}}f)|_{(2,0)} &= \nabla f|_{(2,0)} \cdot \mathbf{u} \quad \text{Equation (4)} \\ &= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1. \end{aligned}$$

Evaluating the dot product in the formula

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f , reveals the following properties.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\mathbf{u}$ is the direction of ∇f . That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P . The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f|\cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f|\cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f|\cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

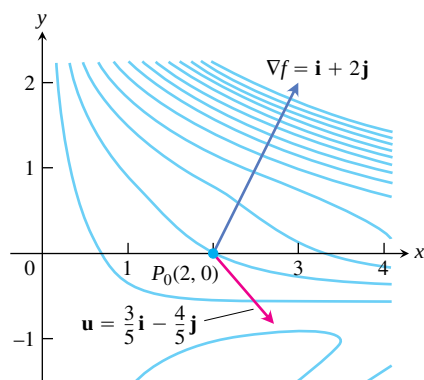


FIGURE 14.26 Picture ∇f as a vector in the domain of f . In the case of $f(x, y) = xe^y + \cos(xy)$, the domain is the entire plane. The rate at which f changes at $(2, 0)$ in the direction $\mathbf{u} = (3/5)\mathbf{i} - (4/5)\mathbf{j}$ is $\nabla f \cdot \mathbf{u} = -1$ (Example 2).

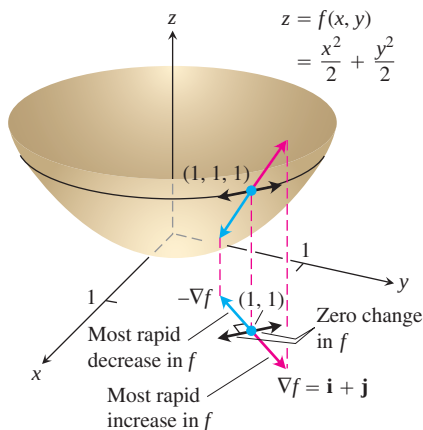


FIGURE 14.27 The direction in which $f(x, y) = (x^2/2) + (y^2/2)$ increases most rapidly at $(1, 1)$ is the direction of $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at $(1, 1, 1)$ (Example 3).

As we discuss later, these properties hold in three dimensions as well as two.

EXAMPLE 3 Finding Directions of Maximal, Minimal, and Zero Change

Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

- (a) Increases most rapidly at the point $(1, 1)$
- (b) Decreases most rapidly at $(1, 1)$.
- (c) What are the directions of zero change in f at $(1, 1)$?

Solution

- (a) The function increases most rapidly in the direction of ∇f at $(1, 1)$. The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

- (b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

- (c) The directions of zero change at $(1, 1)$ are the directions orthogonal to ∇f :

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

See Figure 14.27.

Gradients and Tangents to Level Curves

If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ (making the curve a level curve of f), then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\begin{aligned} \frac{d}{dt} f(g(t), h(t)) &= \frac{d}{dt}(c) \\ \frac{\partial f}{\partial x} \frac{dg}{dt} + \frac{\partial f}{\partial y} \frac{dh}{dt} &= 0 && \text{Chain Rule} \\ \underbrace{\left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt} \mathbf{i} + \frac{dh}{dt} \mathbf{j} \right)}_{\frac{d\mathbf{r}}{dt}} &= 0. \end{aligned} \tag{5}$$

Equation (5) says that ∇f is normal to the tangent vector $d\mathbf{r}/dt$, so it is normal to the curve.

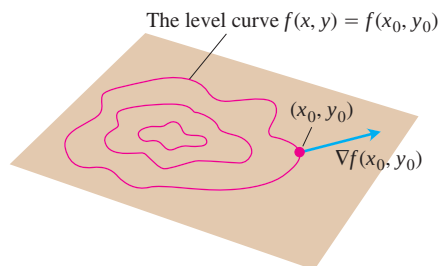


FIGURE 14.28 The gradient of a differentiable function of two variables at a point is always normal to the function's level curve through that point.

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure 14.28).

Equation (5) validates our observation that streams flow perpendicular to the contours in topographical maps (see Figure 14.23). Since the downflowing stream will reach its destination in the fastest way, it must flow in the direction of the negative gradient vectors from Property 2 for the directional derivative. Equation (5) tells us these directions are perpendicular to the level curves.

This observation also enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients. The line through a point $P_0(x_0, y_0)$ normal to a vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

$$A(x - x_0) + B(y - y_0) = 0$$

(Exercise 35). If $\mathbf{N}$ is the gradient $(\nabla f)_{(x_0, y_0)} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$, the equation is the tangent line given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0. \quad (6)$$

EXAMPLE 4 Finding the Tangent Line to an Ellipse

Find an equation for the tangent to the ellipse

$$\frac{x^2}{4} + y^2 = 2$$

(Figure 14.29) at the point $(-2, 1)$.

Solution The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2, 1)$ is

$$\nabla f|_{(-2, 1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j} \right)_{(-2, 1)} = -\mathbf{i} + 2\mathbf{j}.$$

The tangent is the line

$$\begin{aligned} (-1)(x + 2) + (2)(y - 1) &= 0 && \text{Equation (6)} \\ x - 2y &= -4. \end{aligned}$$

If we know the gradients of two functions f and g , we automatically know the gradients of their constant multiples, sum, difference, product, and quotient. You are asked to establish the following rules in Exercise 36. Notice that these rules have the same form as the corresponding rules for derivatives of single-variable functions.

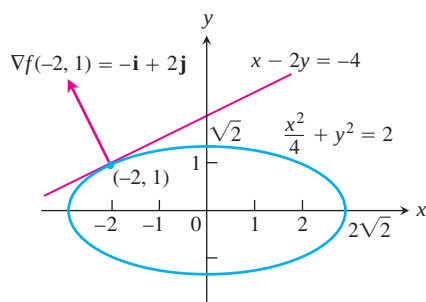


FIGURE 14.29 We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$ (Example 4).

Algebra Rules for Gradients

1. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
2. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
3. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

EXAMPLE 5 Illustrating the Gradient Rules

We illustrate the rules with

$$\begin{aligned} f(x, y) &= x - y & g(x, y) &= 3y \\ \nabla f &= \mathbf{i} - \mathbf{j} & \nabla g &= 3\mathbf{j}. \end{aligned}$$

We have

1. $\nabla(2f) = \nabla(2x - 2y) = 2\mathbf{i} - 2\mathbf{j} = 2\nabla f$
2. $\nabla(f + g) = \nabla(x + 2y) = \mathbf{i} + 2\mathbf{j} = \nabla f + \nabla g$
3. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$
4. $\begin{aligned} \nabla(fg) &= \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j} \\ &= 3y(\mathbf{i} - \mathbf{j}) + 3y\mathbf{j} + (3x - 6y)\mathbf{j} \\ &= 3y(\mathbf{i} - \mathbf{j}) + (3x - 3y)\mathbf{j} \\ &= 3y(\mathbf{i} - \mathbf{j}) + (x - y)3\mathbf{j} = g\nabla f + f\nabla g \end{aligned}$
5. $\begin{aligned} \nabla\left(\frac{f}{g}\right) &= \nabla\left(\frac{x - y}{3y}\right) = \nabla\left(\frac{x}{3y} - \frac{1}{3}\right) \\ &= \frac{1}{3y}\mathbf{i} - \frac{x}{3y^2}\mathbf{j} \\ &= \frac{3y\mathbf{i} - 3x\mathbf{j}}{9y^2} = \frac{3y(\mathbf{i} - \mathbf{j}) - (3x - 3y)\mathbf{j}}{9y^2} \\ &= \frac{3y(\mathbf{i} - \mathbf{j}) - (x - y)3\mathbf{j}}{9y^2} = \frac{g\nabla f - f\nabla g}{g^2}. \end{aligned}$

Functions of Three Variables

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta = |\nabla f| \cos \theta,$$

so the properties listed earlier for functions of two variables continue to hold. At any given point, f increases most rapidly in the direction of ∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f , the derivative is zero.



EXAMPLE 6 Finding Directions of Maximal, Minimal, and Zero Change

- (a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.
- (b) In what directions does f change most rapidly at P_0 , and what are the rates of change in these directions?

Solution

- (a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \quad f_y = -2xy|_{(1,1,0)} = -2, \quad f_z = -1|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$\begin{aligned} (D_{\mathbf{u}}f)_{(1,1,0)} &= \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k} \right) \\ &= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}. \end{aligned}$$

- (b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \quad \text{and} \quad -|\nabla f| = -3. \quad \blacksquare$$

EXERCISES 14.5

Calculating Gradients at Points

In Exercises 1–4, find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $f(x, y) = y - x$, $(2, 1)$ 2. $f(x, y) = \ln(x^2 + y^2)$, $(1, 1)$

3. $g(x, y) = y - x^2$, $(-1, 0)$ 4. $g(x, y) = \frac{x^2}{2} - \frac{y^2}{2}$, $(\sqrt{2}, 1)$

In Exercises 5–8, find ∇f at the given point.

5. $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x$, $(1, 1, 1)$
6. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \tan^{-1}xz$, $(1, 1, 1)$





7. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz)$, $(-1, 2, -2)$
 8. $f(x, y, z) = e^{x+y} \cos z + (y + 1) \sin^{-1} x$, $(0, 0, \pi/6)$

Finding Directional Derivatives

In Exercises 9–16, find the derivative of the function at P_0 in the direction of $\mathbf{A}$.



9. $f(x, y) = 2xy - 3y^2$, $P_0(5, 5)$, $\mathbf{A} = 4\mathbf{i} + 3\mathbf{j}$
 10. $f(x, y) = 2x^2 + y^2$, $P_0(-1, 1)$, $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$
 11. $g(x, y) = x - (y^2/x) + \sqrt{3} \sec^{-1}(2xy)$, $P_0(1, 1)$, $\mathbf{A} = 12\mathbf{i} + 5\mathbf{j}$
 12. $h(x, y) = \tan^{-1}(y/x) + \sqrt{3} \sin^{-1}(xy/2)$, $P_0(1, 1)$, $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j}$
 13. $f(x, y, z) = xy + yz + zx$, $P_0(1, -1, 2)$, $\mathbf{A} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$
 14. $f(x, y, z) = x^2 + 2y^2 - 3z^2$, $P_0(1, 1, 1)$, $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
 15. $g(x, y, z) = 3e^x \cos yz$, $P_0(0, 0, 0)$, $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
 16. $h(x, y, z) = \cos xy + e^{yz} + \ln xz$, $P_0(1, 0, 1/2)$, $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

Directions of Most Rapid Increase and Decrease

In Exercises 17–22, find the directions in which the functions increase and decrease most rapidly at P_0 . Then find the derivatives of the functions in these directions.



17. $f(x, y) = x^2 + xy + y^2$, $P_0(-1, 1)$
 18. $f(x, y) = x^2y + e^{xy} \sin y$, $P_0(1, 0)$
 19. $f(x, y, z) = (x/y) - yz$, $P_0(4, 1, 1)$
 20. $g(x, y, z) = xe^{yz} + z^2$, $P_0(1, \ln 2, 1/2)$
 21. $f(x, y, z) = \ln xy + \ln yz + \ln xz$, $P_0(1, 1, 1)$
 22. $h(x, y, z) = \ln(x^2 + y^2 - 1) + y + 6z$, $P_0(1, 1, 0)$

Tangent Lines to Curves

In Exercises 23–26, sketch the curve $f(x, y) = c$ together with ∇f and the tangent line at the given point. Then write an equation for the tangent line.



23. $x^2 + y^2 = 4$, $(\sqrt{2}, \sqrt{2})$ 24. $x^2 - y = 1$, $(\sqrt{2}, 1)$
 25. $xy = -4$, $(2, -2)$ 26. $x^2 - xy + y^2 = 7$, $(-1, 2)$

Theory and Examples

27. **Zero directional derivative** In what direction is the derivative of $f(x, y) = xy + y^2$ at $P(3, 2)$ equal to zero?
 28. **Zero directional derivative** In what directions is the derivative of $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at $P(1, 1)$ equal to zero?
 29. Is there a direction $\mathbf{u}$ in which the rate of change of $f(x, y) = x^2 - 3xy + 4y^2$ at $P(1, 2)$ equals 14? Give reasons for your answer.

30. **Changing temperature along a circle** Is there a direction $\mathbf{u}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ (temperature in degrees Celsius, distance in feet) at $P(1, -1, 1)$ is $-3^\circ\text{C}/\text{ft}$? Give reasons for your answer.
 31. The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of $\mathbf{i} + \mathbf{j}$ is $2\sqrt{2}$ and in the direction of $-2\mathbf{j}$ is -3 . What is the derivative of f in the direction of $-\mathbf{i} - 2\mathbf{j}$? Give reasons for your answer.
 32. The derivative of $f(x, y, z)$ at a point P is greatest in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. In this direction, the value of the derivative is $2\sqrt{3}$.
 a. What is ∇f at P ? Give reasons for your answer.
 b. What is the derivative of f at P in the direction of $\mathbf{i} + \mathbf{j}$?
 33. **Directional derivatives and scalar components** How is the derivative of a differentiable function $f(x, y, z)$ at a point P_0 in the direction of a unit vector $\mathbf{u}$ related to the scalar component of $(\nabla f)_{P_0}$ in the direction of $\mathbf{u}$? Give reasons for your answer.
 34. **Directional derivatives and partial derivatives** Assuming that the necessary derivatives of $f(x, y, z)$ are defined, how are $D_{\mathbf{i}}f$, $D_{\mathbf{j}}f$, and $D_{\mathbf{k}}f$ related to f_x , f_y , and f_z ? Give reasons for your answer.
 35. **Lines in the xy -plane** Show that $A(x - x_0) + B(y - y_0) = 0$ is an equation for the line in the xy -plane through the point (x_0, y_0) normal to the vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$.
 36. **The algebra rules for gradients** Given a constant k and the gradients

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

and

$$\nabla g = \frac{\partial g}{\partial x} \mathbf{i} + \frac{\partial g}{\partial y} \mathbf{j} + \frac{\partial g}{\partial z} \mathbf{k},$$

use the scalar equations

$$\frac{\partial}{\partial x}(kf) = k \frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x}(f \pm g) = \frac{\partial f}{\partial x} \pm \frac{\partial g}{\partial x},$$

$$\frac{\partial}{\partial x}(fg) = f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x}, \quad \frac{\partial}{\partial x} \left(\frac{f}{g} \right) = \frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2},$$

and so on, to establish the following rules.

- a. $\nabla(kf) = k\nabla f$
 b. $\nabla(f + g) = \nabla f + \nabla g$
 c. $\nabla(f - g) = \nabla f - \nabla g$
 d. $\nabla(fg) = f\nabla g + g\nabla f$
 e. $\nabla \left(\frac{f}{g} \right) = \frac{g\nabla f - f\nabla g}{g^2}$

14.6

Tangent Planes and Differentials

In this section we define the tangent plane at a point on a smooth surface in space. We calculate an equation of the tangent plane from the partial derivatives of the function defining the surface. This idea is similar to the definition of the tangent line at a point on a curve in the coordinate plane for single-variable functions (Section 2.7). We then study the total differential and linearization of functions of several variables.

Tangent Planes and Normal Lines

If $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f , then $f(g(t), h(t), k(t)) = c$. Differentiating both sides of this equation with respect to t leads to

$$\frac{d}{dt} f(g(t), h(t), k(t)) = \frac{d}{dt} (c)$$

$$\frac{\partial f}{\partial x} \frac{dg}{dt} + \frac{\partial f}{\partial y} \frac{dh}{dt} + \frac{\partial f}{\partial z} \frac{dk}{dt} = 0 \quad \text{Chain Rule}$$

$$\underbrace{\left(\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt} \mathbf{i} + \frac{dh}{dt} \mathbf{j} + \frac{dk}{dt} \mathbf{k} \right)}_{d\mathbf{r}/dt} = 0. \quad (1)$$

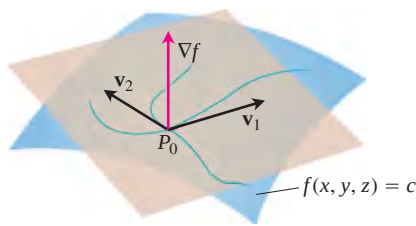


FIGURE 14.30 The gradient ∇f is orthogonal to the velocity vector of every smooth curve in the surface through P_0 . The velocity vectors at P_0 therefore lie in a common plane, which we call the tangent plane at P_0 .

At every point along the curve, ∇f is orthogonal to the curve's velocity vector.

Now let us restrict our attention to the curves that pass through P_0 (Figure 14.30). All the velocity vectors at P_0 are orthogonal to ∇f at P_0 , so the curves' tangent lines all lie in the plane through P_0 normal to ∇f . We call this plane the tangent plane of the surface at P_0 . The line through P_0 perpendicular to the plane is the surface's normal line at P_0 .

DEFINITIONS Tangent Plane, Normal Line

The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

Thus, from Section 12.5, the tangent plane and normal line have the following equations:

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \quad (2)$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t \quad (3)$$

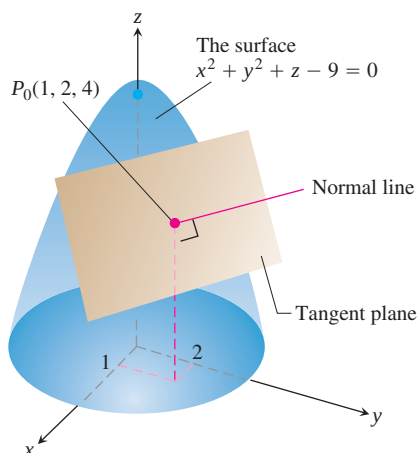


FIGURE 14.31 The tangent plane and normal line to the surface $x^2 + y^2 + z - 9 = 0$ at $P_0(1, 2, 4)$ (Example 1).

EXAMPLE 1 Finding the Tangent Plane and Normal Line

Find the tangent plane and normal line of the surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \quad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.

Solution The surface is shown in Figure 14.31.

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0 . The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \quad \text{or} \quad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t. \quad \blacksquare$$

To find an equation for the plane tangent to a smooth surface $z = f(x, y)$ at a point $P_0(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$, we first observe that the equation $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$. The surface $z = f(x, y)$ is therefore the zero level surface of the function $F(x, y, z) = f(x, y) - z$. The partial derivatives of F are

$$F_x = \frac{\partial}{\partial x}(f(x, y) - z) = f_x - 0 = f_x$$

$$F_y = \frac{\partial}{\partial y}(f(x, y) - z) = f_y - 0 = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x, y) - z) = 0 - 1 = -1.$$

The formula

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

for the plane tangent to the level surface at P_0 therefore reduces to

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0. \quad (4)$$



EXAMPLE 2 Finding a Plane Tangent to a Surface $z = f(x, y)$

Find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.



Solution We calculate the partial derivatives of $f(x, y) = x \cos y - ye^x$ and use Equation (4):

$$f_x(0, 0) = (\cos y - ye^x)_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x \sin y - e^x)_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \quad \text{Equation (4)}$$

or

$$x - y - z = 0.$$



EXAMPLE 3 Tangent Line to the Curve of Intersection of Two Surfaces

The surfaces

$$f(x, y, z) = x^2 + y^2 - z = 0 \quad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \quad \text{A plane}$$

meet in an ellipse E (Figure 14.32). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution The tangent line is orthogonal to both ∇f and ∇g at P_0 , and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\nabla f|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t.$$

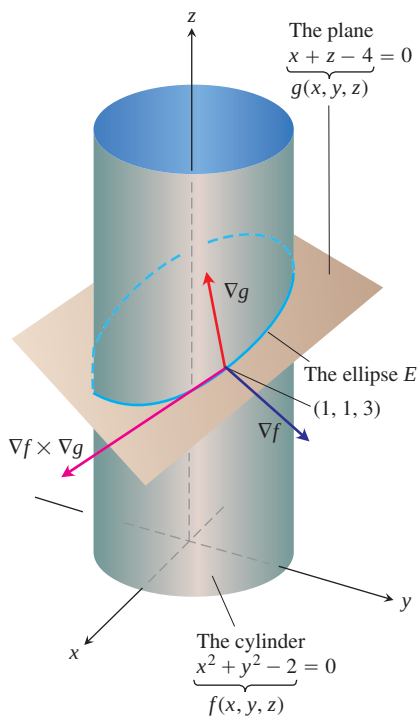


FIGURE 14.32 The cylinder $f(x, y, z) = x^2 + y^2 - z = 0$ and the plane $g(x, y, z) = x + z - 4 = 0$ intersect in an ellipse E (Example 3).

Estimating Change in a Specific Direction

The directional derivative plays the role of an ordinary derivative when we want to estimate how much the value of a function f changes if we move a small distance ds from a point P_0 to another point nearby. If f were a function of a single variable, we would have

$$df = f'(P_0) ds. \quad \text{Ordinary derivative} \times \text{increment}$$

For a function of two or more variables, we use the formula

$$df = (\nabla f|_{P_0} \cdot \mathbf{u}) ds, \quad \text{Directional derivative} \times \text{increment}$$

where $\mathbf{u}$ is the direction of the motion away from P_0 .

Estimating the Change in f in a Direction $\mathbf{u}$

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\text{Directional derivative}} \cdot \underbrace{ds}_{\text{Distance increment}}$$

**EXAMPLE 4** Estimating Change in the Value of $f(x, y, z)$

Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Solution We first find the derivative of f at P_0 in the direction of the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of f at P_0 is

$$\nabla f|_{(0,1,0)} = ((y \cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

Therefore,

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} \right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

The change df in f that results from moving $ds = 0.1$ unit away from P_0 in the direction of $\mathbf{u}$ is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3} \right)(0.1) \approx -0.067 \text{ unit.} \quad \blacksquare$$

How to Linearize a Function of Two Variables

Functions of two variables can be complicated, and we sometimes need to replace them with simpler ones that give the accuracy required for specific applications without being so difficult to work with. We do this in a way that is similar to the way we find linear replacements for functions of a single variable (Section 3.8).

Suppose the function we wish to replace is $z = f(x, y)$ and that we want the replacement to be effective near a point (x_0, y_0) at which we know the values of f , f_x , and f_y and at which f is differentiable. If we move from (x_0, y_0) to any point (x, y) by increments $\Delta x = x - x_0$ and $\Delta y = y - y_0$, then the definition of differentiability from Section 14.3 gives the change

$$f(x, y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

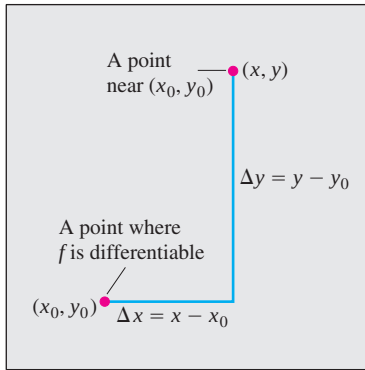


FIGURE 14.33 If f is differentiable at (x_0, y_0) , then the value of f at any point (x, y) nearby is approximately $f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$.

where $\epsilon_1, \epsilon_2 \rightarrow 0$ as $\Delta x, \Delta y \rightarrow 0$. If the increments Δx and Δy are small, the products $\epsilon_1\Delta x$ and $\epsilon_2\Delta y$ will eventually be smaller still and we will have

$$f(x, y) \approx \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x, y)}.$$

In other words, as long as Δx and Δy are small, f will have approximately the same value as the linear function L . If f is hard to use, and our work can tolerate the error involved, we may approximate f by L (Figure 14.33).

DEFINITIONS Linearization, Standard Linear Approximation

The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \quad (5)$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0) .

From Equation (4), we see that the plane $z = L(x, y)$ is tangent to the surface $z = f(x, y)$ at the point (x_0, y_0) . Thus, the linearization of a function of two variables is a *tangent-plane* approximation in the same way that the linearization of a function of a single variable is a *tangent-line* approximation.



EXAMPLE 5 Finding a Linearization

Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point $(3, 2)$.

Solution We first evaluate f , f_x , and f_y at the point $(x_0, y_0) = (3, 2)$:

$$f(3, 2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3 \right)_{(3,2)} = 8$$

$$f_x(3, 2) = \frac{\partial}{\partial x} \left(x^2 - xy + \frac{1}{2}y^2 + 3 \right)_{(3,2)} = (2x - y)_{(3,2)} = 4$$

$$f_y(3, 2) = \frac{\partial}{\partial y} \left(x^2 - xy + \frac{1}{2}y^2 + 3 \right)_{(3,2)} = (-x + y)_{(3,2)} = -1,$$

giving

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of f at $(3, 2)$ is $L(x, y) = 4x - y - 2$. ■

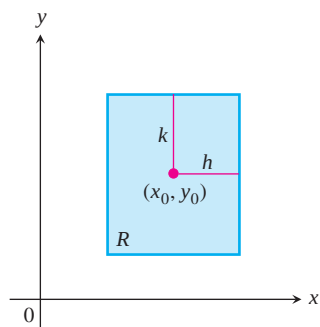


FIGURE 14.34 The rectangular region R : $|x - x_0| \leq h$, $|y - y_0| \leq k$ in the xy -plane.

When approximating a differentiable function $f(x, y)$ by its linearization $L(x, y)$ at (x_0, y_0) , an important question is how accurate the approximation might be.

If we can find a common upper bound M for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on a rectangle R centered at (x_0, y_0) (Figure 14.34), then we can bound the error E throughout R by using a simple formula (derived in Section 14.10). The **error** is defined by $E(x, y) = f(x, y) - L(x, y)$.

The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R , then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2} M(|x - x_0| + |y - y_0|)^2.$$

To make $|E(x, y)|$ small for a given M , we just make $|x - x_0|$ and $|y - y_0|$ small.



EXAMPLE 6 Bounding the Error in Example 5

Find an upper bound for the error in the approximation $f(x, y) \approx L(x, y)$ in Example 5 over the rectangle

$$R: |x - 3| \leq 0.1, \quad |y - 2| \leq 0.1.$$

Express the upper bound as a percentage of $f(3, 2)$, the value of f at the center of the rectangle.

Solution We use the inequality

$$|E(x, y)| \leq \frac{1}{2} M(|x - x_0| + |y - y_0|)^2.$$

To find a suitable value for M , we calculate f_{xx} , f_{xy} , and f_{yy} , finding, after a routine differentiation, that all three derivatives are constant, with values

$$|f_{xx}| = |2| = 2, \quad |f_{xy}| = |-1| = 1, \quad |f_{yy}| = |1| = 1.$$

The largest of these is 2, so we may safely take M to be 2. With $(x_0, y_0) = (3, 2)$, we then know that, throughout R ,

$$|E(x, y)| \leq \frac{1}{2} (2)(|x - 3| + |y - 2|)^2 = (|x - 3| + |y - 2|)^2.$$

Finally, since $|x - 3| \leq 0.1$ and $|y - 2| \leq 0.1$ on R , we have

$$|E(x, y)| \leq (0.1 + 0.1)^2 = 0.04.$$

As a percentage of $f(3, 2) = 8$, the error is no greater than

$$\frac{0.04}{8} \times 100 = 0.5\%.$$

Differentials

Recall from Section 3.8 that for a function of a single variable, $y = f(x)$, we defined the change in f as x changes from a to $a + \Delta x$ by

$$\Delta f = f(a + \Delta x) - f(a)$$

and the differential of f as

$$df = f'(a)\Delta x.$$

We now consider a function of two variables.

Suppose a differentiable function $f(x, y)$ and its partial derivatives exist at a point (x_0, y_0) . If we move to a nearby point $(x_0 + \Delta x, y_0 + \Delta y)$, the change in f is

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

A straightforward calculation from the definition of $L(x, y)$, using the notation $x - x_0 = \Delta x$ and $y - y_0 = \Delta y$, shows that the corresponding change in L is

$$\begin{aligned}\Delta L &= L(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0) \\ &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.\end{aligned}$$

The **differentials** dx and dy are independent variables, so they can be assigned any values. Often we take $dx = \Delta x = x - x_0$, and $dy = \Delta y = y - y_0$. We then have the following definition of the differential or *total* differential of f .

DEFINITION Total Differential

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy$$

in the linearization of f is called the **total differential of f** .

EXAMPLE 7 Estimating Change in Volume

Suppose that a cylindrical can is designed to have a radius of 1 in. and a height of 5 in., but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Solution To estimate the absolute change in $V = \pi r^2 h$, we use

$$\Delta V \approx dV = V_r(r_0, h_0) dr + V_h(r_0, h_0) dh.$$

With $V_r = 2\pi r h$ and $V_h = \pi r^2$, we get

$$\begin{aligned}dV &= 2\pi r_0 h_0 dr + \pi r_0^2 dh = 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1) \\ &= 0.3\pi - 0.1\pi = 0.2\pi \approx 0.63 \text{ in.}^3\end{aligned}$$

Instead of absolute change in the value of a function $f(x, y)$, we can estimate *relative change* or *percentage change* by

$$\frac{df}{f(x_0, y_0)} \quad \text{and} \quad \frac{df}{f(x_0, y_0)} \times 100,$$

respectively. In Example 7, the relative change is estimated by

$$\frac{dV}{V(r_0, h_0)} = \frac{0.2\pi}{\pi r_0^2 h_0} = \frac{0.2\pi}{\pi(1)^2(5)} = 0.04,$$

giving 4% as an estimate of the percentage change.

EXAMPLE 8 Sensitivity to Change

Your company manufactures right circular cylindrical molasses storage tanks that are 25 ft high with a radius of 5 ft. How sensitive are the tanks' volumes to small variations in height and radius?

Solution With $V = \pi r^2 h$, we have the approximation for the change in volume as

$$\begin{aligned} dV &= V_r(5, 25) dr + V_h(5, 25) dh \\ &= (2\pi rh)_{(5,25)} dr + (\pi r^2)_{(5,25)} dh \\ &= 250\pi dr + 25\pi dh. \end{aligned}$$

Thus, a 1-unit change in r will change V by about 250π units. A 1-unit change in h will change V by about 25π units. The tank's volume is 10 times more sensitive to a small change in r than it is to a small change of equal size in h . As a quality control engineer concerned with being sure the tanks have the correct volume, you would want to pay special attention to their radii.

In contrast, if the values of r and h are reversed to make $r = 25$ and $h = 5$, then the total differential in V becomes

$$dV = (2\pi rh)_{(25,5)} dr + (\pi r^2)_{(25,5)} dh = 250\pi dr + 625\pi dh.$$

Now the volume is more sensitive to changes in h than to changes in r (Figure 14.35).

The general rule is that functions are most sensitive to small changes in the variables that generate the largest partial derivatives. ■

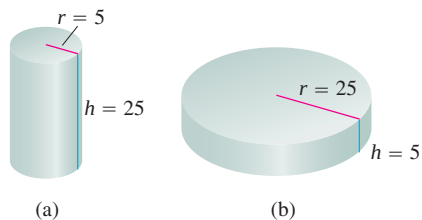


FIGURE 14.35 The volume of cylinder (a) is more sensitive to a small change in r than it is to an equally small change in h . The volume of cylinder (b) is more sensitive to small changes in h than it is to small changes in r (Example 8).



EXAMPLE 9 Estimating Percentage Error

The volume $V = \pi r^2 h$ of a right circular cylinder is to be calculated from measured values of r and h . Suppose that r is measured with an error of no more than 2% and h with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of V .

Solution We are told that

$$\left| \frac{dr}{r} \times 100 \right| \leq 2 \quad \text{and} \quad \left| \frac{dh}{h} \times 100 \right| \leq 0.5.$$

Since

$$\frac{dV}{V} = \frac{2\pi rh dr + \pi r^2 dh}{\pi r^2 h} = \frac{2 dr}{r} + \frac{dh}{h},$$

we have

$$\begin{aligned}\left|\frac{dV}{V}\right| &= \left|2\frac{dr}{r} + \frac{dh}{h}\right| \\ &\leq \left|2\frac{dr}{r}\right| + \left|\frac{dh}{h}\right| \\ &\leq 2(0.02) + 0.005 = 0.045.\end{aligned}$$

We estimate the error in the volume calculation to be at most 4.5%. ■

Functions of More Than Two Variables

Analogous results hold for differentiable functions of more than two variables.

1. The **linearization** of $f(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0).$$

2. Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of f are continuous. Suppose also that $|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R . Then the **error** $E(x, y, z) = f(x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by the inequality

$$|E| \leq \frac{1}{2}M(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$

3. If the second partial derivatives of f are continuous and if x, y , and z change from x_0, y_0 , and z_0 by small amounts dx, dy , and dz , the **total differential**

$$df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz$$

gives a good approximation of the resulting change in f .



EXAMPLE 10 Finding a Linear Approximation in 3-Space

Find the linearization $L(x, y, z)$ of

$$f(x, y, z) = x^2 - xy + 3 \sin z$$

at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangle

$$R: |x - 2| \leq 0.01, \quad |y - 1| \leq 0.02, \quad |z| \leq 0.01.$$

Solution A routine evaluation gives

$$f(2, 1, 0) = 2, \quad f_x(2, 1, 0) = 3, \quad f_y(2, 1, 0) = -2, \quad f_z(2, 1, 0) = 3.$$

Thus,

$$L(x, y, z) = 2 + 3(x - 2) + (-2)(y - 1) + 3(z - 0) = 3x - 2y + 3z - 2.$$

Since

$$\begin{aligned}f_{xx} &= 2, & f_{yy} &= 0, & f_{zz} &= -3 \sin z, \\ f_{xy} &= -1, & f_{xz} &= 0, & f_{yz} &= 0,\end{aligned}$$

we may safely take M to be $\max | -3 \sin z | = 3$. Hence, the error incurred by replacing f by L on R satisfies

$$|E| \leq \frac{1}{2} (3)(0.01 + 0.02 + 0.01)^2 = 0.0024.$$

The error will be no greater than 0.0024. ■

EXERCISES 14.6

Tangent Planes and Normal Lines to Surfaces

In Exercises 1–8, find equations for the

- (a) tangent plane and (b) normal line at the point P_0 on the given surface.

1. $x^2 + y^2 + z^2 = 3$, $P_0(1, 1, 1)$
2. $x^2 + y^2 - z^2 = 18$, $P_0(3, 5, -4)$
3. $2z - x^2 = 0$, $P_0(2, 0, 2)$
4. $x^2 + 2xy - y^2 + z^2 = 7$, $P_0(1, -1, 3)$
5. $\cos \pi x - x^2 y + e^{xz} + yz = 4$, $P_0(0, 1, 2)$
6. $x^2 - xy - y^2 - z = 0$, $P_0(1, 1, -1)$
7. $x + y + z = 1$, $P_0(0, 1, 0)$
8. $x^2 + y^2 - 2xy - x + 3y - z = -4$, $P_0(2, -3, 18)$

In Exercises 9–12, find an equation for the plane that is tangent to the given surface at the given point.

9. $z = \ln(x^2 + y^2)$, $(1, 0, 0)$
10. $z = e^{-(x^2 + y^2)}$, $(0, 0, 1)$
11. $z = \sqrt{y - x}$, $(1, 2, 1)$
12. $z = 4x^2 + y^2$, $(1, 1, 5)$

Tangent Lines to Curves

In Exercises 13–18, find parametric equations for the line tangent to the curve of intersection of the surfaces at the given point.

13. Surfaces: $x + y^2 + 2z = 4$, $x = 1$
Point: $(1, 1, 1)$
14. Surfaces: $xyz = 1$, $x^2 + 2y^2 + 3z^2 = 6$
Point: $(1, 1, 1)$
15. Surfaces: $x^2 + 2y + 2z = 4$, $y = 1$
Point: $(1, 1, 1/2)$
16. Surfaces: $x + y^2 + z = 2$, $y = 1$
Point: $(1/2, 1, 1/2)$
17. Surfaces: $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0$, $x^2 + y^2 + z^2 = 11$
Point: $(1, 1, 3)$
18. Surfaces: $x^2 + y^2 = 4$, $x^2 + y^2 - z = 0$
Point: $(\sqrt{2}, \sqrt{2}, 4)$

Estimating Change

19. By about how much will

$$f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$$

change if the point $P(x, y, z)$ moves from $P_0(3, 4, 12)$ a distance of $ds = 0.1$ unit in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$?

20. By about how much will

$$f(x, y, z) = e^x \cos yz$$

change as the point $P(x, y, z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction of $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$?

21. By about how much will

$$g(x, y, z) = x + x \cos z - y \sin z + y$$

change if the point $P(x, y, z)$ moves from $P_0(2, -1, 0)$ a distance of $ds = 0.2$ unit toward the point $P_1(0, 1, 2)$?

22. By about how much will

$$h(x, y, z) = \cos(\pi xy) + xz^2$$

change if the point $P(x, y, z)$ moves from $P_0(-1, -1, -1)$ a distance of $ds = 0.1$ unit toward the origin?

23. **Temperature change along a circle** Suppose that the Celsius temperature at the point (x, y) in the xy -plane is $T(x, y) = x \sin 2y$ and that distance in the xy -plane is measured in meters. A particle is moving *clockwise* around the circle of radius 1 m centered at the origin at the constant rate of 2 m/sec.

- a. How fast is the temperature experienced by the particle changing in degrees Celsius per meter at the point $P(1/2, \sqrt{3}/2)$?
- b. How fast is the temperature experienced by the particle changing in degrees Celsius per second at P ?

24. **Changing temperature along a space curve** The Celsius temperature in a region in space is given by $T(x, y, z) = 2x^2 - xyz$. A particle is moving in this region and its position at time t is given by $x = 2t^2$, $y = 3t$, $z = -t^2$, where time is measured in seconds and distance in meters.





- a. How fast is the temperature experienced by the particle changing in degrees Celsius per meter when the particle is at the point $P(8, 6, -4)$?
- b. How fast is the temperature experienced by the particle changing in degrees Celsius per second at P ?

Finding Linearizations

In Exercises 25–30, find the linearization $L(x, y)$ of the function at each point.

25. $f(x, y) = x^2 + y^2 + 1$ at a. $(0, 0)$, b. $(1, 1)$
 26. $f(x, y) = (x + y + 2)^2$ at a. $(0, 0)$, b. $(1, 2)$
 27. $f(x, y) = 3x - 4y + 5$ at a. $(0, 0)$, b. $(1, 1)$
 28. $f(x, y) = x^3 y^4$ at a. $(1, 1)$, b. $(0, 0)$
 29. $f(x, y) = e^x \cos y$ at a. $(0, 0)$, b. $(0, \pi/2)$
 30. $f(x, y) = e^{2y-x}$ at a. $(0, 0)$, b. $(1, 2)$

Upper Bounds for Errors in Linear Approximations

In Exercises 31–36, find the linearization $L(x, y)$ of the function $f(x, y)$ at P_0 . Then find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x, y) \approx L(x, y)$ over the rectangle R .

31. $f(x, y) = x^2 - 3xy + 5$ at $P_0(2, 1)$,
 $R: |x - 2| \leq 0.1, |y - 1| \leq 0.1$
 32. $f(x, y) = (1/2)x^2 + xy + (1/4)y^2 + 3x - 3y + 4$ at $P_0(2, 2)$,
 $R: |x - 2| \leq 0.1, |y - 2| \leq 0.1$
 33. $f(x, y) = 1 + y + x \cos y$ at $P_0(0, 0)$,
 $R: |x| \leq 0.2, |y| \leq 0.2$
 (Use $|\cos y| \leq 1$ and $|\sin y| \leq 1$ in estimating E .)
 34. $f(x, y) = xy^2 + y \cos(x - 1)$ at $P_0(1, 2)$,
 $R: |x - 1| \leq 0.1, |y - 2| \leq 0.1$
 35. $f(x, y) = e^x \cos y$ at $P_0(0, 0)$,
 $R: |x| \leq 0.1, |y| \leq 0.1$
 (Use $e^x \leq 1.11$ and $|\cos y| \leq 1$ in estimating E .)
 36. $f(x, y) = \ln x + \ln y$ at $P_0(1, 1)$,
 $R: |x - 1| \leq 0.2, |y - 1| \leq 0.2$

Functions of Three Variables

Find the linearizations $L(x, y, z)$ of the functions in Exercises 37–42 at the given points.

37. $f(x, y, z) = xy + yz + xz$ at a. $(1, 1, 1)$ b. $(1, 0, 0)$ c. $(0, 0, 0)$
 38. $f(x, y, z) = x^2 + y^2 + z^2$ at a. $(1, 1, 1)$ b. $(0, 1, 0)$ c. $(1, 0, 0)$
 39. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at a. $(1, 0, 0)$ b. $(1, 1, 0)$ c. $(1, 2, 2)$

40. $f(x, y, z) = (\sin xy)/z$ at a. $(\pi/2, 1, 1)$ b. $(2, 0, 1)$
 41. $f(x, y, z) = e^x + \cos(y + z)$ at a. $(0, 0, 0)$ b. $(0, \frac{\pi}{2}, 0)$ c. $(0, \frac{\pi}{4}, \frac{\pi}{4})$
 42. $f(x, y, z) = \tan^{-1}(xyz)$ at a. $(1, 0, 0)$ b. $(1, 1, 0)$ c. $(1, 1, 1)$

In Exercises 43–46, find the linearization $L(x, y, z)$ of the function $f(x, y, z)$ at P_0 . Then find an upper bound for the magnitude of the error E in the approximation $f(x, y, z) \approx L(x, y, z)$ over the region R .

43. $f(x, y, z) = xz - 3yz + 2$ at $P_0(1, 1, 2)$
 $R: |x - 1| \leq 0.01, |y - 1| \leq 0.01, |z - 2| \leq 0.02$
 44. $f(x, y, z) = x^2 + xy + yz + (1/4)z^2$ at $P_0(1, 1, 2)$
 $R: |x - 1| \leq 0.01, |y - 1| \leq 0.01, |z - 2| \leq 0.08$
 45. $f(x, y, z) = xy + 2yz - 3xz$ at $P_0(1, 1, 0)$
 $R: |x - 1| \leq 0.01, |y - 1| \leq 0.01, |z| \leq 0.01$
 46. $f(x, y, z) = \sqrt{2} \cos x \sin(y + z)$ at $P_0(0, 0, \pi/4)$
 $R: |x| \leq 0.01, |y| \leq 0.01, |z - \pi/4| \leq 0.01$

Estimating Error; Sensitivity to Change

47. **Estimating maximum error** Suppose that T is to be found from the formula $T = x(e^y + e^{-y})$, where x and y are found to be 2 and $\ln 2$ with maximum possible errors of $|dx| = 0.1$ and $|dy| = 0.02$. Estimate the maximum possible error in the computed value of T .
48. **Estimating volume of a cylinder** About how accurately may $V = \pi r^2 h$ be calculated from measurements of r and h that are in error by 1%?
49. **Maximum percentage error** If $r = 5.0$ cm and $h = 12.0$ cm to the nearest millimeter, what should we expect the maximum percentage error in calculating $V = \pi r^2 h$ to be?
50. **Variation in electrical resistance** The resistance R produced by wiring resistors of R_1 and R_2 ohms in parallel (see accompanying figure) can be calculated from the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

- a. Show that

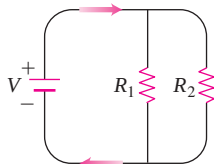
$$dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2.$$

- b. You have designed a two-resistor circuit like the one shown on the next page to have resistances of $R_1 = 100$ ohms and $R_2 = 400$ ohms, but there is always some variation in manufacturing and the resistors received by your firm will probably not have these exact values. Will the value of R be

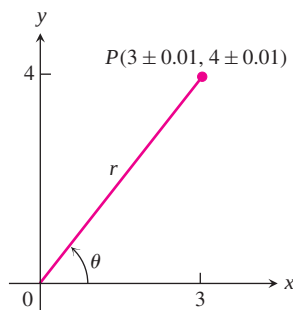




more sensitive to variation in R_1 or to variation in R_2 ? Give reasons for your answer.



- c. In another circuit like the one shown you plan to change R_1 from 20 to 20.1 ohms and R_2 from 25 to 24.9 ohms. By about what percentage will this change R ?
51. You plan to calculate the area of a long, thin rectangle from measurements of its length and width. Which dimension should you measure more carefully? Give reasons for your answer.
52. a. Around the point $(1, 0)$, is $f(x, y) = x^2(y + 1)$ more sensitive to changes in x or to changes in y ? Give reasons for your answer.
b. What ratio of dx to dy will make df equal zero at $(1, 0)$?
53. **Error carryover in coordinate changes**



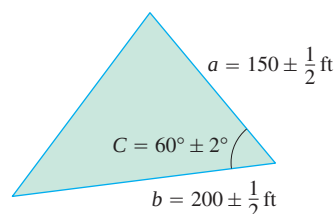
- a. If $x = 3 \pm 0.01$ and $y = 4 \pm 0.01$, as shown here, with approximately what accuracy can you calculate the polar coordinates r and θ of the point $P(x, y)$ from the formulas $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$? Express your estimates as percentage changes of the values that r and θ have at the point $(x_0, y_0) = (3, 4)$.
- b. At the point $(x_0, y_0) = (3, 4)$, are the values of r and θ more sensitive to changes in x or to changes in y ? Give reasons for your answer.
54. **Designing a soda can** A standard 12-fl oz can of soda is essentially a cylinder of radius $r = 1$ in. and height $h = 5$ in.
- a. At these dimensions, how sensitive is the can's volume to a small change in radius versus a small change in height?
- b. Could you design a soda can that *appears* to hold more soda but in fact holds the same 12-fl oz? What might its dimensions be? (There is more than one correct answer.)

55. **Value of a 2×2 determinant** If $|a|$ is much greater than $|b|$, $|c|$, and $|d|$, to which of a , b , c , and d is the value of the determinant

$$f(a, b, c, d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

most sensitive? Give reasons for your answer.

56. **Estimating maximum error** Suppose that $u = xe^y + y \sin z$ and that x , y , and z can be measured with maximum possible errors of ± 0.2 , ± 0.6 , and $\pm \pi/180$, respectively. Estimate the maximum possible error in calculating u from the measured values $x = 2$, $y = \ln 3$, $z = \pi/2$.
57. **The Wilson lot size formula** The Wilson lot size formula in economics says that the most economical quantity Q of goods (radios, shoes, brooms, whatever) for a store to order is given by the formula $Q = \sqrt{2KM/h}$, where K is the cost of placing the order, M is the number of items sold per week, and h is the weekly holding cost for each item (cost of space, utilities, security, and so on). To which of the variables K , M , and h is Q most sensitive near the point $(K_0, M_0, h_0) = (2, 20, 0.05)$? Give reasons for your answer.
58. **Surveying a triangular field** The area of a triangle is $(1/2)ab \sin C$, where a and b are the lengths of two sides of the triangle and C is the measure of the included angle. In surveying a triangular plot, you have measured a , b , and C to be 150 ft, 200 ft, and 60° , respectively. By about how much could your area calculation be in error if your values of a and b are off by half a foot each and your measurement of C is off by 2° ? See the accompanying figure. Remember to use radians.



Theory and Examples

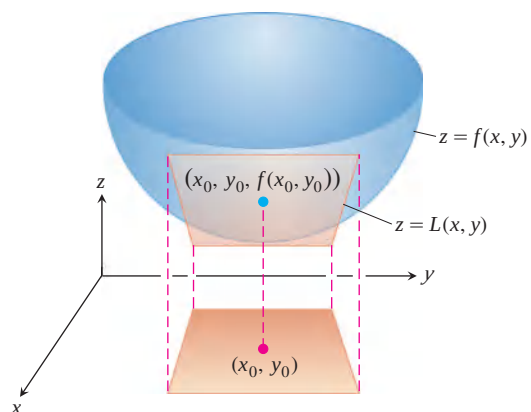
59. **The linearization of $f(x, y)$ is a tangent-plane approximation** Show that the tangent plane at the point $P_0(x_0, y_0)$, $f(x_0, y_0)$ on the surface $z = f(x, y)$ defined by a differentiable function f is the plane

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

or

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Thus, the tangent plane at P_0 is the graph of the linearization of f at P_0 (see accompanying figure).



- 60. Change along the involute of a circle** Find the derivative of $f(x, y) = x^2 + y^2$ in the direction of the unit tangent vector of the curve

$$\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad t > 0.$$

- 61. Change along a helix** Find the derivative of $f(x, y, z) = x^2 + y^2 + z^2$ in the direction of the unit tangent vector of the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

at the points where $t = -\pi/4, 0$, and $\pi/4$. The function f gives the square of the distance from a point $P(x, y, z)$ on the helix to the origin. The derivatives calculated here give the rates at which the square of the distance is changing with respect to t as P moves through the points where $t = -\pi/4, 0$, and $\pi/4$.

- 62. Normal curves** A smooth curve is *normal* to a surface $f(x, y, z) = c$ at a point of intersection if the curve's velocity vector is a nonzero scalar multiple of ∇f at the point.

Show that the curve

$$\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - \frac{1}{4}(t + 3)\mathbf{k}$$

is normal to the surface $x^2 + y^2 - z = 3$ when $t = 1$.

- 63. Tangent curves** A smooth curve is *tangent* to the surface at a point of intersection if its velocity vector is orthogonal to ∇f there.

Show that the curve

$$\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} + (2t - 1)\mathbf{k}$$

is tangent to the surface $x^2 + y^2 - z = 1$ when $t = 1$.

14.7

Extreme Values and Saddle Points



Project



Project

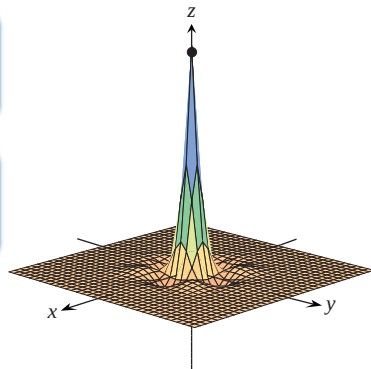


FIGURE 14.36 The function

$$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$$

has a maximum value of 1 and a minimum value of about -0.067 on the square region $|x| \leq 3\pi/2, |y| \leq 3\pi/2$.

Continuous functions of two variables assume extreme values on closed, bounded domains (see Figures 14.36 and 14.37). We see in this section that we can narrow the search for these extreme values by examining the functions' first partial derivatives. A function of two variables can assume extreme values only at domain boundary points or at interior domain points where both first partial derivatives are zero or where one or both of the first partial derivatives fails to exist. However, the vanishing of derivatives at an interior point (a, b) does not always signal the presence of an extreme value. The surface that is the graph of the function might be shaped like a saddle right above (a, b) and cross its tangent plane there.

Derivative Tests for Local Extreme Values

To find the local extreme values of a function of a single variable, we look for points where the graph has a horizontal tangent line. At such points, we then look for local maxima, local minima, and points of inflection. For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a horizontal tangent *plane*. At such points, we then look for local maxima, local minima, and saddle points (more about saddle points in a moment).

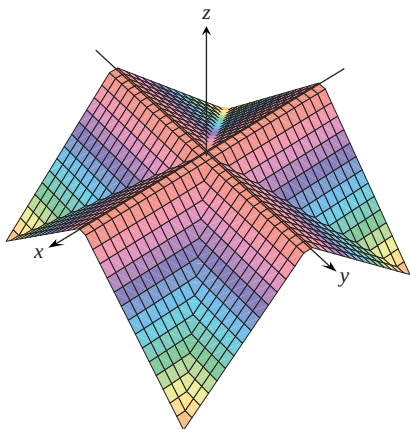


FIGURE 14.37 The “roof surface”

$$z = \frac{1}{2}(|x| - |y| - |x| - |y|)$$

viewed from the point (10, 15, 20). The defining function has a maximum value of 0 and a minimum value of $-a$ on the square region $|x| \leq a, |y| \leq a$.

HISTORICAL BIOGRAPHY

Siméon-Denis Poisson
(1781–1840)

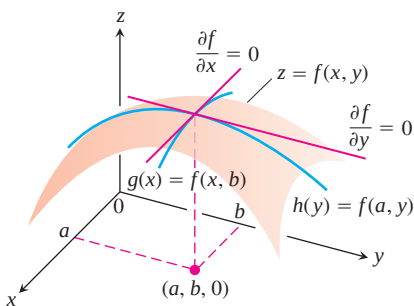


FIGURE 14.39 If a local maximum of f occurs at $x = a, y = b$, then the first partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are both zero.

DEFINITIONS Local Maximum, Local Minimum

Let $f(x, y)$ be defined on a region R containing the point (a, b) . Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b) .
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b) .

Local maxima correspond to mountain peaks on the surface $z = f(x, y)$ and local minima correspond to valley bottoms (Figure 14.38). At such points the tangent planes, when they exist, are horizontal. Local extrema are also called **relative extrema**.

As with functions of a single variable, the key to identifying the local extrema is a first derivative test.

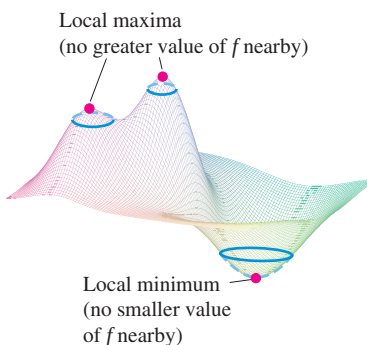


FIGURE 14.38 A local maximum is a mountain peak and a local minimum is a valley low.

THEOREM 10 First Derivative Test for Local Extreme Values

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proof If f has a local extremum at (a, b) , then the function $g(x) = f(x, b)$ has a local extremum at $x = a$ (Figure 14.39). Therefore, $g'(a) = 0$ (Chapter 4, Theorem 2). Now $g'(a) = f_x(a, b)$, so $f_x(a, b) = 0$. A similar argument with the function $h(y) = f(a, y)$ shows that $f_y(a, b) = 0$. ■

If we substitute the values $f_x(a, b) = 0$ and $f_y(a, b) = 0$ into the equation

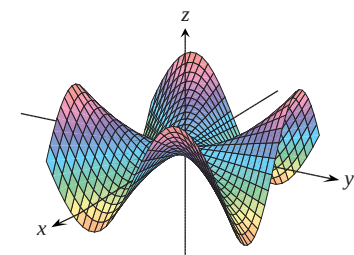
$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

for the tangent plane to the surface $z = f(x, y)$ at (a, b) , the equation reduces to

$$0 \cdot (x - a) + 0 \cdot (y - b) - z + f(a, b) = 0$$

or

$$z = f(a, b).$$



$$z = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

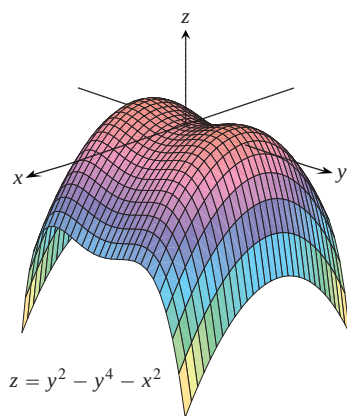


FIGURE 14.40 Saddle points at the origin.

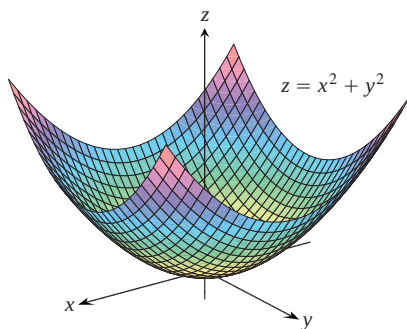


FIGURE 14.41 The graph of the function $f(x, y) = x^2 + y^2$ is the paraboloid $z = x^2 + y^2$. The function has a local minimum value of 0 at the origin (Example 1).

Thus, Theorem 10 says that the surface does indeed have a horizontal tangent plane at a local extremum, provided there is a tangent plane there.

DEFINITION Critical Point

An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f .

Theorem 10 says that the only points where a function $f(x, y)$ can assume extreme values are critical points and boundary points. As with differentiable functions of a single variable, not every critical point gives rise to a local extremum. A differentiable function of a single variable might have a point of inflection. A differentiable function of two variables might have a *saddle point*.

DEFINITION Saddle Point

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.40).

EXAMPLE 1 Finding Local Extreme Values

Find the local extreme values of $f(x, y) = x^2 + y^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \quad \text{and} \quad f_y = 2y = 0.$$

The only possibility is the origin, where the value of f is zero. Since f is never negative, we see that the origin gives a local minimum (Figure 14.41).

EXAMPLE 2 Identifying a Saddle Point

Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = -2x$ and $f_y = 2y$ exist everywhere. Therefore, local extrema can occur only at the origin $(0, 0)$. Along the positive x -axis, however, f has the value $f(x, 0) = -x^2 < 0$; along the positive y -axis, f has the value $f(0, y) = y^2 > 0$. Therefore, every open disk in the xy -plane centered at $(0, 0)$ contains points where the function is positive and points where it is negative. The function has a saddle point at the origin (Figure 14.42) instead of a local extreme value. We conclude that the function has no local extreme values.

That $f_x = f_y = 0$ at an interior point (a, b) of R does not guarantee f has a local extreme value there. If f and its first and second partial derivatives are continuous on R , however, we may be able to learn more from the following theorem, proved in Section 14.10.



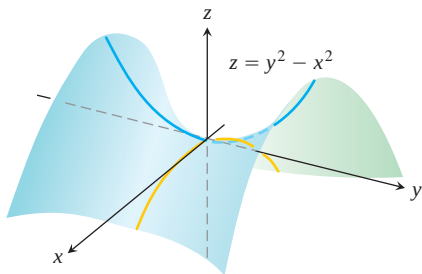


FIGURE 14.42 The origin is a saddle point of the function $f(x, y) = y^2 - x^2$. There are no local extreme values (Example 2).

THEOREM 11 Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

- i. f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) .
- ii. f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) .
- iii. f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) .
- iv. **The test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) . In this case, we must find some other way to determine the behavior of f at (a, b) .

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of f . It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

Theorem 11 says that if the discriminant is positive at the point (a, b) , then the surface curves the same way in all directions: downward if $f_{xx} < 0$, giving rise to a local maximum, and upward if $f_{xx} > 0$, giving a local minimum. On the other hand, if the discriminant is negative at (a, b) , then the surface curves up in some directions and down in others, so we have a saddle point.



EXAMPLE 3 Finding Local Extreme Values

Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \quad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \quad \text{and} \quad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$. ■

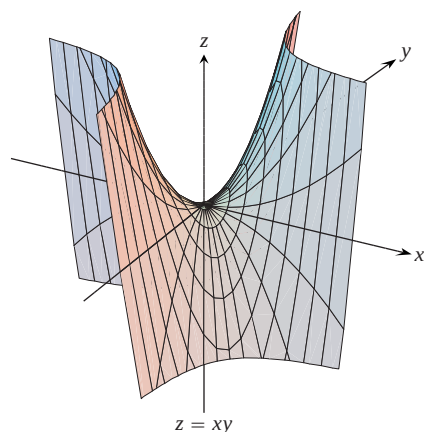


FIGURE 14.43 The surface $z = xy$ has a saddle point at the origin (Example 4).

EXAMPLE 4 Searching for Local Extreme Values

Find the local extreme values of $f(x, y) = xy$.

Solution Since f is differentiable everywhere (Figure 14.43), it can assume extreme values only where

$$f_x = y = 0 \quad \text{and} \quad f_y = x = 0.$$

Thus, the origin is the only point where f might have an extreme value. To see what happens there, we calculate

$$f_{xx} = 0, \quad f_{yy} = 0, \quad f_{xy} = 1.$$

The discriminant,

$$f_{xx}f_{yy} - f_{xy}^2 = -1,$$

is negative. Therefore, the function has a saddle point at $(0, 0)$. We conclude that $f(x, y) = xy$ has no local extreme values. ■

Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

1. List the interior points of R where f may have local maxima and minima and evaluate f at these points. These are the critical points of f .
2. List the boundary points of R where f has local maxima and minima and evaluate f at these points. We show how to do this shortly.
3. Look through the lists for the maximum and minimum values of f . These will be the absolute maximum and minimum values of f on R . Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists made in Steps 1 and 2.

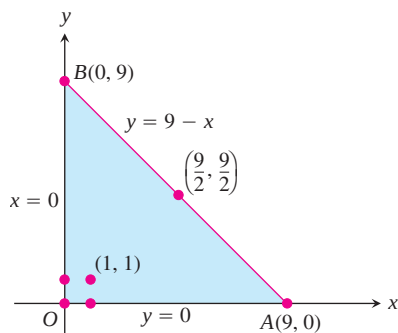


FIGURE 14.44 This triangular region is the domain of the function in Example 5.

EXAMPLE 5 Finding Absolute Extrema

Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.

Solution Since f is differentiable, the only places where f can assume these values are points inside the triangle (Figure 14.44) where $f_x = f_y = 0$ and points on the boundary.

(a) **Interior points.** For these we have

$$f_x = 2 - 2x = 0, \quad f_y = 2 - 2y = 0,$$

yielding the single point $(x, y) = (1, 1)$. The value of f there is

$$f(1, 1) = 4.$$



(b) Boundary points. We take the triangle one side at a time:

(i) On the segment OA , $y = 0$. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of x defined on the closed interval $0 \leq x \leq 9$. Its extreme values (we know from Chapter 4) may occur at the endpoints

$$x = 0 \quad \text{where} \quad f(0, 0) = 2$$

$$x = 9 \quad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61$$

and at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$

(ii) On the segment OB , $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 2y - y^2.$$

We know from the symmetry of f in x and y and from the analysis we just carried out that the candidates on this segment are

$$f(0, 0) = 2, \quad f(0, 9) = -61, \quad f(0, 1) = 3.$$

(iii) We have already accounted for the values of f at the endpoints of AB , so we need only look at the interior points of AB . With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2 = -61 + 18x - 2x^2.$$

Setting $f'(x, 9 - x) = 18 - 4x = 0$ gives

$$x = \frac{18}{4} = \frac{9}{2}.$$

At this value of x ,

$$y = 9 - \frac{9}{2} = \frac{9}{2} \quad \text{and} \quad f(x, y) = f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}.$$

Summary We list all the candidates: 4, 2, -61 , 3, $-(41/2)$. The maximum is 4, which f assumes at $(1, 1)$. The minimum is -61 , which f assumes at $(0, 9)$ and $(9, 0)$. ■

Solving extreme value problems with algebraic constraints on the variables usually requires the method of Lagrange multipliers in the next section. But sometimes we can solve such problems directly, as in the next example.

EXAMPLE 6 Solving a Volume Problem with a Constraint

A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume.

Solution Let x , y , and z represent the length, width, and height of the rectangular box, respectively. Then the girth is $2y + 2z$. We want to maximize the volume $V = xyz$ of the

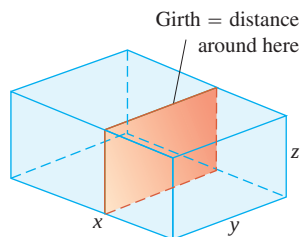


FIGURE 14.45 The box in Example 6.

box (Figure 14.45) satisfying $x + 2y + 2z = 108$ (the largest box accepted by the delivery company). Thus, we can write the volume of the box as a function of two variables.

$$\begin{aligned} V(y, z) &= (108 - 2y - 2z)yz && \begin{array}{l} V = xyz \text{ and} \\ x = 108 - 2y - 2z \end{array} \\ &= 108yz - 2y^2z - 2yz^2 \end{aligned}$$

Setting the first partial derivatives equal to zero,

$$V_y(y, z) = 108z - 4yz - 2z^2 = (108 - 4y - 2z)z = 0$$

$$V_z(y, z) = 108y - 2y^2 - 4yz = (108 - 2y - 4z)y = 0,$$

gives the critical points $(0, 0)$, $(0, 54)$, $(54, 0)$, and $(18, 18)$. The volume is zero at $(0, 0)$, $(0, 54)$, $(54, 0)$, which are not maximum values. At the point $(18, 18)$, we apply the Second Derivative Test (Theorem 11):

$$V_{yy} = -4z, \quad V_{zz} = -4y, \quad V_{yz} = 108 - 4y - 4z.$$

Then

$$V_{yy}V_{zz} - V_{yz}^2 = 16yz - 16(27 - y - z)^2.$$

Thus,

$$V_{yy}(18, 18) = -4(18) < 0$$

and

$$[V_{yy}V_{zz} - V_{yz}^2]_{(18,18)} = 16(18)(18) - 16(-9)^2 > 0$$

imply that $(18, 18)$ gives a maximum volume. The dimensions of the package are $x = 108 - 2(18) - 2(18) = 36$ in., $y = 18$ in., and $z = 18$ in. The maximum volume is $V = (36)(18)(18) = 11,664$ in.³, or 6.75 ft³. ■

Despite the power of Theorem 10, we urge you to remember its limitations. It does not apply to boundary points of a function's domain, where it is possible for a function to have extreme values along with nonzero derivatives. Also, it does not apply to points where either f_x or f_y fails to exist.

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

- i. **boundary points** of the domain of f
- ii. **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

- i. $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ **local maximum**
- ii. $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ **local minimum**
- iii. $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a, b) \Rightarrow$ **saddle point**
- iv. $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Rightarrow$ **test is inconclusive.**

EXERCISES 14.7

Finding Local Extrema

Find all the local maxima, local minima, and saddle points of the functions in Exercises 1–30.

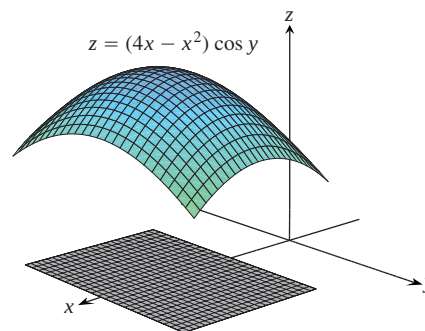
1. $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$
2. $f(x, y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$
3. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$
4. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x - 4$
5. $f(x, y) = x^2 + xy + 3x + 2y + 5$
6. $f(x, y) = y^2 + xy - 2x - 2y + 2$
7. $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$
8. $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$
9. $f(x, y) = x^2 - 4xy + y^2 + 6y + 2$
10. $f(x, y) = 3x^2 + 6xy + 7y^2 - 2x + 4y$
11. $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$
12. $f(x, y) = 4x^2 - 6xy + 5y^2 - 20x + 26y$
13. $f(x, y) = x^2 - y^2 - 2x + 4y + 6$
14. $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$
15. $f(x, y) = x^2 + 2xy$
16. $f(x, y) = 3 + 2x + 2y - 2x^2 - 2xy - y^2$
17. $f(x, y) = x^3 - y^3 - 2xy + 6$
18. $f(x, y) = x^3 + 3xy + y^3$
19. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$
20. $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$
21. $f(x, y) = 9x^3 + y^3/3 - 4xy$
22. $f(x, y) = 8x^3 + y^3 + 6xy$
23. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$
24. $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$
25. $f(x, y) = 4xy - x^4 - y^4$
26. $f(x, y) = x^4 + y^4 + 4xy$
27. $f(x, y) = \frac{1}{x^2 + y^2 - 1}$
28. $f(x, y) = \frac{1}{x} + xy + \frac{1}{y}$
29. $f(x, y) = y \sin x$
30. $f(x, y) = e^{2x} \cos y$

Finding Absolute Extrema

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0$, $y = 2$, $y = 2x$ in the first quadrant
32. $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 4$, $y = x$

33. $f(x, y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0$, $y = 0$, $y + 2x = 2$ in the first quadrant
34. $T(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 3$
35. $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \leq x \leq 5$, $-3 \leq y \leq 0$
36. $f(x, y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \leq x \leq 1$, $0 \leq y \leq 1$
37. $f(x, y) = (4x - x^2) \cos y$ on the rectangular plate $1 \leq x \leq 3$, $-\pi/4 \leq y \leq \pi/4$ (see accompanying figure).



38. $f(x, y) = 4x - 8xy + 2y + 1$ on the triangular plate bounded by the lines $x = 0$, $y = 0$, $x + y = 1$ in the first quadrant
39. Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (6 - x - x^2) dx$$

has its largest value.

40. Find two numbers a and b with $a \leq b$ such that

$$\int_a^b (24 - 2x - x^2)^{1/3} dx$$

has its largest value.

41. **Temperatures** The flat circular plate in Figure 14.46 has the shape of the region $x^2 + y^2 \leq 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x, y) is

$$T(x, y) = x^2 + 2y^2 - x.$$

Find the temperatures at the hottest and coldest points on the plate.

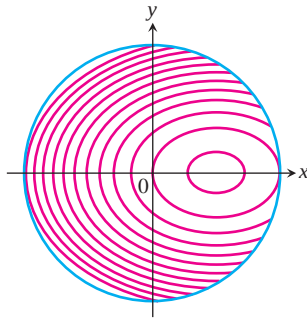


FIGURE 14.46 Curves of constant temperature are called isotherms. The figure shows isotherms of the temperature function $T(x, y) = x^2 + 2y^2 - x$ on the disk $x^2 + y^2 \leq 1$ in the xy -plane. Exercise 41 asks you to locate the extreme temperatures.

42. Find the critical point of

$$f(x, y) = xy + 2x - \ln x^2 y$$

in the open first quadrant ($x > 0, y > 0$) and show that f takes on a minimum there (Figure 14.47).

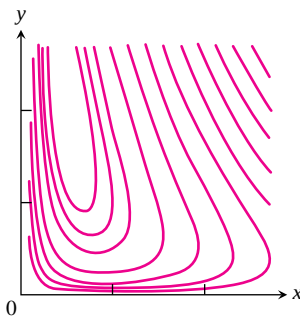


FIGURE 14.47 The function $f(x, y) = xy + 2x - \ln x^2 y$ (selected level curves shown here) takes on a minimum value somewhere in the open first quadrant $x > 0, y > 0$ (Exercise 42).

Theory and Examples

43. Find the maxima, minima, and saddle points of $f(x, y)$, if any, given that
- $f_x = 2x - 4y$ and $f_y = 2y - 4x$
 - $f_x = 2x - 2$ and $f_y = 2y - 4$
 - $f_x = 9x^2 - 9$ and $f_y = 2y + 4$
- Describe your reasoning in each case.
44. The discriminant $f_{xx}f_{yy} - f_{xy}^2$ is zero at the origin for each of the following functions, so the Second Derivative Test fails there. Determine whether the function has a maximum, a minimum, or neither at the origin by imagining what the surface $z = f(x, y)$ looks like. Describe your reasoning in each case.
- $f(x, y) = x^2 y^2$
 - $f(x, y) = 1 - x^2 y^2$
 - $f(x, y) = xy^2$
 - $f(x, y) = x^3 y^2$
 - $f(x, y) = x^3 y^3$
 - $f(x, y) = x^4 y^4$
45. Show that $(0, 0)$ is a critical point of $f(x, y) = x^2 + kxy + y^2$ no matter what value the constant k has. (Hint: Consider two cases: $k = 0$ and $k \neq 0$.)
46. For what values of the constant k does the Second Derivative Test guarantee that $f(x, y) = x^2 + kxy + y^2$ will have a saddle point at $(0, 0)$? A local minimum at $(0, 0)$? For what values of k is the Second Derivative Test inconclusive? Give reasons for your answers.
47. If $f_x(a, b) = f_y(a, b) = 0$, must f have a local maximum or minimum value at (a, b) ? Give reasons for your answer.
48. Can you conclude anything about $f(a, b)$ if f and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign? Give reasons for your answer.
49. Among all the points on the graph of $z = 10 - x^2 - y^2$ that lie above the plane $x + 2y + 3z = 0$, find the point farthest from the plane.
50. Find the point on the graph of $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.
51. The function $f(x, y) = x + y$ fails to have an absolute maximum value in the closed first quadrant $x \geq 0$ and $y \geq 0$. Does this contradict the discussion on finding absolute extrema given in the text? Give reasons for your answer.
52. Consider the function $f(x, y) = x^2 + y^2 + 2xy - x - y + 1$ over the square $0 \leq x \leq 1$ and $0 \leq y \leq 1$.
- Show that f has an absolute minimum along the line segment $2x + 2y = 1$ in this square. What is the absolute minimum value?
 - Find the absolute maximum value of f over the square.

Extreme Values on Parametrized Curves

To find the extreme values of a function $f(x, y)$ on a curve $x = x(t), y = y(t)$, we treat f as a function of the single variable t and

use the Chain Rule to find where df/dt is zero. As in any other single-variable case, the extreme values of f are then found among the values at the

- critical points (points where df/dt is zero or fails to exist), and
- endpoints of the parameter domain.

Find the absolute maximum and minimum values of the following functions on the given curves.

53. Functions:

- $f(x, y) = x + y$
- $g(x, y) = xy$
- $h(x, y) = 2x^2 + y^2$

Curves:

- The semicircle $x^2 + y^2 = 4$, $y \geq 0$
 - The quarter circle $x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$
- Use the parametric equations $x = 2 \cos t$, $y = 2 \sin t$.

54. Functions:

- $f(x, y) = 2x + 3y$
- $g(x, y) = xy$
- $h(x, y) = x^2 + 3y^2$

Curves:

- The semi-ellipse $(x^2/9) + (y^2/4) = 1$, $y \geq 0$
 - The quarter ellipse $(x^2/9) + (y^2/4) = 1$, $x \geq 0$, $y \geq 0$
- Use the parametric equations $x = 3 \cos t$, $y = 2 \sin t$.

55. Function: $f(x, y) = xy$

Curves:

- The line $x = 2t$, $y = t + 1$
- The line segment $x = 2t$, $y = t + 1$, $-1 \leq t \leq 0$
- The line segment $x = 2t$, $y = t + 1$, $0 \leq t \leq 1$

56. Functions:

- $f(x, y) = x^2 + y^2$
- $g(x, y) = 1/(x^2 + y^2)$

Curves:

- The line $x = t$, $y = 2 - 2t$
- The line segment $x = t$, $y = 2 - 2t$, $0 \leq t \leq 1$

Least Squares and Regression Lines

When we try to fit a line $y = mx + b$ to a set of numerical data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ (Figure 14.48), we usually choose the line that minimizes the sum of the squares of the vertical distances from the points to the line. In theory, this means finding the values of m and b that minimize the value of the function

$$w = (mx_1 + b - y_1)^2 + \cdots + (mx_n + b - y_n)^2. \quad (1)$$

The values of m and b that do this are found with the First and Second Derivative Tests to be

$$m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n \sum x_k y_k}{\left(\sum x_k\right)^2 - n \sum x_k^2}, \quad (2)$$

$$b = \frac{1}{n} \left(\sum y_k - m \sum x_k \right), \quad (3)$$

with all sums running from $k = 1$ to $k = n$. Many scientific calculators have these formulas built in, enabling you to find m and b with only a few key strokes after you have entered the data.

The line $y = mx + b$ determined by these values of m and b is called the **least squares line**, **regression line**, or **trend line** for the data under study. Finding a least squares line lets you

- summarize data with a simple expression,
- predict values of y for other, experimentally untried values of x ,
- handle data analytically.

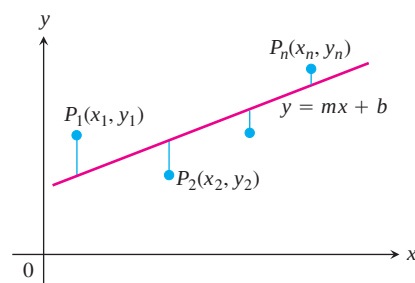


FIGURE 14.48 To fit a line to noncollinear points, we choose the line that minimizes the sum of the squares of the deviations.

EXAMPLE Find the least squares line for the points $(0, 1)$, $(1, 3)$, $(2, 2)$, $(3, 4)$, $(4, 5)$.

Solution We organize the calculations in a table:

k	x_k	y_k	x_k^2	$x_k y_k$
1	0	1	0	0
2	1	3	1	3
3	2	2	4	4
4	3	4	9	12
5	4	5	16	20
Σ	10	15	30	39

Then we find

$$m = \frac{(10)(15) - 5(39)}{(10)^2 - 5(30)} = 0.9 \quad \text{Equation (2) with } n = 5 \text{ and data from the table}$$

and use the value of m to find

$$b = \frac{1}{5} (15 - (0.9)(10)) = 1.2. \quad \text{Equation (3) with } n = 5, m = 0.9$$

The least squares line is $y = 0.9x + 1.2$ (Figure 14.49). ■

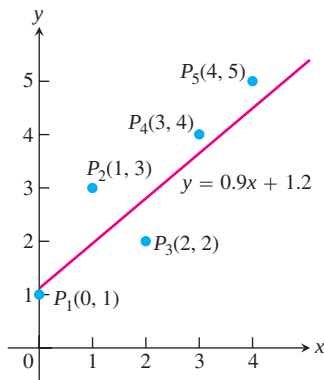


FIGURE 14.49 The least squares line for the data in the example.

In Exercises 57–60, use Equations (2) and (3) to find the least squares line for each set of data points. Then use the linear equation you obtain to predict the value of y that would correspond to $x = 4$.

57. $(-1, 2), (0, 1), (3, -4)$ 58. $(-2, 0), (0, 2), (2, 3)$

59. $(0, 0), (1, 2), (2, 3)$ 60. $(0, 1), (2, 2), (3, 2)$

T 61. Write a linear equation for the effect of irrigation on the yield of alfalfa by fitting a least squares line to the data in Table 14.1 (from the University of California Experimental Station, *Bulletin* No. 450, p. 8). Plot the data and draw the line.

TABLE 14.1 Growth of alfalfa

x (total seasonal depth of water applied, in.)	y (average alfalfa yield, tons/acre)
12	5.27
18	5.68
24	6.25
30	7.21
36	8.20
42	8.71

T 62. Craters of Mars One theory of crater formation suggests that the frequency of large craters should fall off as the square of the diameter (Marcus, *Science*, June 21, 1968, p. 1334). Pictures from *Mariner IV* show the frequencies listed in Table 14.2. Fit a line of the form $F = m(1/D^2) + b$ to the data. Plot the data and draw the line.

TABLE 14.2 Crater sizes on Mars

Diameter in km, D	$1/D^2$ (for left value of class interval)	Frequency, F
32–45	0.001	51
45–64	0.0005	22
64–90	0.00024	14
90–128	0.000123	4

T 63. Köchel numbers In 1862, the German musicologist Ludwig von Köchel made a chronological list of the musical works of Wolfgang Amadeus Mozart. This list is the source of the Köchel numbers, or “K numbers,” that now accompany the titles of Mozart’s pieces (Sinfonia Concertante in E-flat major, K.364, for example). Table 14.3 gives the Köchel numbers and composition dates (y) of ten of Mozart’s works.

- Plot y vs. K to show that y is close to being a linear function of K .
- Find a least squares line $y = mK + b$ for the data and add the line to your plot in part (a).
- K.364 was composed in 1779. What date is predicted by the least squares line?

TABLE 14.3 Compositions by Mozart

Köchel number, K	Year composed, y
1	1761
75	1771
155	1772
219	1775
271	1777
351	1780
425	1783
503	1786
575	1789
626	1791

T 64. Submarine sinkings The data in Table 14.4 show the results of a historical study of German submarines sunk by the U.S. Navy during 16 consecutive months of World War II. The data given for each month are the number of reported sinkings and the number of actual sinkings. The number of submarines sunk was slightly greater than the Navy’s reports implied. Find a least squares line for estimating the number of actual sinkings from the number of reported sinkings.

TABLE 14.4 Sinkings of German submarines by U.S. during 16 consecutive months of WWII

Month	Guesses by U.S. (reported sinkings) x	Actual number y
1	3	3
2	2	2
3	4	6
4	2	3
5	5	4
6	5	3
7	9	11
8	12	9
9	8	10
10	13	16
11	14	13
12	3	5
13	4	6
14	13	19
15	10	15
16	16	15
123		140

COMPUTER EXPLORATIONS**Exploring Local Extrema at Critical Points**

In Exercises 65–70, you will explore functions to identify their local extrema. Use a CAS to perform the following steps:

- Plot the function over the given rectangle.
 - Plot some level curves in the rectangle.
 - Calculate the function's first partial derivatives and use the CAS equation solver to find the critical points. How do the critical points relate to the level curves plotted in part (b)? Which critical points, if any, appear to give a saddle point? Give reasons for your answer.
 - Calculate the function's second partial derivatives and find the discriminant $f_{xx}f_{yy} - f_{xy}^2$.
 - Using the max-min tests, classify the critical points found in part (c). Are your findings consistent with your discussion in part (c)?
- $f(x, y) = x^2 + y^3 - 3xy$, $-5 \leq x \leq 5$, $-5 \leq y \leq 5$
 - $f(x, y) = x^3 - 3xy^2 + y^2$, $-2 \leq x \leq 2$, $-2 \leq y \leq 2$
 - $f(x, y) = x^4 + y^2 - 8x^2 - 6y + 16$, $-3 \leq x \leq 3$, $-6 \leq y \leq 6$
 - $f(x, y) = 2x^4 + y^4 - 2x^2 - 2y^2 + 3$, $-3/2 \leq x \leq 3/2$, $-3/2 \leq y \leq 3/2$
 - $f(x, y) = 5x^6 + 18x^5 - 30x^4 + 30xy^2 - 120x^3$, $-4 \leq x \leq 3$, $-2 \leq y \leq 2$
 - $f(x, y) = \begin{cases} x^5 \ln(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$, $-2 \leq x \leq 2$, $-2 \leq y \leq 2$

14.8 Lagrange Multipliers

HISTORICAL BIOGRAPHY

Joseph Louis Lagrange
(1736–1813)



Sometimes we need to find the extreme values of a function whose domain is constrained to lie within some particular subset of the plane—a disk, for example, a closed triangular region, or along a curve. In this section, we explore a powerful method for finding extreme values of constrained functions: the method of *Lagrange multipliers*.

Constrained Maxima and Minima

EXAMPLE 1 Finding a Minimum with Constraint

Find the point $P(x, y, z)$ closest to the origin on the plane $2x + y - z - 5 = 0$.

Solution The problem asks us to find the minimum value of the function

$$\begin{aligned} |\vec{OP}| &= \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} \\ &= \sqrt{x^2 + y^2 + z^2} \end{aligned}$$



subject to the constraint that

$$2x + y - z - 5 = 0.$$

Since $|\overrightarrow{OP}|$ has a minimum value wherever the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

has a minimum value, we may solve the problem by finding the minimum value of $f(x, y, z)$ subject to the constraint $2x + y - z - 5 = 0$ (thus avoiding square roots). If we regard x and y as the independent variables in this equation and write z as

$$z = 2x + y - 5,$$

our problem reduces to one of finding the points (x, y) at which the function

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2$$

has its minimum value or values. Since the domain of h is the entire xy -plane, the First Derivative Test of Section 14.7 tells us that any minima that h might have must occur at points where

$$h_x = 2x + 2(2x + y - 5)(2) = 0, \quad h_y = 2y + 2(2x + y - 5) = 0.$$

This leads to

$$10x + 4y = 20, \quad 4x + 4y = 10,$$

and the solution

$$x = \frac{5}{3}, \quad y = \frac{5}{6}.$$

We may apply a geometric argument together with the Second Derivative Test to show that these values minimize h . The z -coordinate of the corresponding point on the plane $z = 2x + y - 5$ is

$$z = 2\left(\frac{5}{3}\right) + \frac{5}{6} - 5 = -\frac{5}{6}.$$

Therefore, the point we seek is

$$\text{Closest point: } P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

The distance from P to the origin is $5/\sqrt{6} \approx 2.04$. ■

Attempts to solve a constrained maximum or minimum problem by substitution, as we might call the method of Example 1, do not always go smoothly. This is one of the reasons for learning the new method of this section.

EXAMPLE 2 Finding a Minimum with Constraint

Find the points closest to the origin on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$.

Solution 1 The cylinder is shown in Figure 14.50. We seek the points on the cylinder closest to the origin. These are the points whose coordinates minimize the value of the function

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{Square of the distance}$$

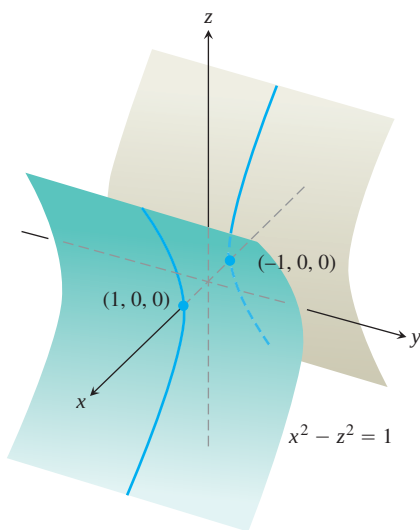


FIGURE 14.50 The hyperbolic cylinder $x^2 - z^2 - 1 = 0$ in Example 2.

subject to the constraint that $x^2 - z^2 - 1 = 0$. If we regard x and y as independent variables in the constraint equation, then

$$z^2 = x^2 - 1$$

and the values of $f(x, y, z) = x^2 + y^2 + z^2$ on the cylinder are given by the function

$$h(x, y) = x^2 + y^2 + (x^2 - 1) = 2x^2 + y^2 - 1.$$

To find the points on the cylinder whose coordinates minimize f , we look for the points in the xy -plane whose coordinates minimize h . The only extreme value of h occurs where

$$h_x = 4x = 0 \quad \text{and} \quad h_y = 2y = 0,$$

that is, at the point $(0, 0)$. But there are no points on the cylinder where both x and y are zero. What went wrong?

What happened was that the First Derivative Test found (as it should have) the point *in the domain of h* where h has a minimum value. We, on the other hand, want the points *on the cylinder* where h has a minimum value. Although the domain of h is the entire xy -plane, the domain from which we can select the first two coordinates of the points (x, y, z) on the cylinder is restricted to the “shadow” of the cylinder on the xy -plane; it does not include the band between the lines $x = -1$ and $x = 1$ (Figure 14.51).

We can avoid this problem if we treat y and z as independent variables (instead of x and y) and express x in terms of y and z as

$$x^2 = z^2 + 1.$$

With this substitution, $f(x, y, z) = x^2 + y^2 + z^2$ becomes

$$k(y, z) = (z^2 + 1) + y^2 + z^2 = 1 + y^2 + 2z^2$$

and we look for the points where k takes on its smallest value. The domain of k in the yz -plane now matches the domain from which we select the y - and z -coordinates of the points (x, y, z) on the cylinder. Hence, the points that minimize k in the plane will have corresponding points on the cylinder. The smallest values of k occur where

$$k_y = 2y = 0 \quad \text{and} \quad k_z = 4z = 0,$$

or where $y = z = 0$. This leads to

$$x^2 = z^2 + 1 = 1, \quad x = \pm 1.$$

The corresponding points on the cylinder are $(\pm 1, 0, 0)$. We can see from the inequality

$$k(y, z) = 1 + y^2 + 2z^2 \geq 1$$

that the points $(\pm 1, 0, 0)$ give a minimum value for k . We can also see that the minimum distance from the origin to a point on the cylinder is 1 unit.

Solution 2 Another way to find the points on the cylinder closest to the origin is to imagine a small sphere centered at the origin expanding like a soap bubble until it just touches the cylinder (Figure 14.52). At each point of contact, the cylinder and sphere have the same tangent plane and normal line. Therefore, if the sphere and cylinder are represented as the level surfaces obtained by setting

$$f(x, y, z) = x^2 + y^2 + z^2 = a^2 \quad \text{and} \quad g(x, y, z) = x^2 - z^2 = 1$$

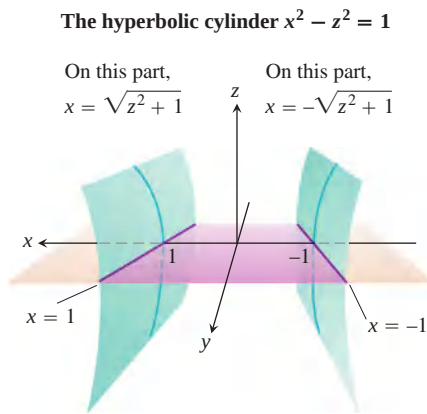


FIGURE 14.51 The region in the xy -plane from which the first two coordinates of the points (x, y, z) on the hyperbolic cylinder $x^2 - z^2 = 1$ are selected excludes the band $-1 < x < 1$ in the xy -plane (Example 2).

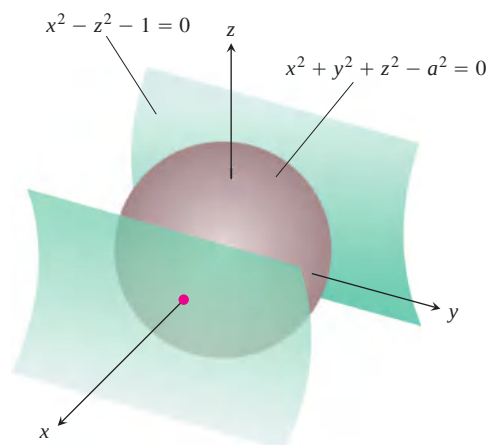


FIGURE 14.52 A sphere expanding like a soap bubble centered at the origin until it just touches the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ (Example 2).

equal to 0, then the gradients ∇f and ∇g will be parallel where the surfaces touch. At any point of contact, we should therefore be able to find a scalar λ (“lambda”) such that

$$\nabla f = \lambda \nabla g,$$

or

$$2xi + 2yj + 2zk = \lambda(2xi - 2zk).$$

Thus, the coordinates x , y , and z of any point of tangency will have to satisfy the three scalar equations

$$2x = 2\lambda x, \quad 2y = 0, \quad 2z = -2\lambda z.$$

For what values of λ will a point (x, y, z) whose coordinates satisfy these scalar equations also lie on the surface $x^2 - z^2 - 1 = 0$? To answer this question, we use our knowledge that no point on the surface has a zero x -coordinate to conclude that $x \neq 0$. Hence, $2x = 2\lambda x$ only if

$$2 = 2\lambda, \quad \text{or} \quad \lambda = 1.$$

For $\lambda = 1$, the equation $2z = -2\lambda z$ becomes $2z = -2z$. If this equation is to be satisfied as well, z must be zero. Since $y = 0$ also (from the equation $2y = 0$), we conclude that the points we seek all have coordinates of the form

$$(x, 0, 0).$$

What points on the surface $x^2 - z^2 = 1$ have coordinates of this form? The answer is the points $(x, 0, 0)$ for which

$$x^2 - (0)^2 = 1, \quad x^2 = 1, \quad \text{or} \quad x = \pm 1.$$

The points on the cylinder closest to the origin are the points $(\pm 1, 0, 0)$. ■

The Method of Lagrange Multipliers

In Solution 2 of Example 2, we used the **method of Lagrange multipliers**. The method says that the extreme values of a function $f(x, y, z)$ whose variables are subject to a constraint $g(x, y, z) = 0$ are to be found on the surface $g = 0$ at the points where

$$\nabla f = \lambda \nabla g$$

for some scalar λ (called a **Lagrange multiplier**).

To explore the method further and see why it works, we first make the following observation, which we state as a theorem.

THEOREM 12 The Orthogonal Gradient Theorem

Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}.$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C , then ∇f is orthogonal to C at P_0 .

Proof We show that ∇f is orthogonal to the curve's velocity vector at P_0 . The values of f on C are given by the composite $f(g(t), h(t), k(t))$, whose derivative with respect to t is

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dg}{dt} + \frac{\partial f}{\partial y} \frac{dh}{dt} + \frac{\partial f}{\partial z} \frac{dk}{dt} = \nabla f \cdot \mathbf{v}.$$

At any point P_0 where f has a local maximum or minimum relative to its values on the curve, $df/dt = 0$, so

$$\nabla f \cdot \mathbf{v} = 0. \quad \blacksquare$$

By dropping the z -terms in Theorem 12, we obtain a similar result for functions of two variables.

COROLLARY OF THEOREM 12

At the points on a smooth curve $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{v} = 0$, where $\mathbf{v} = d\mathbf{r}/dt$.

Theorem 12 is the key to the method of Lagrange multipliers. Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and that P_0 is a point on the surface $g(x, y, z) = 0$ where f has a local maximum or minimum value relative to its other values on the surface. Then f takes on a local maximum or minimum at P_0 relative to its values on every differentiable curve through P_0 on the surface $g(x, y, z) = 0$. Therefore, ∇f is orthogonal to the velocity vector of every such differentiable curve through P_0 . So is ∇g , moreover (because ∇g is orthogonal to the level surface $g = 0$, as we saw in Section 14.5). Therefore, at P_0 , ∇f is some scalar multiple λ of ∇g .

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$, find the values of x, y, z , and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0. \quad (1)$$

For functions of two independent variables, the condition is similar, but without the variable z .

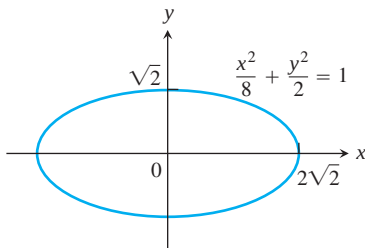


FIGURE 14.53 Example 3 shows how to find the largest and smallest values of the product xy on this ellipse.

EXAMPLE 3 Using the Method of Lagrange Multipliers

Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.53)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Solution We want the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of x, y , and λ for which

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4}x, \quad x = \lambda y, \quad \text{and} \quad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \quad 4y^2 + 4y^2 = 8 \quad \text{and} \quad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

The Geometry of the Solution

The level curves of the function $f(x, y) = xy$ are the hyperbolas $xy = c$ (Figure 14.54). The farther the hyperbolas lie from the origin, the larger the absolute value of f . We want



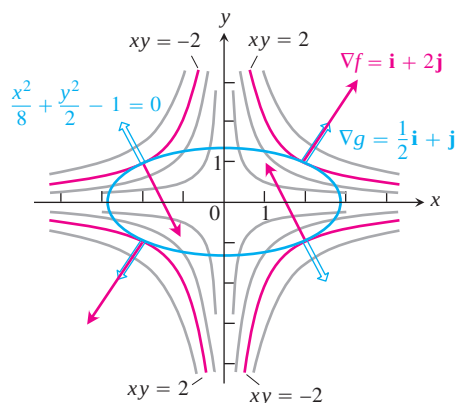


FIGURE 14.54 When subjected to the constraint $g(x, y) = x^2/8 + y^2/2 - 1 = 0$, the function $f(x, y) = xy$ takes on extreme values at the four points $(\pm 2, \pm 1)$. These are the points on the ellipse when ∇f (red) is a scalar multiple of ∇g (blue) (Example 3).

to find the extreme values of $f(x, y)$, given that the point (x, y) also lies on the ellipse $x^2 + 4y^2 = 8$. Which hyperbolas intersecting the ellipse lie farthest from the origin? The hyperbolas that just graze the ellipse, the ones that are tangent to it, are farthest. At these points, any vector normal to the hyperbola is normal to the ellipse, so $\nabla f = y\mathbf{i} + x\mathbf{j}$ is a multiple ($\lambda = \pm 2$) of $\nabla g = (x/4)\mathbf{i} + y\mathbf{j}$. At the point $(2, 1)$, for example,

$$\nabla f = \mathbf{i} + 2\mathbf{j}, \quad \nabla g = \frac{1}{2}\mathbf{i} + \mathbf{j}, \quad \text{and} \quad \nabla f = 2\nabla g.$$

At the point $(-2, 1)$,

$$\nabla f = \mathbf{i} - 2\mathbf{j}, \quad \nabla g = -\frac{1}{2}\mathbf{i} + \mathbf{j}, \quad \text{and} \quad \nabla f = -2\nabla g. \quad \blacksquare$$



EXAMPLE 4 Finding Extreme Function Values on a Circle

Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \quad g(x, y) = x^2 + y^2 - 1$$

and look for the values of x , y , and λ that satisfy the equations

$$\nabla f = \lambda \nabla g: \quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j}$$

$$g(x, y) = 0: \quad x^2 + y^2 - 1 = 0.$$

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \quad y = \frac{2}{\lambda}.$$

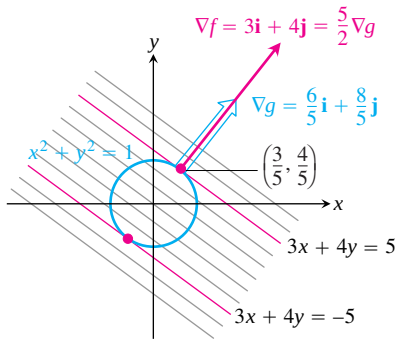


FIGURE 14.55 The function $f(x, y) = 3x + 4y$ takes on its largest value on the unit circle $g(x, y) = x^2 + y^2 - 1 = 0$ at the point $(3/5, 4/5)$ and its smallest value at the point $(-3/5, -4/5)$ (Example 4). At each of these points, ∇f is a scalar multiple of ∇g . The figure shows the gradients at the first point but not the second.

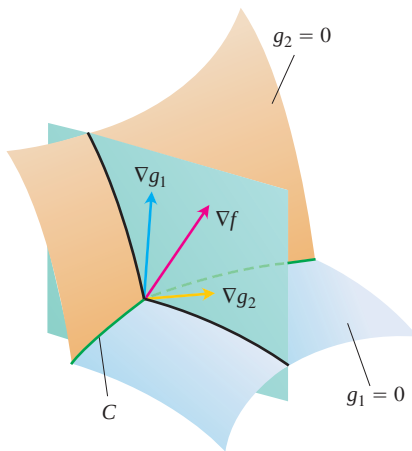


FIGURE 14.56 The vectors ∇g_1 and ∇g_2 lie in a plane perpendicular to the curve C because ∇g_1 is normal to the surface $g_1 = 0$ and ∇g_2 is normal to the surface $g_2 = 0$.

These equations tell us, among other things, that x and y have the same sign. With these values for x and y , the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \quad 9 + 16 = 4\lambda^2, \quad 4\lambda^2 = 25, \quad \text{and} \quad \lambda = \pm \frac{5}{2}.$$

Thus,

$$x = \frac{3}{2\lambda} = \pm \frac{3}{5}, \quad y = \frac{2}{\lambda} = \pm \frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \quad \text{and} \quad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

The Geometry of the Solution

The level curves of $f(x, y) = 3x + 4y$ are the lines $3x + 4y = c$ (Figure 14.55). The farther the lines lie from the origin, the larger the absolute value of f . We want to find the extreme values of $f(x, y)$ given that the point (x, y) also lies on the circle $x^2 + y^2 = 1$. Which lines intersecting the circle lie farthest from the origin? The lines tangent to the circle are farthest. At the points of tangency, any vector normal to the line is normal to the circle, so the gradient $\nabla f = 3\mathbf{i} + 4\mathbf{j}$ is a multiple ($\lambda = \pm 5/2$) of the gradient $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$. At the point $(3/5, 4/5)$, for example,

$$\nabla f = 3\mathbf{i} + 4\mathbf{j}, \quad \nabla g = \frac{6}{5}\mathbf{i} + \frac{8}{5}\mathbf{j}, \quad \text{and} \quad \nabla f = \frac{5}{2}\nabla g. \quad \blacksquare$$

Lagrange Multipliers with Two Constraints

Many problems require us to find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0 \quad \text{and} \quad g_2(x, y, z) = 0$$

and g_1 and g_2 are differentiable, with ∇g_1 not parallel to ∇g_2 , we find the constrained local maxima and minima of f by introducing two Lagrange multipliers λ and μ (mu, pronounced “mew”). That is, we locate the points $P(x, y, z)$ where f takes on its constrained extreme values by finding the values of x, y, z, λ , and μ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0 \quad (2)$$

Equations (2) have a nice geometric interpretation. The surfaces $g_1 = 0$ and $g_2 = 0$ (usually) intersect in a smooth curve, say C (Figure 14.56). Along this curve we seek the points where f has local maximum and minimum values relative to its other values on the curve.

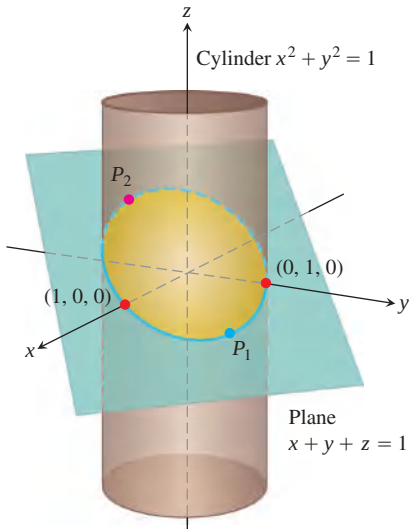


FIGURE 14.57 On the ellipse where the plane and cylinder meet, what are the points closest to and farthest from the origin? (Example 5)

These are the points where ∇f is normal to C , as we saw in Theorem 12. But ∇g_1 and ∇g_2 are also normal to C at these points because C lies in the surfaces $g_1 = 0$ and $g_2 = 0$. Therefore, ∇f lies in the plane determined by ∇g_1 and ∇g_2 , which means that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ for some λ and μ . Since the points we seek also lie in both surfaces, their coordinates must satisfy the equations $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, which are the remaining requirements in Equations (2).

EXAMPLE 5 Finding Extremes of Distance on an Ellipse

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse (Figure 14.57). Find the points on the ellipse that lie closest to and farthest from the origin.

Solution We find the extreme values of

$$f(x, y, z) = x^2 + y^2 + z^2$$

(the square of the distance from (x, y, z) to the origin) subject to the constraints

$$g_1(x, y, z) = x^2 + y^2 - 1 = 0 \quad (3)$$

$$g_2(x, y, z) = x + y + z - 1 = 0. \quad (4)$$

The gradient equation in Equations (2) then gives

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = (2\lambda x + \mu)\mathbf{i} + (2\lambda y + \mu)\mathbf{j} + \mu\mathbf{k}$$

or

$$2x = 2\lambda x + \mu, \quad 2y = 2\lambda y + \mu, \quad 2z = \mu. \quad (5)$$

The scalar equations in Equations (5) yield

$$\begin{aligned} 2x &= 2\lambda x + 2z \Rightarrow (1 - \lambda)x = z, \\ 2y &= 2\lambda y + 2z \Rightarrow (1 - \lambda)y = z. \end{aligned} \quad (6)$$

Equations (6) are satisfied simultaneously if either $\lambda = 1$ and $z = 0$ or $\lambda \neq 1$ and $x = y = z/(1 - \lambda)$.

If $z = 0$, then solving Equations (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points $(1, 0, 0)$ and $(0, 1, 0)$. This makes sense when you look at Figure 14.57.

If $x = y$, then Equations (3) and (4) give

$$\begin{aligned} x^2 + x^2 - 1 &= 0 & x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm \frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2} \right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2} \right).$$



Here we need to be careful, however. Although P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1 .

The points on the ellipse closest to the origin are $(1, 0, 0)$ and $(0, 1, 0)$. The point on the ellipse farthest from the origin is P_2 . ■

EXERCISES 14.8

Two Independent Variables with One Constraint



1. **Extrema on an ellipse** Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ as its extreme values.
2. **Extrema on a circle** Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.
3. **Maximum on a line** Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.
4. **Extrema on a line** Find the local extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.
5. **Constrained minimum** Find the points on the curve $xy^2 = 54$ nearest the origin.
6. **Constrained minimum** Find the points on the curve $x^2y = 2$ nearest the origin.
7. Use the method of Lagrange multipliers to find
 - a. **Minimum on a hyperbola** The minimum value of $x + y$, subject to the constraints $xy = 16$, $x > 0$, $y > 0$
 - b. **Maximum on a line** The maximum value of xy , subject to the constraint $x + y = 16$.

Comment on the geometry of each solution.
8. **Extrema on a curve** Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy -plane that are nearest to and farthest from the origin.
9. **Minimum surface area with fixed volume** Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is $16\pi \text{ cm}^3$.
10. **Cylinder in a sphere** Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a . What is the largest surface area?
11. **Rectangle of greatest area in an ellipse** Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16 + y^2/9 = 1$ with sides parallel to the coordinate axes.
12. **Rectangle of longest perimeter in an ellipse** Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What is the largest perimeter?
13. **Extrema on a circle** Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.
14. **Extrema on a circle** Find the maximum and minimum values of $3x - y + 6$ subject to the constraint $x^2 + y^2 = 4$.

15. **Ant on a metal plate** The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?
16. **Cheapest storage tank** Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold 8000 m^3 of gas. The customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Three Independent Variables with One Constraint



17. **Minimum distance to a point** Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.
18. **Maximum distance to a point** Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1, -1, 1)$.
19. **Minimum distance to the origin** Find the minimum distance from the surface $x^2 + y^2 - z^2 = 1$ to the origin.
20. **Minimum distance to the origin** Find the point on the surface $z = xy + 1$ nearest the origin.
21. **Minimum distance to the origin** Find the points on the surface $z^2 = xy + 4$ closest to the origin.
22. **Minimum distance to the origin** Find the point(s) on the surface $xyz = 1$ closest to the origin.
23. **Extrema on a sphere** Find the maximum and minimum values of

$$f(x, y, z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$.

24. **Extrema on a sphere** Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.
25. **Minimizing a sum of squares** Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.
26. **Maximizing a product** Find the largest product the positive numbers x, y , and z can have if $x + y + z^2 = 16$.
27. **Rectangular box of longest volume in a sphere** Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.



- 28. Box with vertex on a plane** Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane $x/a + y/b + z/c = 1$, where $a > 0$, $b > 0$, and $c > 0$.
- 29. Hottest point on a space probe** A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

- 30. Extreme temperatures on a sphere** Suppose that the Celsius temperature at the point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ is $T = 400xyz^2$. Locate the highest and lowest temperatures on the sphere.
- 31. Maximizing a utility function: an example from economics** In economics, the usefulness or *utility* of amounts x and y of two capital goods G_1 and G_2 is sometimes measured by a function $U(x, y)$. For example, G_1 and G_2 might be two chemicals a pharmaceutical company needs to have on hand and $U(x, y)$ the gain from manufacturing a product whose synthesis requires different amounts of the chemicals depending on the process used. If G_1 costs a dollars per kilogram, G_2 costs b dollars per kilogram, and the total amount allocated for the purchase of G_1 and G_2 together is c dollars, then the company's managers want to maximize $U(x, y)$ given that $ax + by = c$. Thus, they need to solve a typical Lagrange multiplier problem.

Suppose that

$$U(x, y) = xy + 2x$$

and that the equation $ax + by = c$ simplifies to

$$2x + y = 30.$$

Find the maximum value of U and the corresponding values of x and y subject to this latter constraint.

- 32. Locating a radio telescope** You are in charge of erecting a radio telescope on a newly discovered planet. To minimize interference, you want to place it where the magnetic field of the planet is weakest. The planet is spherical, with a radius of 6 units. Based on a coordinate system whose origin is at the center of the planet, the strength of the magnetic field is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Where should you locate the radio telescope?

Extreme Values Subject to Two Constraints



- 33.** Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.
- 34.** Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.



- 35. Minimum distance to the origin** Find the point closest to the origin on the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$.
- 36. Maximum value on line of intersection** Find the maximum value that $f(x, y, z) = x^2 + 2y - z^2$ can have on the line of intersection of the planes $2x - y = 0$ and $y + z = 0$.
- 37. Extrema on a curve of intersection** Find the extreme values of $f(x, y, z) = x^2yz + 1$ on the intersection of the plane $z = 1$ with the sphere $x^2 + y^2 + z^2 = 10$.
- 38. a. Maximum on line of intersection** Find the maximum value of $w = xyz$ on the line of intersection of the two planes $x + y + z = 40$ and $x + y - z = 0$.
- b.** Give a geometric argument to support your claim that you have found a maximum, and not a minimum, value of w .
- 39. Extrema on a circle of intersection** Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.
- 40. Minimum distance to the origin** Find the point closest to the origin on the curve of intersection of the plane $2y + 4z = 5$ and the cone $z^2 = 4x^2 + 4y^2$.

Theory and Examples

- 41. The condition $\nabla f = \lambda \nabla g$ is not sufficient** Although $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x, y)$ subject to the condition $g(x, y) = 0$, it does not in itself guarantee that one exists. As a case in point, try using the method of Lagrange multipliers to find a maximum value of $f(x, y) = x + y$ subject to the constraint that $xy = 16$. The method will identify the two points $(4, 4)$ and $(-4, -4)$ as candidates for the location of extreme values. Yet the sum $(x + y)$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x, y) = x + y$ becomes.
- 42. A least squares plane** The plane $z = Ax + By + C$ is to be "fitted" to the following points (x_k, y_k, z_k) :

$$(0, 0, 0), \quad (0, 1, 1), \quad (1, 1, 1), \quad (1, 0, -1).$$

Find the values of A , B , and C that minimize

$$\sum_{k=1}^4 (Ax_k + By_k + C - z_k)^2,$$

the sum of the squares of the deviations.

- 43. a. Maximum on a sphere** Show that the maximum value of $a^2b^2c^2$ on a sphere of radius r centered at the origin of a Cartesian abc -coordinate system is $(r^2/3)^3$.
- b. Geometric and arithmetic means** Using part (a), show that for nonnegative numbers a , b , and c ,

$$(abc)^{1/3} \leq \frac{a + b + c}{3};$$

that is, the *geometric mean* of three nonnegative numbers is less than or equal to their *arithmetic mean*.

- 44. Sum of products** Let $a_1, a_2, \dots, a_n$ be n positive numbers. Find the maximum of $\sum_{i=1}^n a_i x_i$ subject to the constraint $\sum_{i=1}^n x_i^2 = 1$.

COMPUTER EXPLORATIONS

Implementing the Method of Lagrange Multipliers

In Exercises 45–50, use a CAS to perform the following steps implementing the method of Lagrange multipliers for finding constrained extrema:

- a. Form the function $h = f - \lambda_1 g_1 - \lambda_2 g_2$, where f is the function to optimize subject to the constraints $g_1 = 0$ and $g_2 = 0$.
 - b. Determine all the first partial derivatives of h , including the partials with respect to λ_1 and λ_2 , and set them equal to 0.
 - c. Solve the system of equations found in part (b) for all the unknowns, including λ_1 and λ_2 .
 - d. Evaluate f at each of the solution points found in part (c) and select the extreme value subject to the constraints asked for in the exercise.
45. Minimize $f(x, y, z) = xy + yz$ subject to the constraints $x^2 + y^2 - 2 = 0$ and $x^2 + z^2 - 2 = 0$.
 46. Minimize $f(x, y, z) = xyz$ subject to the constraints $x^2 + y^2 - 1 = 0$ and $x - z = 0$.
 47. Maximize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $2y + 4z - 5 = 0$ and $4x^2 + 4y^2 - z^2 = 0$.
 48. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x^2 - xy + y^2 - z^2 - 1 = 0$ and $x^2 + y^2 - 1 = 0$.
 49. Minimize $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ subject to the constraints $2x - y + z - w - 1 = 0$ and $x + y - z + w - 1 = 0$.
 50. Determine the distance from the line $y = x + 1$ to the parabola $y^2 = x$. (*Hint:* Let (x, y) be a point on the line and (w, z) a point on the parabola. You want to minimize $(x - w)^2 + (y - z)^2$.)

14.9

Partial Derivatives with Constrained Variables

In finding partial derivatives of functions like $w = f(x, y)$, we have assumed x and y to be independent. In many applications, however, this is not the case. For example, the internal energy U of a gas may be expressed as a function $U = f(P, V, T)$ of pressure P , volume V , and temperature T . If the individual molecules of the gas do not interact, however, P , V , and T obey (and are constrained by) the ideal gas law

$$PV = nRT \quad (n \text{ and } R \text{ constant}),$$

and fail to be independent. In this section we learn how to find partial derivatives in situations like this, which you may encounter in studying economics, engineering, or physics.†

Decide Which Variables Are Dependent and Which Are Independent

If the variables in a function $w = f(x, y, z)$ are constrained by a relation like the one imposed on x , y , and z by the equation $z = x^2 + y^2$, the geometric meanings and the numerical values of the partial derivatives of f will depend on which variables are chosen to be dependent and which are chosen to be independent. To see how this choice can affect the outcome, we consider the calculation of $\partial w / \partial x$ when $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$.



EXAMPLE 1 Finding a Partial Derivative with Constrained Independent Variables

Find $\partial w / \partial x$ if $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$.

†This section is based on notes written for MIT by Arthur P. Mattuck.

Solution We are given two equations in the four unknowns x , y , z , and w . Like many such systems, this one can be solved for two of the unknowns (the dependent variables) in terms of the others (the independent variables). In being asked for $\partial w/\partial x$, we are told that w is to be a dependent variable and x an independent variable. The possible choices for the other variables come down to

Dependent	Independent
w, z	x, y
w, y	x, z

In either case, we can express w explicitly in terms of the selected independent variables. We do this by using the second equation $z = x^2 + y^2$ to eliminate the remaining dependent variable in the first equation.

In the first case, the remaining dependent variable is z . We eliminate it from the first equation by replacing it by $x^2 + y^2$. The resulting expression for w is

$$\begin{aligned} w &= x^2 + y^2 + z^2 = x^2 + y^2 + (x^2 + y^2)^2 \\ &= x^2 + y^2 + x^4 + 2x^2y^2 + y^4 \end{aligned}$$

and

$$\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2. \quad (1)$$

This is the formula for $\partial w/\partial x$ when x and y are the independent variables.

In the second case, where the independent variables are x and z and the remaining dependent variable is y , we eliminate the dependent variable y in the expression for w by replacing y^2 in the second equation by $z - x^2$. This gives

$$w = x^2 + y^2 + z^2 = x^2 + (z - x^2) + z^2 = z + z^2$$

and

$$\frac{\partial w}{\partial x} = 0. \quad (2)$$

This is the formula for $\partial w/\partial x$ when x and z are the independent variables.

The formulas for $\partial w/\partial x$ in Equations (1) and (2) are genuinely different. We cannot change either formula into the other by using the relation $z = x^2 + y^2$. There is not just one $\partial w/\partial x$, there are two, and we see that the original instruction to find $\partial w/\partial x$ was incomplete. *Which $\partial w/\partial x$?* we ask.

The geometric interpretations of Equations (1) and (2) help to explain why the equations differ. The function $w = x^2 + y^2 + z^2$ measures the square of the distance from the point (x, y, z) to the origin. The condition $z = x^2 + y^2$ says that the point (x, y, z) lies on the paraboloid of revolution shown in Figure 14.58. What does it mean to calculate $\partial w/\partial x$ at a point $P(x, y, z)$ that can move only on this surface? What is the value of $\partial w/\partial x$ when the coordinates of P are, say, $(1, 0, 1)$?

If we take x and y to be independent, then we find $\partial w/\partial x$ by holding y fixed (at $y = 0$ in this case) and letting x vary. Hence, P moves along the parabola $z = x^2$ in the xz -plane. As P moves on this parabola, w , which is the square of the distance from P to the origin, changes. We calculate $\partial w/\partial x$ in this case (our first solution above) to be

$$\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2.$$

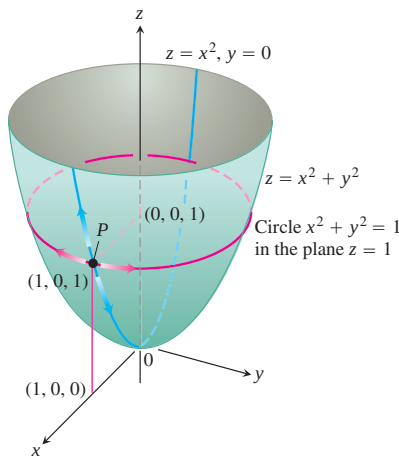


FIGURE 14.58 If P is constrained to lie on the paraboloid $z = x^2 + y^2$, the value of the partial derivative of $w = x^2 + y^2 + z^2$ with respect to x at P depends on the direction of motion (Example 1). (1) As x changes, with $y = 0$, P moves up or down the surface on the parabola $z = x^2$ in the xz -plane with $\partial w/\partial x = 2x + 4x^3$. (2) As x changes, with $z = 1$, P moves on the circle $x^2 + y^2 = 1$, $z = 1$, and $\partial w/\partial x = 0$.

At the point $P(1, 0, 1)$, the value of this derivative is

$$\frac{\partial w}{\partial x} = 2 + 4 + 0 = 6.$$

If we take x and z to be independent, then we find $\partial w/\partial x$ by holding z fixed while x varies. Since the z -coordinate of P is 1, varying x moves P along a circle in the plane $z = 1$. As P moves along this circle, its distance from the origin remains constant, and w , being the square of this distance, does not change. That is,

$$\frac{\partial w}{\partial x} = 0,$$

as we found in our second solution. ■

How to Find $\partial w/\partial x$ When the Variables in $w = f(x, y, z)$ Are Constrained by Another Equation

As we saw in Example 1, a typical routine for finding $\partial w/\partial x$ when the variables in the function $w = f(x, y, z)$ are related by another equation has three steps. These steps apply to finding $\partial w/\partial y$ and $\partial w/\partial z$ as well.

1. *Decide* which variables are to be dependent and which are to be independent. (In practice, the decision is based on the physical or theoretical context of our work. In the exercises at the end of this section, we say which variables are which.)
2. *Eliminate* the other dependent variable(s) in the expression for w .
3. *Differentiate* as usual.

If we cannot carry out Step 2 after deciding which variables are dependent, we differentiate the equations as they are and try to solve for $\partial w/\partial x$ afterward. The next example shows how this is done.



EXAMPLE 2 Finding a Partial Derivative with Identified Constrained Independent Variables

Find $\partial w/\partial x$ at the point $(x, y, z) = (2, -1, 1)$ if

$$w = x^2 + y^2 + z^2, \quad z^3 - xy + yz + y^3 = 1,$$

and x and y are the independent variables.

Solution It is not convenient to eliminate z in the expression for w . We therefore differentiate both equations implicitly with respect to x , treating x and y as independent variables and w and z as dependent variables. This gives

$$\frac{\partial w}{\partial x} = 2x + 2z \frac{\partial z}{\partial x} \tag{3}$$

and

$$3z^2 \frac{\partial z}{\partial x} - y + y \frac{\partial z}{\partial x} + 0 = 0. \quad (4)$$

These equations may now be combined to express $\partial w/\partial x$ in terms of x , y , and z . We solve Equation (4) for $\partial z/\partial x$ to get

$$\frac{\partial z}{\partial x} = \frac{y}{y + 3z^2}$$

and substitute into Equation (3) to get

$$\frac{\partial w}{\partial x} = 2x + \frac{2yz}{y + 3z^2}.$$

The value of this derivative at $(x, y, z) = (2, -1, 1)$ is

$$\left(\frac{\partial w}{\partial x}\right)_{(2,-1,1)} = 2(2) + \frac{2(-1)(1)}{-1 + 3(1)^2} = 4 + \frac{-2}{2} = 3. \quad \blacksquare$$

HISTORICAL BIOGRAPHY

Sonya Kovalevsky
(1850–1891)



Notation

To show what variables are assumed to be independent in calculating a derivative, we can use the following notation:

$$\left(\frac{\partial w}{\partial x}\right)_y \quad \partial w/\partial x \text{ with } x \text{ and } y \text{ independent}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,t} \quad \partial f/\partial y \text{ with } y, x \text{ and } t \text{ independent}$$

EXAMPLE 3 Finding a Partial Derivative with Constrained Variables Notationally Identified

Find $(\partial w/\partial x)_{y,z}$ if $w = x^2 + y - z + \sin t$ and $x + y = t$.

Solution With x, y, z independent, we have

$$t = x + y, \quad w = x^2 + y - z + \sin(x + y)$$

$$\begin{aligned} \left(\frac{\partial w}{\partial x}\right)_{y,z} &= 2x + 0 - 0 + \cos(x + y) \frac{\partial}{\partial x}(x + y) \\ &= 2x + \cos(x + y). \end{aligned} \quad \blacksquare$$

Arrow Diagrams

In solving problems like the one in Example 3, it often helps to start with an arrow diagram that shows how the variables and functions are related. If

$$w = x^2 + y - z + \sin t \quad \text{and} \quad x + y = t$$

and we are asked to find $\partial w/\partial x$ when x, y , and z are independent, the appropriate diagram is one like this:

$$\begin{array}{ccc} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \rightarrow & \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \rightarrow w \\ \text{Independent} & & \text{Intermediate} \quad \text{Dependent} \\ \text{variables} & & \text{variables} \quad \text{variable} \end{array} \quad (5)$$

To avoid confusion between the independent and intermediate variables with the same symbolic names in the diagram, it is helpful to rename the intermediate variables (so they are seen as *functions* of the independent variables). Thus, let $u = x$, $v = y$, and $s = z$ denote the renamed intermediate variables. With this notation, the arrow diagram becomes

$$\begin{array}{ccc} \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \rightarrow & \begin{pmatrix} u \\ v \\ s \\ t \end{pmatrix} \rightarrow w \\ \text{Independent} & & \text{Intermediate} \quad \text{Dependent} \\ \text{variables} & & \text{variables and} \quad \text{variable} \\ & & \text{relations} \\ & & u = x \\ & & v = y \\ & & s = z \\ & & t = x + y \end{array} \quad (6)$$

The diagram shows the independent variables on the left, the intermediate variables and their relation to the independent variables in the middle, and the dependent variable on the right. The function w now becomes

$$w = u^2 + v - s + \sin t,$$

where

$$u = x, \quad v = y, \quad s = z, \quad \text{and} \quad t = x + y.$$

To find $\partial w/\partial x$, we apply the four-variable form of the Chain Rule to w , guided by the arrow diagram in Equation (6):

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial w}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial x} \\ &= (2u)(1) + (1)(0) + (-1)(0) + (\cos t)(1) \\ &= 2u + \cos t \\ &= 2x + \cos(x + y). \end{aligned} \quad \begin{array}{l} \text{Substituting the original independent} \\ \text{variables } u = x \text{ and } t = x + y. \end{array}$$

EXERCISES 14.9

Finding Partial Derivatives with Constrained Variables

In Exercises 1–3, begin by drawing a diagram that shows the relations among the variables.

1. If $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$, find

a. $\left(\frac{\partial w}{\partial y}\right)_z$ b. $\left(\frac{\partial w}{\partial z}\right)_x$ c. $\left(\frac{\partial w}{\partial z}\right)_y$.

2. If $w = x^2 + y - z + \sin t$ and $x + y = t$, find

a. $\left(\frac{\partial w}{\partial y}\right)_{x,z}$ b. $\left(\frac{\partial w}{\partial y}\right)_{z,t}$ c. $\left(\frac{\partial w}{\partial z}\right)_{x,y}$
d. $\left(\frac{\partial w}{\partial z}\right)_{y,t}$ e. $\left(\frac{\partial w}{\partial t}\right)_{x,z}$ f. $\left(\frac{\partial w}{\partial t}\right)_{y,z}$.





3. Let $U = f(P, V, T)$ be the internal energy of a gas that obeys the ideal gas law $PV = nRT$ (n and R constant). Find

a. $\left(\frac{\partial U}{\partial P}\right)_V$ b. $\left(\frac{\partial U}{\partial T}\right)_V$.

4. Find

a. $\left(\frac{\partial w}{\partial x}\right)_y$ b. $\left(\frac{\partial w}{\partial z}\right)_y$

at the point $(x, y, z) = (0, 1, \pi)$ if

$$w = x^2 + y^2 + z^2 \quad \text{and} \quad y \sin z + z \sin x = 0.$$

5. Find

a. $\left(\frac{\partial w}{\partial y}\right)_x$ b. $\left(\frac{\partial w}{\partial y}\right)_z$

at the point $(w, x, y, z) = (4, 2, 1, -1)$ if

$$w = x^2 y^2 + yz - z^3 \quad \text{and} \quad x^2 + y^2 + z^2 = 6.$$

6. Find $(\partial u / \partial y)_x$ at the point $(u, v) = (\sqrt{2}, 1)$, if $x = u^2 + v^2$ and $y = uv$.

7. Suppose that $x^2 + y^2 = r^2$ and $x = r \cos \theta$, as in polar coordinates. Find

$$\left(\frac{\partial x}{\partial r}\right)_\theta \quad \text{and} \quad \left(\frac{\partial r}{\partial x}\right)_y.$$

8. Suppose that

$$w = x^2 - y^2 + 4z + t \quad \text{and} \quad x + 2z + t = 25.$$

Show that the equations

$$\frac{\partial w}{\partial x} = 2x - 1 \quad \text{and} \quad \frac{\partial w}{\partial x} = 2x - 2$$

each give $\partial w / \partial x$, depending on which variables are chosen to be dependent and which variables are chosen to be independent. Identify the independent variables in each case.

Partial Derivatives Without Specific Formulas

9. Establish the fact, widely used in hydrodynamics, that if $f(x, y, z) = 0$, then

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

(Hint: Express all the derivatives in terms of the formal partial derivatives $\partial f / \partial x$, $\partial f / \partial y$, and $\partial f / \partial z$.)

10. If $z = x + f(u)$, where $u = xy$, show that

$$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = x.$$

11. Suppose that the equation $g(x, y, z) = 0$ determines z as a differentiable function of the independent variables x and y and that $g_z \neq 0$. Show that

$$\left(\frac{\partial z}{\partial y}\right)_x = -\frac{\partial g / \partial y}{\partial g / \partial z}.$$

12. Suppose that $f(x, y, z, w) = 0$ and $g(x, y, z, w) = 0$ determine z and w as differentiable functions of the independent variables x and y , and suppose that

$$\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z} \neq 0.$$

Show that

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial f}{\partial x} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial x}}{\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z}}$$

and

$$\left(\frac{\partial w}{\partial y}\right)_x = -\frac{\frac{\partial f}{\partial z} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial z}}{\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z}}.$$

14.10 Taylor's Formula for Two Variables

This section uses Taylor's formula to derive the Second Derivative Test for local extreme values (Section 14.7) and the error formula for linearizations of functions of two independent variables (Section 14.6). The use of Taylor's formula in these derivations leads to an extension of the formula that provides polynomial approximations of all orders for functions of two independent variables.

Derivation of the Second Derivative Test

Let $f(x, y)$ have continuous partial derivatives in an open region R containing a point $P(a, b)$ where $f_x = f_y = 0$ (Figure 14.59). Let h and k be increments small enough to put the

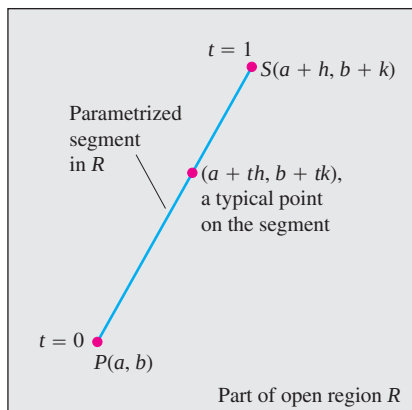


FIGURE 14.59 We begin the derivation of the second derivative test at $P(a, b)$ by parametrizing a typical line segment from P to a point S nearby.

point $S(a + h, b + k)$ and the line segment joining it to P inside R . We parametrize the segment PS as

$$x = a + th, \quad y = b + tk, \quad 0 \leq t \leq 1.$$

If $F(t) = f(a + th, b + tk)$, the Chain Rule gives

$$F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

Since f_x and f_y are differentiable (they have continuous partial derivatives), F' is a differentiable function of t and

$$\begin{aligned} F'' &= \frac{\partial F'}{\partial x} \frac{dx}{dt} + \frac{\partial F'}{\partial y} \frac{dy}{dt} = \frac{\partial}{\partial x} (hf_x + kf_y) \cdot h + \frac{\partial}{\partial y} (hf_x + kf_y) \cdot k \\ &= h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}. \end{aligned} \quad f_{xy} = f_{yx}$$

Since F and F' are continuous on $[0, 1]$ and F' is differentiable on $(0, 1)$, we can apply Taylor's formula with $n = 2$ and $a = 0$ to obtain

$$\begin{aligned} F(1) &= F(0) + F'(0)(1 - 0) + F''(c) \frac{(1 - 0)^2}{2} \\ F(1) &= F(0) + F'(0) + \frac{1}{2} F''(c) \end{aligned} \quad (1)$$

for some c between 0 and 1. Writing Equation (1) in terms of f gives

$$\begin{aligned} f(a + h, b + k) &= f(a, b) + hf_x(a, b) + kf_y(a, b) \\ &\quad + \frac{1}{2} (h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) \Big|_{(a+ch, b+ck)}. \end{aligned} \quad (2)$$

Since $f_x(a, b) = f_y(a, b) = 0$, this reduces to

$$f(a + h, b + k) - f(a, b) = \frac{1}{2} (h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) \Big|_{(a+ch, b+ck)}. \quad (3)$$

The presence of an extremum of f at (a, b) is determined by the sign of $f(a + h, b + k) - f(a, b)$. By Equation (3), this is the same as the sign of

$$Q(c) = (h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) \Big|_{(a+ch, b+ck)}.$$

Now, if $Q(0) \neq 0$, the sign of $Q(c)$ will be the same as the sign of $Q(0)$ for sufficiently small values of h and k . We can predict the sign of

$$Q(0) = h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b) \quad (4)$$

from the signs of f_{xx} and $f_{xx}f_{yy} - f_{xy}^2$ at (a, b) . Multiply both sides of Equation (4) by f_{xx} and rearrange the right-hand side to get

$$f_{xx}Q(0) = (hf_{xx} + kf_{xy})^2 + (f_{xx}f_{yy} - f_{xy}^2)k^2. \quad (5)$$

From Equation (5) we see that

1. If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) , then $Q(0) < 0$ for all sufficiently small nonzero values of h and k , and f has a *local maximum* value at (a, b) .
2. If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) , then $Q(0) > 0$ for all sufficiently small nonzero values of h and k and f has a *local minimum* value at (a, b) .

3. If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) , there are combinations of arbitrarily small nonzero values of h and k for which $Q(0) > 0$, and other values for which $Q(0) < 0$. Arbitrarily close to the point $P_0(a, b, f(a, b))$ on the surface $z = f(x, y)$ there are points above P_0 and points below P_0 , so f has a *saddle point* at (a, b) .
4. If $f_{xx}f_{yy} - f_{xy}^2 = 0$, another test is needed. The possibility that $Q(0)$ equals zero prevents us from drawing conclusions about the sign of $Q(c)$.

The Error Formula for Linear Approximations

We want to show that the difference $E(x, y)$, between the values of a function $f(x, y)$, and its linearization $L(x, y)$ at (x_0, y_0) satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2} M(|x - x_0| + |y - y_0|)^2.$$

The function f is assumed to have continuous second partial derivatives throughout an open set containing a closed rectangular region R centered at (x_0, y_0) . The number M is an upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R .

The inequality we want comes from Equation (2). We substitute x_0 and y_0 for a and b , and $x - x_0$ and $y - y_0$ for h and k , respectively, and rearrange the result as

$$\begin{aligned} f(x, y) &= \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{\text{linearization } L(x, y)} \\ &\quad + \underbrace{\frac{1}{2} \left((x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0) f_{xy} + (y - y_0)^2 f_{yy} \right)}_{\text{error } E(x, y)} \bigg|_{(x_0 + c(x - x_0), y_0 + c(y - y_0))}. \end{aligned}$$

This equation reveals that

$$|E| \leq \frac{1}{2} (|x - x_0|^2 |f_{xx}| + 2|x - x_0||y - y_0| |f_{xy}| + |y - y_0|^2 |f_{yy}|).$$

Hence, if M is an upper bound for the values of $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$ on R ,

$$\begin{aligned} |E| &\leq \frac{1}{2} (|x - x_0|^2 M + 2|x - x_0||y - y_0| M + |y - y_0|^2 M) \\ &= \frac{1}{2} M(|x - x_0| + |y - y_0|)^2. \end{aligned}$$

Taylor's Formula for Functions of Two Variables

The formulas derived earlier for F' and F'' can be obtained by applying to $f(x, y)$ the operators

$$\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) \quad \text{and} \quad \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 = h^2 \frac{\partial^2}{\partial x^2} + 2hk \frac{\partial^2}{\partial x \partial y} + k^2 \frac{\partial^2}{\partial y^2}.$$

These are the first two instances of a more general formula,

$$F^{(n)}(t) = \frac{d^n}{dt^n} F(t) = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x, y), \quad (6)$$

which says that applying d^n/dt^n to $F(t)$ gives the same result as applying the operator

$$\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^n$$

to $f(x, y)$ after expanding it by the Binomial Theorem.

If partial derivatives of f through order $n + 1$ are continuous throughout a rectangular region centered at (a, b) , we may extend the Taylor formula for $F(t)$ to

$$F(t) = F(0) + F'(0)t + \frac{F''(0)}{2!}t^2 + \cdots + \frac{F^{(n)}(0)}{n!}t^n + \text{remainder},$$

and take $t = 1$ to obtain

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \cdots + \frac{F^{(n)}(0)}{n!} + \text{remainder}.$$

When we replace the first n derivatives on the right of this last series by their equivalent expressions from Equation (6) evaluated at $t = 0$ and add the appropriate remainder term, we arrive at the following formula.

Taylor's Formula for $f(x, y)$ at the Point (a, b)

Suppose $f(x, y)$ and its partial derivatives through order $n + 1$ are continuous throughout an open rectangular region R centered at a point (a, b) . Then, throughout R ,

$$\begin{aligned} f(a + h, b + k) &= f(a, b) + (hf_x + kf_y)|_{(a,b)} + \frac{1}{2!}(h^2f_{xx} + 2hkf_{xy} + k^2f_{yy})|_{(a,b)} \\ &\quad + \frac{1}{3!}(h^3f_{xxx} + 3h^2kf_{xxy} + 3hk^2f_{xyy} + k^3f_{yyy})|_{(a,b)} + \cdots + \frac{1}{n!}\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^n f \Big|_{(a,b)} \\ &\quad + \frac{1}{(n+1)!}\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^{n+1} f \Big|_{(a+ch, b+ck)}. \end{aligned} \quad (7)$$

The first n derivative terms are evaluated at (a, b) . The last term is evaluated at some point $(a + ch, b + ck)$ on the line segment joining (a, b) and $(a + h, b + k)$.

If $(a, b) = (0, 0)$ and we treat h and k as independent variables (denoting them now by x and y), then Equation (7) assumes the following simpler form.

Taylor's Formula for $f(x, y)$ at the Origin

$$\begin{aligned} f(x, y) &= f(0, 0) + xf_x + yf_y + \frac{1}{2!}(x^2f_{xx} + 2xyf_{xy} + y^2f_{yy}) \\ &\quad + \frac{1}{3!}(x^3f_{xxx} + 3x^2yf_{xxy} + 3xy^2f_{xyy} + y^3f_{yyy}) + \cdots + \frac{1}{n!}\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)^n f \\ &\quad + \frac{1}{(n+1)!}\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)^{n+1} f \Big|_{(cx, cy)} \end{aligned} \quad (8)$$

The first n derivative terms are evaluated at $(0, 0)$. The last term is evaluated at a point on the line segment joining the origin and (x, y) .

Taylor's formula provides polynomial approximations of two-variable functions. The first n derivative terms give the polynomial; the last term gives the approximation error. The first three terms of Taylor's formula give the function's linearization. To improve on the linearization, we add higher power terms.



EXAMPLE 1 Finding a Quadratic Approximation

Find a quadratic approximation to $f(x, y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$?

Solution We take $n = 2$ in Equation (8):

$$\begin{aligned} f(x, y) &= f(0, 0) + (xf_x + yf_y) + \frac{1}{2}(x^2f_{xx} + 2xyf_{xy} + y^2f_{yy}) \\ &\quad + \frac{1}{6}(x^3f_{xxx} + 3x^2yf_{xxy} + 3xy^2f_{xyy} + y^3f_{yyy})_{(cx,cy)} \end{aligned}$$

with

$$f(0, 0) = \sin x \sin y|_{(0,0)} = 0, \quad f_{xx}(0, 0) = -\sin x \sin y|_{(0,0)} = 0,$$

$$f_x(0, 0) = \cos x \sin y|_{(0,0)} = 0, \quad f_{xy}(0, 0) = \cos x \cos y|_{(0,0)} = 1,$$

$$f_y(0, 0) = \sin x \cos y|_{(0,0)} = 0, \quad f_{yy}(0, 0) = -\sin x \sin y|_{(0,0)} = 0,$$

we have

$$\sin x \sin y \approx 0 + 0 + 0 + \frac{1}{2}(x^2(0) + 2xy(1) + y^2(0)),$$

$$\sin x \sin y \approx xy.$$

The error in the approximation is

$$E(x, y) = \frac{1}{6}(x^3f_{xxx} + 3x^2yf_{xxy} + 3xy^2f_{xyy} + y^3f_{yyy})_{(cx,cy)}.$$

The third derivatives never exceed 1 in absolute value because they are products of sines and cosines. Also, $|x| \leq 0.1$ and $|y| \leq 0.1$. Hence

$$|E(x, y)| \leq \frac{1}{6}((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3) = \frac{8}{6}(0.1)^3 \leq 0.00134$$

(rounded up). The error will not exceed 0.00134 if $|x| \leq 0.1$ and $|y| \leq 0.1$. ■

EXERCISES 14.10

Finding Quadratic and Cubic Approximations

In Exercises 1–10, use Taylor's formula for $f(x, y)$ at the origin to find quadratic and cubic approximations of f near the origin.

1. $f(x, y) = xe^y$

2. $f(x, y) = e^x \cos y$

3. $f(x, y) = y \sin x$

4. $f(x, y) = \sin x \cos y$

5. $f(x, y) = e^x \ln(1 + y)$

6. $f(x, y) = \ln(2x + y + 1)$

7. $f(x, y) = \sin(x^2 + y^2)$

8. $f(x, y) = \cos(x^2 + y^2)$





9. $f(x, y) = \frac{1}{1 - x - y}$

10. $f(x, y) = \frac{1}{1 - x - y + xy}$

11. Use Taylor's formula to find a quadratic approximation of $f(x, y) = \cos x \cos y$ at the origin. Estimate the error in the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.



12. Use Taylor's formula to find a quadratic approximation of $e^x \sin y$ at the origin. Estimate the error in the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

Chapter 14

Questions to Guide Your Review

1. What is a real-valued function of two independent variables? Three independent variables? Give examples.
2. What does it mean for sets in the plane or in space to be open? Closed? Give examples. Give examples of sets that are neither open nor closed.
3. How can you display the values of a function $f(x, y)$ of two independent variables graphically? How do you do the same for a function $f(x, y, z)$ of three independent variables?
4. What does it mean for a function $f(x, y)$ to have limit L as $(x, y) \rightarrow (x_0, y_0)$? What are the basic properties of limits of functions of two independent variables?
5. When is a function of two (three) independent variables continuous at a point in its domain? Give examples of functions that are continuous at some points but not others.
6. What can be said about algebraic combinations and composites of continuous functions?
7. Explain the two-path test for nonexistence of limits.
8. How are the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of a function $f(x, y)$ defined? How are they interpreted and calculated?
9. How does the relation between first partial derivatives and continuity of functions of two independent variables differ from the relation between first derivatives and continuity for real-valued functions of a single independent variable? Give an example.
10. What is the Mixed Derivative Theorem for mixed second-order partial derivatives? How can it help in calculating partial derivatives of second and higher orders? Give examples.
11. What does it mean for a function $f(x, y)$ to be differentiable? What does the Increment Theorem say about differentiability?
12. How can you sometimes decide from examining f_x and f_y that a function $f(x, y)$ is differentiable? What is the relation between the differentiability of f and the continuity of f at a point?
13. What is the Chain Rule? What form does it take for functions of two independent variables? Three independent variables? Functions defined on surfaces? How do you diagram these different forms? Give examples. What pattern enables one to remember all the different forms?
14. What is the derivative of a function $f(x, y)$ at a point P_0 in the direction of a unit vector $\mathbf{u}$? What rate does it describe? What geometric interpretation does it have? Give examples.
15. What is the gradient vector of a differentiable function $f(x, y)$? How is it related to the function's directional derivatives? State the analogous results for functions of three independent variables.
16. How do you find the tangent line at a point on a level curve of a differentiable function $f(x, y)$? How do you find the tangent plane and normal line at a point on a level surface of a differentiable function $f(x, y, z)$? Give examples.
17. How can you use directional derivatives to estimate change?
18. How do you linearize a function $f(x, y)$ of two independent variables at a point (x_0, y_0) ? Why might you want to do this? How do you linearize a function of three independent variables?
19. What can you say about the accuracy of linear approximations of functions of two (three) independent variables?
20. If (x, y) moves from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, how can you estimate the resulting change in the value of a differentiable function $f(x, y)$? Give an example.
21. How do you define local maxima, local minima, and saddle points for a differentiable function $f(x, y)$? Give examples.
22. What derivative tests are available for determining the local extreme values of a function $f(x, y)$? How do they enable you to narrow your search for these values? Give examples.
23. How do you find the extrema of a continuous function $f(x, y)$ on a closed bounded region of the xy -plane? Give an example.
24. Describe the method of Lagrange multipliers and give examples.
25. If $w = f(x, y, z)$, where the variables x, y , and z are constrained by an equation $g(x, y, z) = 0$, what is the meaning of the notation $(\partial w/\partial x)_y$? How can an arrow diagram help you calculate this partial derivative with constrained variables? Give examples.
26. How does Taylor's formula for a function $f(x, y)$ generate polynomial approximations and error estimates?

Chapter 14 Practice Exercises

Domain, Range, and Level Curves

In Exercises 1–4, find the domain and range of the given function and identify its level curves. Sketch a typical level curve.

1. $f(x, y) = 9x^2 + y^2$
2. $f(x, y) = e^{x+y}$
3. $g(x, y) = 1/xy$
4. $g(x, y) = \sqrt{x^2 - y}$

In Exercises 5–8, find the domain and range of the given function and identify its level surfaces. Sketch a typical level surface.

5. $f(x, y, z) = x^2 + y^2 - z$
6. $g(x, y, z) = x^2 + 4y^2 + 9z^2$
7. $h(x, y, z) = \frac{1}{x^2 + y^2 + z^2}$
8. $k(x, y, z) = \frac{1}{x^2 + y^2 + z^2 + 1}$

Evaluating Limits

Find the limits in Exercises 9–14.

9. $\lim_{(x,y) \rightarrow (\pi, \ln 2)} e^y \cos x$
10. $\lim_{(x,y) \rightarrow (0,0)} \frac{2+y}{x + \cos y}$
11. $\lim_{(x,y) \rightarrow (1,1)} \frac{x-y}{x^2 - y^2}$
12. $\lim_{(x,y) \rightarrow (1,1)} \frac{x^3 y^3 - 1}{xy - 1}$
13. $\lim_{P \rightarrow (1, -1, e)} \ln|x + y + z|$
14. $\lim_{P \rightarrow (1, -1, -1)} \tan^{-1}(x + y + z)$

By considering different paths of approach, show that the limits in Exercises 15 and 16 do not exist.

15. $\lim_{(x,y) \rightarrow (0,0)} \frac{y}{x^2 - y}$
 16. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{xy}$
17. **Continuous extension** Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Is it possible to define $f(0, 0)$ in a way that makes f continuous at the origin? Why?

18. **Continuous extension** Let

$$f(x, y) = \begin{cases} \frac{\sin(x - y)}{|x| + |y|}, & |x| + |y| \neq 0 \\ 0, & (x, y) = (0, 0). \end{cases}$$

Is f continuous at the origin? Why?

Partial Derivatives

In Exercises 19–24, find the partial derivative of the function with respect to each variable.

19. $g(r, \theta) = r \cos \theta + r \sin \theta$
20. $f(x, y) = \frac{1}{2} \ln(x^2 + y^2) + \tan^{-1} \frac{y}{x}$
21. $f(R_1, R_2, R_3) = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$
22. $h(x, y, z) = \sin(2\pi x + y - 3z)$

$$23. P(n, R, T, V) = \frac{nRT}{V} \text{ (the ideal gas law)}$$

$$24. f(r, l, T, w) = \frac{1}{2rl} \sqrt{\frac{T}{\pi w}}$$

Second-Order Partial Derivatives

Find the second-order partial derivatives of the functions in Exercises 25–28.

25. $g(x, y) = y + \frac{x}{y}$
26. $g(x, y) = e^x + y \sin x$
27. $f(x, y) = x + xy - 5x^3 + \ln(x^2 + 1)$
28. $f(x, y) = y^2 - 3xy + \cos y + 7e^y$

Chain Rule Calculations

29. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi)$, $x = e^t$, and $y = \ln(t + 1)$.
30. Find dw/dt at $t = 1$ if $w = xe^y + y \sin z - \cos z$, $x = 2\sqrt{t}$, $y = t - 1 + \ln t$, and $z = \pi t$.
31. Find $\partial w/\partial r$ and $\partial w/\partial s$ when $r = \pi$ and $s = 0$ if $w = \sin(2x - y)$, $x = r + \sin s$, $y = rs$.
32. Find $\partial w/\partial u$ and $\partial w/\partial v$ when $u = v = 0$ if $w = \ln \sqrt{1 + x^2} - \tan^{-1} x$ and $x = 2e^u \cos v$.
33. Find the value of the derivative of $f(x, y, z) = xy + yz + xz$ with respect to t on the curve $x = \cos t$, $y = \sin t$, $z = \cos 2t$ at $t = 1$.
34. Show that if $w = f(s)$ is any differentiable function of s and if $s = y + 5x$, then

$$\frac{\partial w}{\partial x} - 5 \frac{\partial w}{\partial y} = 0.$$

Implicit Differentiation

Assuming that the equations in Exercises 35 and 36 define y as a differentiable function of x , find the value of dy/dx at point P .

35. $1 - x - y^2 - \sin xy = 0$, $P(0, 1)$
36. $2xy + e^{x+y} - 2 = 0$, $P(0, \ln 2)$

Directional Derivatives

In Exercises 37–40, find the directions in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\mathbf{v}$.

37. $f(x, y) = \cos x \cos y$, $P_0(\pi/4, \pi/4)$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$
38. $f(x, y) = x^2 e^{-2y}$, $P_0(1, 0)$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$
39. $f(x, y, z) = \ln(2x + 3y + 6z)$, $P_0(-1, -1, 1)$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

40. $f(x, y, z) = x^2 + 3xy - z^2 + 2y + z + 4$, $P_0(0, 0, 0)$,
 $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
41. **Derivative in velocity direction** Find the derivative of $f(x, y, z) = xyz$ in the direction of the velocity vector of the helix
 $\mathbf{r}(t) = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + 3t\mathbf{k}$
 at $t = \pi/3$.
42. **Maximum directional derivative** What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1, 1, 1)$?
43. **Directional derivatives with given values** At the point $(1, 2)$, the function $f(x, y)$ has a derivative of 2 in the direction toward $(2, 2)$ and a derivative of -2 in the direction toward $(1, 1)$.
 a. Find $f_x(1, 2)$ and $f_y(1, 2)$.
 b. Find the derivative of f at $(1, 2)$ in the direction toward the point $(4, 6)$.
44. Which of the following statements are true if $f(x, y)$ is differentiable at (x_0, y_0) ? Give reasons for your answers.
 a. If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u}$.
 b. The derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is a vector.
 c. The directional derivative of f at (x_0, y_0) has its greatest value in the direction of ∇f .
 d. At (x_0, y_0) , vector ∇f is normal to the curve $f(x, y) = f(x_0, y_0)$.

Gradients, Tangent Planes, and Normal Lines

In Exercises 45 and 46, sketch the surface $f(x, y, z) = c$ together with ∇f at the given points.

45. $x^2 + y + z^2 = 0$; $(0, -1, \pm 1)$, $(0, 0, 0)$
 46. $y^2 + z^2 = 4$; $(2, \pm 2, 0)$, $(2, 0, \pm 2)$

In Exercises 47 and 48, find an equation for the plane tangent to the level surface $f(x, y, z) = c$ at the point P_0 . Also, find parametric equations for the line that is normal to the surface at P_0 .

47. $x^2 - y - 5z = 0$, $P_0(2, -1, 1)$
 48. $x^2 + y^2 + z = 4$, $P_0(1, 1, 2)$

In Exercises 49 and 50, find an equation for the plane tangent to the surface $z = f(x, y)$ at the given point.

49. $z = \ln(x^2 + y^2)$, $(0, 1, 0)$
 50. $z = 1/(x^2 + y^2)$, $(1, 1, 1/2)$

In Exercises 51 and 52, find equations for the lines that are tangent and normal to the level curve $f(x, y) = c$ at the point P_0 . Then sketch the lines and level curve together with ∇f at P_0 .

51. $y - \sin x = 1$, $P_0(\pi, 1)$ 52. $\frac{y^2}{2} - \frac{x^2}{2} = \frac{3}{2}$, $P_0(1, 2)$

Tangent Lines to Curves

In Exercises 53 and 54, find parametric equations for the line that is tangent to the curve of intersection of the surfaces at the given point.

53. Surfaces: $x^2 + 2y + 2z = 4$, $y = 1$
 Point: $(1, 1, 1/2)$
 54. Surfaces: $x + y^2 + z = 2$, $y = 1$
 Point: $(1/2, 1, 1/2)$

Linearizations

In Exercises 55 and 56, find the linearization $L(x, y)$ of the function $f(x, y)$ at the point P_0 . Then find an upper bound for the magnitude of the error E in the approximation $f(x, y) \approx L(x, y)$ over the rectangle R .

55. $f(x, y) = \sin x \cos y$, $P_0(\pi/4, \pi/4)$
 $R: \left| x - \frac{\pi}{4} \right| \leq 0.1, \left| y - \frac{\pi}{4} \right| \leq 0.1$
 56. $f(x, y) = xy - 3y^2 + 2$, $P_0(1, 1)$
 $R: |x - 1| \leq 0.1, |y - 1| \leq 0.2$

Find the linearizations of the functions in Exercises 57 and 58 at the given points.

57. $f(x, y, z) = xy + 2yz - 3xz$ at $(1, 0, 0)$ and $(1, 1, 0)$
 58. $f(x, y, z) = \sqrt{2} \cos x \sin(y + z)$ at $(0, 0, \pi/4)$ and $(\pi/4, \pi/4, 0)$

Estimates and Sensitivity to Change

59. **Measuring the volume of a pipeline** You plan to calculate the volume inside a stretch of pipeline that is about 36 in. in diameter and 1 mile long. With which measurement should you be more careful, the length or the diameter? Why?

60. **Sensitivity to change** Near the point $(1, 2)$, is $f(x, y) = x^2 - xy + y^2 - 3$ more sensitive to changes in x or to changes in y ? How do you know?

61. **Change in an electrical circuit** Suppose that the current I (amperes) in an electrical circuit is related to the voltage V (volts) and the resistance R (ohms) by the equation $I = V/R$. If the voltage drops from 24 to 23 volts and the resistance drops from 100 to 80 ohms, will I increase or decrease? By about how much? Is the change in I more sensitive to change in the voltage or to change in the resistance? How do you know?

62. **Maximum error in estimating the area of an ellipse** If $a = 10$ cm and $b = 16$ cm to the nearest millimeter, what should you expect the maximum percentage error to be in the calculated area $A = \pi ab$ of the ellipse $x^2/a^2 + y^2/b^2 = 1$?

63. **Error in estimating a product** Let $y = uv$ and $z = u + v$, where u and v are positive independent variables.

- a. If u is measured with an error of 2% and v with an error of 3%, about what is the percentage error in the calculated value of y ?

- b. Show that the percentage error in the calculated value of z is less than the percentage error in the value of y .

64. Cardiac index To make different people comparable in studies of cardiac output (Section 3.7, Exercise 25), researchers divide the measured cardiac output by the body surface area to find the *cardiac index* C :

$$C = \frac{\text{cardiac output}}{\text{body surface area}}.$$

The body surface area B of a person with weight w and height h is approximated by the formula

$$B = 71.84w^{0.425}h^{0.725},$$

which gives B in square centimeters when w is measured in kilograms and h in centimeters. You are about to calculate the cardiac index of a person with the following measurements:

Cardiac output:	7 L/min
Weight:	70 kg
Height:	180 cm

Which will have a greater effect on the calculation, a 1-kg error in measuring the weight or a 1-cm error in measuring the height?

Local Extrema

Test the functions in Exercises 65–70 for local maxima and minima and saddle points. Find each function's value at these points.

65. $f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4$
 66. $f(x, y) = 5x^2 + 4xy - 2y^2 + 4x - 4y$
 67. $f(x, y) = 2x^3 + 3xy + 2y^3$
 68. $f(x, y) = x^3 + y^3 - 3xy + 15$
 69. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$
 70. $f(x, y) = x^4 - 8x^2 + 3y^2 - 6y$

Absolute Extrema

In Exercises 71–78, find the absolute maximum and minimum values of f on the region R .

71. $f(x, y) = x^2 + xy + y^2 - 3x + 3y$
 R : The triangular region cut from the first quadrant by the line $x + y = 4$
 72. $f(x, y) = x^2 - y^2 - 2x + 4y + 1$
 R : The rectangular region in the first quadrant bounded by the coordinate axes and the lines $x = 4$ and $y = 2$
 73. $f(x, y) = y^2 - xy - 3y + 2x$
 R : The square region enclosed by the lines $x = \pm 2$ and $y = \pm 2$
 74. $f(x, y) = 2x + 2y - x^2 - y^2$
 R : The square region bounded by the coordinate axes and the lines $x = 2, y = 2$ in the first quadrant

75. $f(x, y) = x^2 - y^2 - 2x + 4y$

R : The triangular region bounded below by the x -axis, above by the line $y = x + 2$, and on the right by the line $x = 2$

76. $f(x, y) = 4xy - x^4 - y^4 + 16$

R : The triangular region bounded below by the line $y = -2$, above by the line $y = x$, and on the right by the line $x = 2$

77. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$

R : The square region enclosed by the lines $x = \pm 1$ and $y = \pm 1$

78. $f(x, y) = x^3 + 3xy + y^3 + 1$

R : The square region enclosed by the lines $x = \pm 1$ and $y = \pm 1$

Lagrange Multipliers

79. **Extrema on a circle** Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.
 80. **Extrema on a circle** Find the extreme values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 1$.
 81. **Extrema in a disk** Find the extreme values of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 \leq 1$.
 82. **Extrema in a disk** Find the extreme values of $f(x, y) = x^2 + y^2 - 3x - xy$ on the disk $x^2 + y^2 \leq 9$.
 83. **Extrema on a sphere** Find the extreme values of $f(x, y, z) = x - y + z$ on the unit sphere $x^2 + y^2 + z^2 = 1$.
 84. **Minimum distance to origin** Find the points on the surface $z^2 - xy = 4$ closest to the origin.
 85. **Minimizing cost of a box** A closed rectangular box is to have volume $V \text{ cm}^3$. The cost of the material used in the box is a cents/cm² for top and bottom, b cents/cm² for front and back, and c cents/cm² for the remaining sides. What dimensions minimize the total cost of materials?
 86. **Least volume** Find the plane $x/a + y/b + z/c = 1$ that passes through the point $(2, 1, 2)$ and cuts off the least volume from the first octant.
 87. **Extrema on curve of intersecting surfaces** Find the extreme values of $f(x, y, z) = x(y + z)$ on the curve of intersection of the right circular cylinder $x^2 + y^2 = 1$ and the hyperbolic cylinder $xz = 1$.
 88. **Minimum distance to origin on curve of intersecting plane and cone** Find the point closest to the origin on the curve of intersection of the plane $x + y + z = 1$ and the cone $z^2 = 2x^2 + 2y^2$.

Partial Derivatives with Constrained Variables

In Exercises 89 and 90, begin by drawing a diagram that shows the relations among the variables.

89. If $w = x^2e^{yz}$ and $z = x^2 - y^2$ find

a. $\left(\frac{\partial w}{\partial y}\right)_z$ b. $\left(\frac{\partial w}{\partial z}\right)_x$ c. $\left(\frac{\partial w}{\partial z}\right)_y$.

90. Let $U = f(P, V, T)$ be the internal energy of a gas that obeys the ideal gas law $PV = nRT$ (n and R constant). Find

a. $\left(\frac{\partial U}{\partial T}\right)_P$ b. $\left(\frac{\partial U}{\partial V}\right)_T$.

Theory and Examples

91. Let $w = f(r, \theta)$, $r = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1}(y/x)$. Find $\partial w/\partial x$ and $\partial w/\partial y$ and express your answers in terms of r and θ .
92. Let $z = f(u, v)$, $u = ax + by$, and $v = ax - by$. Express z_x and z_y in terms of f_u, f_v , and the constants a and b .
93. If a and b are constants, $w = u^3 + \tanh u + \cos u$, and $u = ax + by$, show that

$$a \frac{\partial w}{\partial y} = b \frac{\partial w}{\partial x}.$$

94. **Using the Chain Rule** If $w = \ln(x^2 + y^2 + 2z)$, $x = r + s$, $y = r - s$, and $z = 2rs$, find w_r and w_s by the Chain Rule. Then check your answer another way.
95. **Angle between vectors** The equations $e^u \cos v - x = 0$ and $e^u \sin v - y = 0$ define u and v as differentiable functions of x and y . Show that the angle between the vectors

$$\frac{\partial u}{\partial x} \mathbf{i} + \frac{\partial u}{\partial y} \mathbf{j} \quad \text{and} \quad \frac{\partial v}{\partial x} \mathbf{i} + \frac{\partial v}{\partial y} \mathbf{j}$$

is constant.

96. **Polar coordinates and second derivatives** Introducing polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$ changes $f(x, y)$ to $g(r, \theta)$. Find the value of $\partial^2 g/\partial \theta^2$ at the point $(r, \theta) = (2, \pi/2)$, given that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 1$$

at that point.

97. **Normal line parallel to a plane** Find the points on the surface

$$(y + z)^2 + (z - x)^2 = 16$$

where the normal line is parallel to the yz -plane.

98. **Tangent plane parallel to xy -plane** Find the points on the surface

$$xy + yz + zx - x - z^2 = 0$$

where the tangent plane is parallel to the xy -plane.

99. **When gradient is parallel to position vector** Suppose that $\nabla f(x, y, z)$ is always parallel to the position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $f(0, 0, a) = f(0, 0, -a)$ for any a .

100. **Directional derivative in all directions, but no gradient** Show that the directional derivative of

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

at the origin equals 1 in any direction but that f has no gradient vector at the origin.

101. **Normal line through origin** Show that the line normal to the surface $xy + z = 2$ at the point $(1, 1, 1)$ passes through the origin.

102. **Tangent plane and normal line**

- a. Sketch the surface $x^2 - y^2 + z^2 = 4$.
- b. Find a vector normal to the surface at $(2, -3, 3)$. Add the vector to your sketch.
- c. Find equations for the tangent plane and normal line at $(2, -3, 3)$.

Chapter 14 Additional and Advanced Exercises



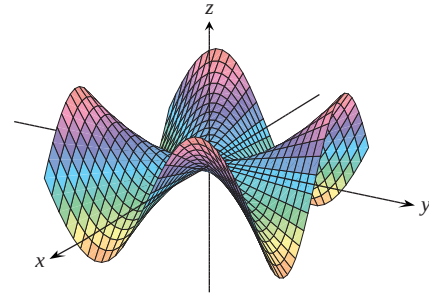
Project

Partial Derivatives

1. **Function with saddle at the origin** If you did Exercise 50 in Section 14.2, you know that the function

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(see the accompanying figure) is continuous at $(0, 0)$. Find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.



- 2. Finding a function from second partials** Find a function $w = f(x, y)$ whose first partial derivatives are $\partial w/\partial x = 1 + e^x \cos y$ and $\partial w/\partial y = 2y - e^x \sin y$ and whose value at the point $(\ln 2, 0)$ is $\ln 2$.
- 3. A proof of Leibniz's Rule** Leibniz's Rule says that if f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}.$$

Prove the rule by setting

$$g(u, v) = \int_u^v f(t) dt, \quad u = u(x), \quad v = v(x)$$

and calculating dg/dx with the Chain Rule.

- 4. Finding a function with constrained second partials** Suppose that f is a twice-differentiable function of r , that $r = \sqrt{x^2 + y^2 + z^2}$, and that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

Show that for some constants a and b ,

$$f(r) = \frac{a}{r} + b.$$

- 5. Homogeneous functions** A function $f(x, y)$ is *homogeneous of degree n* (n a nonnegative integer) if $f(tx, ty) = t^n f(x, y)$ for all t , x , and y . For such a function (sufficiently differentiable), prove that

a. $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f(x, y)$

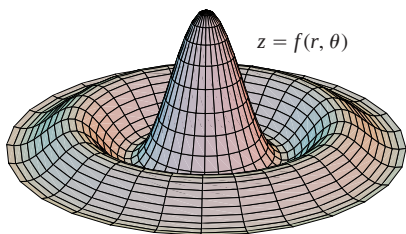
b. $x^2 \left(\frac{\partial^2 f}{\partial x^2} \right) + 2xy \left(\frac{\partial^2 f}{\partial x \partial y} \right) + y^2 \left(\frac{\partial^2 f}{\partial y^2} \right) = n(n-1)f.$

- 6. Surface in polar coordinates** Let

$$f(r, \theta) = \begin{cases} \frac{\sin 6r}{6r}, & r \neq 0 \\ 1, & r = 0, \end{cases}$$

where r and θ are polar coordinates. Find

a. $\lim_{r \rightarrow 0} f(r, \theta)$ b. $f_r(0, 0)$ c. $f_\theta(r, \theta), \quad r \neq 0.$



Gradients and Tangents

- 7. Properties of position vectors** Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $r = |\mathbf{r}|$.

a. Show that $\nabla r = \mathbf{r}/r$.

b. Show that $\nabla(r^n) = nr^{n-2}\mathbf{r}$.

c. Find a function whose gradient equals $\mathbf{r}$.

d. Show that $\mathbf{r} \cdot d\mathbf{r} = r dr$.

e. Show that $\nabla(\mathbf{A} \cdot \mathbf{r}) = \mathbf{A}$ for any constant vector $\mathbf{A}$.

- 8. Gradient orthogonal to tangent** Suppose that a differentiable function $f(x, y)$ has the constant value c along the differentiable curve $x = g(t), y = h(t)$; that is

$$f(g(t), h(t)) = c$$

for all values of t . Differentiate both sides of this equation with respect to t to show that ∇f is orthogonal to the curve's tangent vector at every point on the curve.

- 9. Curve tangent to a surface** Show that the curve

$$\mathbf{r}(t) = (\ln t)\mathbf{i} + (t \ln t)\mathbf{j} + t\mathbf{k}$$

is tangent to the surface

$$xz^2 - yz + \cos xy = 1$$

at $(0, 0, 1)$.

- 10. Curve tangent to a surface** Show that the curve

$$\mathbf{r}(t) = \left(\frac{t^3}{4} - 2 \right) \mathbf{i} + \left(\frac{4}{t} - 3 \right) \mathbf{j} + \cos(t-2)\mathbf{k}$$

is tangent to the surface

$$x^3 + y^3 + z^3 - xyz = 0$$

at $(0, -1, 1)$.

Extreme Values

- 11. Extrema on a surface** Show that the only possible maxima and minima of z on the surface $z = x^3 + y^3 - 9xy + 27$ occur at $(0, 0)$ and $(3, 3)$. Show that neither a maximum nor a minimum occurs at $(0, 0)$. Determine whether z has a maximum or a minimum at $(3, 3)$.

- 12. Maximum in closed first quadrant** Find the maximum value of $f(x, y) = 6xye^{-(2x+3y)}$ in the closed first quadrant (includes the nonnegative axes).

- 13. Minimum volume cut from first octant** Find the minimum volume for a region bounded by the planes $x = 0, y = 0, z = 0$ and a plane tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at a point in the first octant.

- 14. Minimum distance from line to parabola in xy -plane** By minimizing the function $f(x, y, u, v) = (x - u)^2 + (y - v)^2$ subject to the constraints $y = x + 1$ and $u = v^2$, find the minimum distance in the xy -plane from the line $y = x + 1$ to the parabola $y^2 = x$.

Theory and Examples

- 15. Boundedness of first partials implies continuity** Prove the following theorem: If $f(x, y)$ is defined in an open region R of the xy -plane and if f_x and f_y are bounded on R , then $f(x, y)$ is continuous on R . (The assumption of boundedness is essential.)
- 16.** Suppose that $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ is a smooth curve in the domain of a differentiable function $f(x, y, z)$. Describe the relation between df/dt , ∇f , and $\mathbf{v} = d\mathbf{r}/dt$. What can be said about ∇f and $\mathbf{v}$ at interior points of the curve where f has extreme values relative to its other values on the curve? Give reasons for your answer.
- 17. Finding functions from partial derivatives** Suppose that f and g are functions of x and y such that

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y},$$

and suppose that

$$\frac{\partial f}{\partial x} = 0, \quad f(1, 2) = g(1, 2) = 5 \quad \text{and} \quad f(0, 0) = 4.$$

Find $f(x, y)$ and $g(x, y)$.

- 18. Rate of change of the rate of change** We know that if $f(x, y)$ is a function of two variables and if $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$ is the rate of change of $f(x, y)$ at (x, y) in the direction of $\mathbf{u}$. Give a similar formula for the rate of change of the rate of change of $f(x, y)$ at (x, y) in the direction $\mathbf{u}$.
- 19. Path of a heat-seeking particle** A heat-seeking particle has the property that at any point (x, y) in the plane it moves in the direction of maximum temperature increase. If the temperature at (x, y) is $T(x, y) = -e^{-2y} \cos x$, find an equation $y = f(x)$ for the path of a heat-seeking particle at the point $(\pi/4, 0)$.
- 20. Velocity after a ricochet** A particle traveling in a straight line with constant velocity $\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ passes through the point $(0, 0, 30)$ and hits the surface $z = 2x^2 + 3y^2$. The particle ricochets off the surface, the angle of reflection being equal to the angle of incidence. Assuming no loss of speed, what is the velocity of the particle after the ricochet? Simplify your answer.
- 21. Directional derivatives tangent to a surface** Let S be the surface that is the graph of $f(x, y) = 10 - x^2 - y^2$. Suppose that the temperature in space at each point (x, y, z) is $T(x, y, z) = x^2y + y^2z + 4x + 14y + z$.
- a. Among all the possible directions tangential to the surface S at the point $(0, 0, 10)$, which direction will make the rate of change of temperature at $(0, 0, 10)$ a maximum?

- b. Which direction tangential to S at the point $(1, 1, 8)$ will make the rate of change of temperature a maximum?

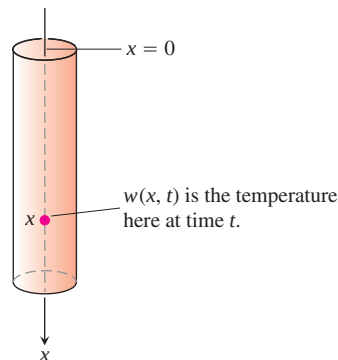
- 22. Drilling another borehole** On a flat surface of land, geologists drilled a borehole straight down and hit a mineral deposit at 1000 ft. They drilled a second borehole 100 ft to the north of the first and hit the mineral deposit at 950 ft. A third borehole 100 ft east of the first borehole struck the mineral deposit at 1025 ft. The geologists have reasons to believe that the mineral deposit is in the shape of a dome, and for the sake of economy, they would like to find where the deposit is closest to the surface. Assuming the surface to be the xy -plane, in what direction from the first borehole would you suggest the geologists drill their fourth borehole?

The One-Dimensional Heat Equation

If $w(x, t)$ represents the temperature at position x at time t in a uniform conducting rod with perfectly insulated sides (see the accompanying figure), then the partial derivatives w_{xx} and w_t satisfy a differential equation of the form

$$w_{xx} = \frac{1}{c^2} w_t.$$

This equation is called the **one-dimensional heat equation**. The value of the positive constant c^2 is determined by the material from which the rod is made. It has been determined experimentally for a broad range of materials. For a given application, one finds the appropriate value in a table. For dry soil, for example, $c^2 = 0.19$ ft²/day.



In chemistry and biochemistry, the heat equation is known as the **diffusion equation**. In this context, $w(x, t)$ represents the concentration of a dissolved substance, a salt for instance, diffusing along a tube filled with liquid. The value of $w(x, t)$ is the concentration at point x at time t . In other applications, $w(x, t)$ represents the diffusion of a gas down a long, thin pipe.

In electrical engineering, the heat equation appears in the forms

$$v_{xx} = RCv_t$$

and

$$i_{xx} = RCi_t.$$

These equations describe the voltage v and the flow of current i in a coaxial cable or in any other cable in which leakage and inductance are negligible. The functions and constants in these equations are

$v(x, t)$ = voltage at point x at time t

R = resistance per unit length

C = capacitance to ground per unit of cable length

$i(x, t)$ = current at point x at time t .

23. Find all solutions of the one-dimensional heat equation of the form $w = e^{rt} \sin \pi x$, where r is a constant.
24. Find all solutions of the one-dimensional heat equation that have the form $w = e^{rt} \sin kx$ and satisfy the conditions that $w(0, t) = 0$ and $w(L, t) = 0$. What happens to these solutions as $t \rightarrow \infty$?

Chapter 14 Technology Application Projects

Mathematica/Maple Module

Plotting Surfaces

Efficiently generate plots of surfaces, contours, and level curves.



Mathematica/Maple Module

Exploring the Mathematics Behind Skateboarding: Analysis of the Directional Derivative

The path of a skateboarder is introduced, first on a level plane, then on a ramp, and finally on a paraboloid. Compute, plot, and analyze the directional derivative in terms of the skateboarder.



Mathematica/Maple Module

Looking for Patterns and Applying the Method of Least Squares to Real Data

Fit a line to a set of numerical data points by choosing the line that minimizes the sum of the squares of the vertical distances from the points to the line.

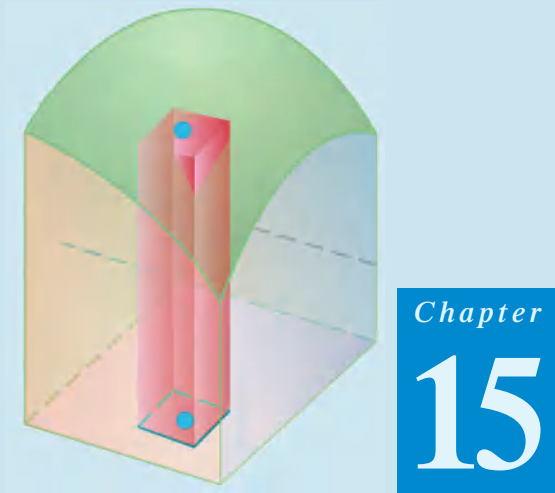


Mathematica/Maple Module

Lagrange Goes Skateboarding: How High Does He Go?

Revisit and analyze the skateboarders' adventures for maximum and minimum heights from both a graphical and analytic perspective using Lagrange multipliers.





Chapter

15

MULTIPLE INTEGRALS

OVERVIEW In this chapter we consider the integral of a function of two variables $f(x, y)$ over a region in the plane and the integral of a function of three variables $f(x, y, z)$ over a region in space. These integrals are called *multiple integrals* and are defined as the limit of approximating Riemann sums, much like the single-variable integrals presented in Chapter 5. We can use multiple integrals to calculate quantities that vary over two or three dimensions, such as the total mass or the angular momentum of an object of varying density and the volumes of solids with general curved boundaries.

15.1

Double Integrals

In Chapter 5 we defined the definite integral of a continuous function $f(x)$ over an interval $[a, b]$ as a limit of Riemann sums. In this section we extend this idea to define the integral of a continuous function of two variables $f(x, y)$ over a bounded region R in the plane. In both cases the integrals are limits of approximating Riemann sums. The Riemann sums for the integral of a single-variable function $f(x)$ are obtained by partitioning a finite interval into thin subintervals, multiplying the width of each subinterval by the value of f at a point c_k inside that subinterval, and then adding together all the products. A similar method of partitioning, multiplying, and summing is used to construct double integrals. However, this time we pack a planar region R with small rectangles, rather than small subintervals. We then take the product of each small rectangle's area with the value of f at a point inside that rectangle, and finally sum together all these products. When f is continuous, these sums converge to a single number as each of the small rectangles shrinks in both width and height. The limit is the *double integral* of f over R . As with single integrals, we can evaluate multiple integrals via antiderivatives, which frees us from the formidable task of calculating a double integral directly from its definition as a limit of Riemann sums. The major practical problem that arises in evaluating multiple integrals lies in determining the limits of integration. While the integrals of Chapter 5 were evaluated over an interval, which is determined by its two endpoints, multiple integrals are evaluated over a region in the plane or in space. This gives rise to limits of integration which often involve variables, not just constants. Describing the regions of integration is the main new issue that arises in the calculation of multiple integrals.

Double Integrals over Rectangles

We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function $f(x, y)$ defined on a rectangular region R ,

$$R: a \leq x \leq b, \quad c \leq y \leq d.$$

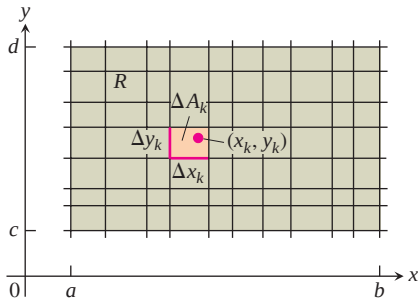


FIGURE 15.1 Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \Delta y_k$.

We subdivide R into small rectangles using a network of lines parallel to the x - and y -axes (Figure 15.1). The lines divide R into n rectangular pieces, where the number of such pieces n gets large as the width and height of each piece gets small. These rectangles form a **partition** of R . A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$. If we number the small pieces partitioning R in some order, then their areas are given by numbers $\Delta A_1, \Delta A_2, \dots, \Delta A_n$, where ΔA_k is the area of the k th small rectangle.

To form a Riemann sum over R , we choose a point (x_k, y_k) in the k th small rectangle, multiply the value of f at that point by the area ΔA_k , and add together the products:

$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k.$$

Depending on how we pick (x_k, y_k) in the k th small rectangle, we may get different values for S_n .

We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the partition of R approach zero. The **norm** of a partition P , written $\|P\|$, is the largest width or height of any rectangle in the partition. If $\|P\| = 0.1$ then all the rectangles in the partition of R have width at most 0.1 and height at most 0.1. Sometimes the Riemann sums converge as the norm of P goes to zero, written $\|P\| \rightarrow 0$. The resulting limit is then written as

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k.$$

As $\|P\| \rightarrow 0$ and the rectangles get narrow and short, their number n increases, so we can also write this limit as

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k, y_k) \Delta A_k.$$

with the understanding that $\Delta A_k \rightarrow 0$ as $n \rightarrow \infty$ and $\|P\| \rightarrow 0$.

There are many choices involved in a limit of this kind. The collection of small rectangles is determined by the grid of vertical and horizontal lines that determine a rectangular partition of R . In each of the resulting small rectangles there is a choice of an arbitrary point (x_k, y_k) at which f is evaluated. These choices together determine a single Riemann sum. To form a limit, we repeat the whole process again and again, choosing partitions whose rectangle widths and heights both go to zero and whose number goes to infinity.

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be **integrable** and the limit is called the **double integral** of f over R , written as

$$\iint_R f(x, y) \, dA \quad \text{or} \quad \iint_R f(x, y) \, dx \, dy.$$

It can be shown that if $f(x, y)$ is a continuous function throughout R , then f is integrable, as in the single-variable case discussed in Chapter 5. Many discontinuous functions are also integrable, including functions which are discontinuous only on a finite number of points or smooth curves. We leave the proof of these facts to a more advanced text.

Double Integrals as Volumes

When $f(x, y)$ is a positive function over a rectangular region R in the xy -plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy -plane bounded below by R and above by the surface $z = f(x, y)$ (Figure 15.2). Each term $f(x_k, y_k) \Delta A_k$ in the sum $S_n = \sum f(x_k, y_k) \Delta A_k$ is the volume of a vertical

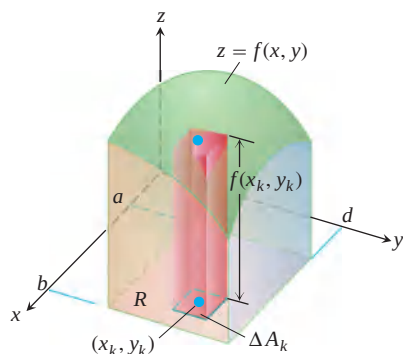


FIGURE 15.2 Approximating solids with rectangular boxes leads us to define the volumes of more general solids as double integrals. The volume of the solid shown here is the double integral of $f(x, y)$ over the base region R .

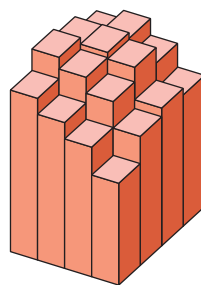


rectangular box that approximates the volume of the portion of the solid that stands directly above the base ΔA_k . The sum S_n thus approximates what we want to call the total volume of the solid. We *define* this volume to be

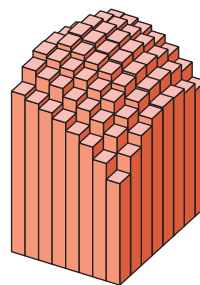
$$\text{Volume} = \lim_{n \rightarrow \infty} S_n = \iint_R f(x, y) \, dA,$$

where $\Delta A_k \rightarrow 0$ as $n \rightarrow \infty$.

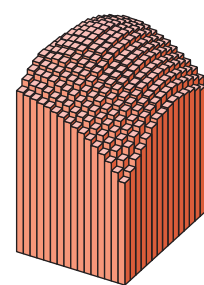
As you might expect, this more general method of calculating volume agrees with the methods in Chapter 6, but we do not prove this here. Figure 15.3 shows Riemann sum approximations to the volume becoming more accurate as the number n of boxes increases.



(a) $n = 16$



(b) $n = 64$



(c) $n = 256$

FIGURE 15.3 As n increases, the Riemann sum approximations approach the total volume of the solid shown in Figure 15.2.

Fubini's Theorem for Calculating Double Integrals

Suppose that we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region R : $0 \leq x \leq 2$, $0 \leq y \leq 1$ in the xy -plane. If we apply the method of slicing from Section 6.1, with slices perpendicular to the x -axis (Figure 15.4), then the volume is

$$\int_{x=0}^{x=2} A(x) \, dx, \quad (1)$$

where $A(x)$ is the cross-sectional area at x . For each value of x , we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y) \, dy, \quad (2)$$

which is the area under the curve $z = 4 - x - y$ in the plane of the cross-section at x . In calculating $A(x)$, x is held fixed and the integration takes place with respect to y . Combining Equations (1) and (2), we see that the volume of the entire solid is

$$\begin{aligned} \text{Volume} &= \int_{x=0}^{x=2} A(x) \, dx = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y) \, dy \right) dx \\ &= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx = \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx \\ &= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 = 5. \end{aligned} \quad (3)$$

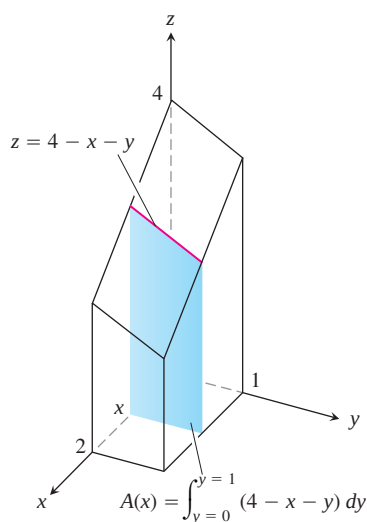


FIGURE 15.4 To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y .

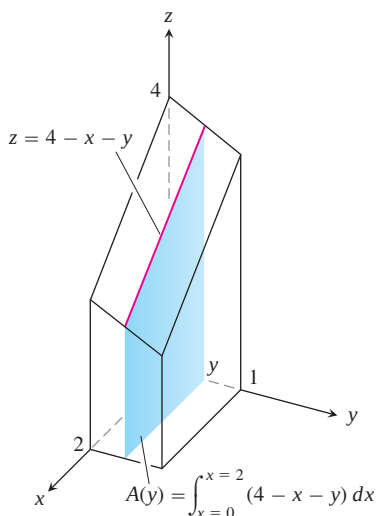


FIGURE 15.5 To obtain the cross-sectional area $A(y)$, we hold y fixed and integrate with respect to x .

If we just wanted to write a formula for the volume, without carrying out any of the integrations, we could write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y) dy dx.$$

The expression on the right, called an **iterated** or **repeated integral**, says that the volume is obtained by integrating $4 - x - y$ with respect to y from $y = 0$ to $y = 1$, holding x fixed, and then integrating the resulting expression in x with respect to x from $x = 0$ to $x = 2$. The limits of integration 0 and 1 are associated with y , so they are placed on the integral closest to dy . The other limits of integration, 0 and 2, are associated with the variable x , so they are placed on the outside integral symbol that is paired with dx .

What would have happened if we had calculated the volume by slicing with planes perpendicular to the y -axis (Figure 15.5)? As a function of y , the typical cross-sectional area is

$$A(y) = \int_{x=0}^{x=2} (4 - x - y) dx = \left[4x - \frac{x^2}{2} - xy \right]_{x=0}^{x=2} = 6 - 2y. \quad (4)$$

The volume of the entire solid is therefore

$$\text{Volume} = \int_{y=0}^{y=1} A(y) dy = \int_{y=0}^{y=1} (6 - 2y) dy = [6y - y^2]_0^1 = 5,$$

in agreement with our earlier calculation.

Again, we may give a formula for the volume as an iterated integral by writing

$$\text{Volume} = \int_0^1 \int_0^2 (4 - x - y) dx dy.$$

The expression on the right says we can find the volume by integrating $4 - x - y$ with respect to x from $x = 0$ to $x = 2$ as in Equation (4) and integrating the result with respect to y from $y = 0$ to $y = 1$. In this iterated integral, the order of integration is first x and then y , the reverse of the order in Equation (3).

What do these two volume calculations with iterated integrals have to do with the double integral

$$\iint_R (4 - x - y) dA$$

over the rectangle $R: 0 \leq x \leq 2, 0 \leq y \leq 1$? The answer is that both iterated integrals give the value of the double integral. This is what we would reasonably expect, since the double integral measures the volume of the same region as the two iterated integrals. A theorem published in 1907 by Guido Fubini says that the double integral of any continuous function over a rectangle can be calculated as an iterated integral in either order of integration. (Fubini proved his theorem in greater generality, but this is what it says in our setting.)

HISTORICAL BIOGRAPHY

Guido Fubini
(1879–1943)



THEOREM 1 Fubini's Theorem (First Form)

If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx.$$

Fubini's Theorem says that double integrals over rectangles can be calculated as iterated integrals. Thus, we can evaluate a double integral by integrating with respect to one variable at a time.

Fubini's Theorem also says that we may calculate the double integral by integrating in *either* order, a genuine convenience, as we see in Example 3. When we calculate a volume by slicing, we may use either planes perpendicular to the x -axis or planes perpendicular to the y -axis.



EXAMPLE 1 Evaluating a Double Integral

Calculate $\iint_R f(x, y) \, dA$ for

$$f(x, y) = 1 - 6x^2y \quad \text{and} \quad R: 0 \leq x \leq 2, \quad -1 \leq y \leq 1.$$

Solution By Fubini's Theorem,

$$\begin{aligned} \iint_R f(x, y) \, dA &= \int_{-1}^1 \int_0^2 (1 - 6x^2y) \, dx \, dy = \int_{-1}^1 [x - 2x^3y]_{x=0}^{x=2} \, dy \\ &= \int_{-1}^1 (2 - 16y) \, dy = [2y - 8y^2]_{-1}^1 = 4. \end{aligned}$$

Reversing the order of integration gives the same answer:

$$\begin{aligned} \int_0^2 \int_{-1}^1 (1 - 6x^2y) \, dy \, dx &= \int_0^2 [y - 3x^2y^2]_{y=-1}^{y=1} \, dx \\ &= \int_0^2 [(1 - 3x^2) - (-1 - 3x^2)] \, dx \\ &= \int_0^2 2 \, dx = 4. \end{aligned}$$

USING TECHNOLOGY Multiple Integration

Most CAS can calculate both multiple and iterated integrals. The typical procedure is to apply the CAS integrate command in nested iterations according to the order of integration you specify.

Integral	Typical CAS Formulation
$\iint x^2y \, dx \, dy$	<code>int (int (x ^ 2 * y, x), y) ;</code>
$\int_{-\pi/3}^{\pi/4} \int_0^1 x \cos y \, dx \, dy$	<code>int (int (x * cos (y), x = 0 .. 1), y = -Pi/3 .. Pi/4);</code>

If a CAS cannot produce an exact value for a definite integral, it can usually find an approximate value numerically. Setting up a multiple integral for a CAS to solve can be a highly nontrivial task, and requires an understanding of how to describe the boundaries of the region and set up an appropriate integral.

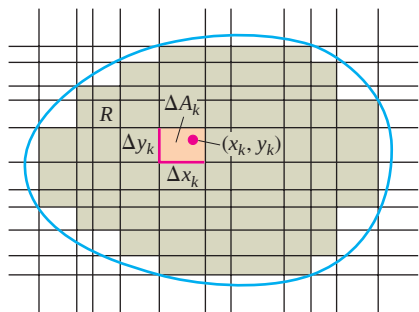


FIGURE 15.6 A rectangular grid partitioning a bounded nonrectangular region into rectangular cells.

Double Integrals over Bounded Nonrectangular Regions

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region R , such as the one in Figure 15.6, we again begin by covering R with a grid of small rectangular cells whose union contains all points of R . This time, however, we cannot exactly fill R with a finite number of rectangles lying inside R , since its boundary is curved, and some of the small rectangles in the grid lie partly outside R . A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of R is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero.

Once we have a partition of R , we number the rectangles in some order from 1 to n and let ΔA_k be the area of the k th rectangle. We then choose a point (x_k, y_k) in the k th rectangle and form the Riemann sum

$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k.$$

As the norm of the partition forming S_n goes to zero, $\|P\| \rightarrow 0$, the width and height of each enclosed rectangle goes to zero and their number goes to infinity. If $f(x, y)$ is a continuous function, then these Riemann sums converge to a limiting value, not dependent on any of the choices we made. This limit is called the **double integral** of $f(x, y)$ over R :

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k) \Delta A_k = \iint_R f(x, y) \, dA.$$

The nature of the boundary of R introduces issues not found in integrals over an interval. When R has a curved boundary, the n rectangles of a partition lie inside R but do not cover all of R . In order for a partition to approximate R well, the parts of R covered by small rectangles lying partly outside R must become negligible as the norm of the partition approaches zero. This property of being nearly filled in by a partition of small norm is satisfied by all the regions that we will encounter. There is no problem with boundaries made from polygons, circles, ellipses, and from continuous graphs over an interval, joined end to end. A curve with a “fractal” type of shape would be problematic, but such curves are not relevant for most applications. A careful discussion of which type of regions R can be used for computing double integrals is left to a more advanced text.

Double integrals of continuous functions over nonrectangular regions have the same algebraic properties (summarized further on) as integrals over rectangular regions. The domain Additivity Property says that if R is decomposed into nonoverlapping regions R_1 and R_2 with boundaries that are again made of a finite number of line segments or smooth curves (see Figure 15.7 for an example), then

$$\iint_R f(x, y) \, dA = \iint_{R_1} f(x, y) \, dA + \iint_{R_2} f(x, y) \, dA.$$

If $f(x, y)$ is positive and continuous over R we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y) \, dA$, as before (Figure 15.8).

If R is a region like the one shown in the xy -plane in Figure 15.9, bounded “above” and “below” by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a$, $x = b$, we may again calculate the volume by the method of slicing. We first calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) \, dy$$

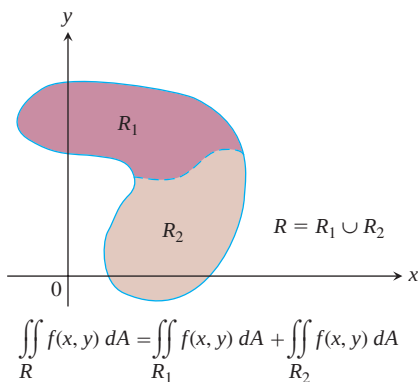
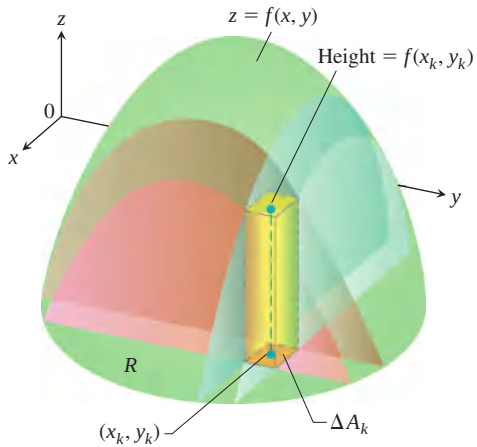


FIGURE 15.7 The Additivity Property for rectangular regions holds for regions bounded by continuous curves.



$$\text{Volume} = \lim \sum f(x_k, y_k) \Delta A_k = \iint_R f(x, y) dA$$

FIGURE 15.8 We define the volumes of solids with curved bases the same way we define the volumes of solids with rectangular bases.

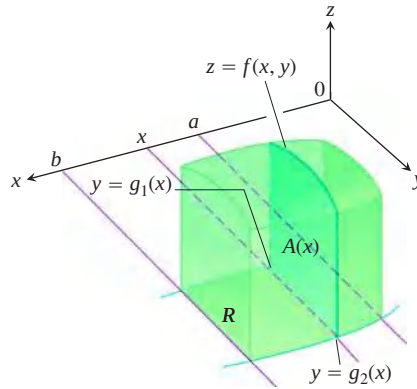


FIGURE 15.9 The area of the vertical slice shown here is

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) dy.$$

To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$.

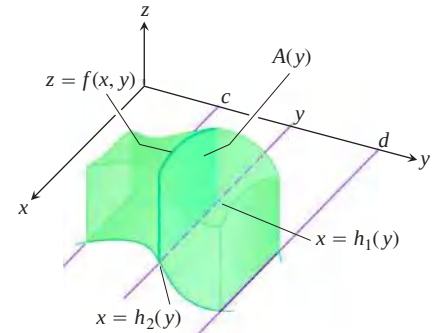


FIGURE 15.10 The volume of the solid shown here is

$$\int_c^d A(y) dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$V = \int_a^b A(x) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx. \quad (5)$$

Similarly, if R is a region like the one shown in Figure 15.10, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines $y = c$ and $y = d$, then the volume calculated by slicing is given by the iterated integral

$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy. \quad (6)$$

That the iterated integrals in Equations (5) and (6) both give the volume that we defined to be the double integral of f over R is a consequence of the following stronger form of Fubini's Theorem.

THEOREM 2 Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region R .

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

**EXAMPLE 2** Finding Volume

Find the volume of the prism whose base is the triangle in the xy -plane bounded by the x -axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure 15.11 on page 1075. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.11b). Hence,

$$\begin{aligned} V &= \int_0^1 \int_0^x (3 - x - y) \, dy \, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx \\ &= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1. \end{aligned}$$

When the order of integration is reversed (Figure 15.11c), the integral for the volume is

$$\begin{aligned} V &= \int_0^1 \int_y^1 (3 - x - y) \, dx \, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy \\ &= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy \\ &= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1. \end{aligned}$$

The two integrals are equal, as they should be. ■

Although Fubini's Theorem assures us that a double integral may be calculated as an iterated integral in either order of integration, the value of one integral may be easier to find than the value of the other. The next example shows how this can happen.

**EXAMPLE 3** Evaluating a Double Integral

Calculate

$$\iint_R \frac{\sin x}{x} \, dA,$$

where R is the triangle in the xy -plane bounded by the x -axis, the line $y = x$, and the line $x = 1$.

Solution The region of integration is shown in Figure 15.12. If we integrate first with respect to y and then with respect to x , we find

$$\begin{aligned} \int_0^1 \left(\int_0^x \frac{\sin x}{x} \, dy \right) dx &= \int_0^1 \left(y \frac{\sin x}{x} \right)_{y=0}^{y=x} dx = \int_0^1 \sin x \, dx \\ &= -\cos(1) + 1 \approx 0.46. \end{aligned}$$

If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy,$$

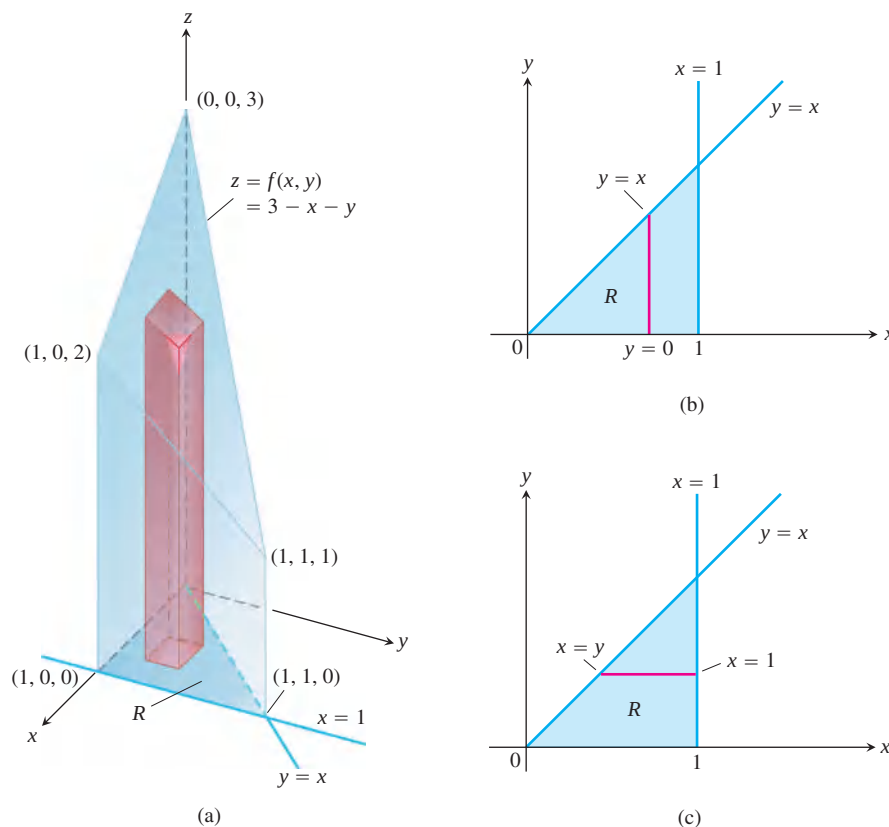


FIGURE 15.11 (a) Prism with a triangular base in the xy -plane. The volume of this prism is defined as a double integral over R . To evaluate it as an iterated integral, we may integrate first with respect to y and then with respect to x , or the other way around (Example 2).
(b) Integration limits of

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} f(x, y) dy dx.$$

If we integrate first with respect to y , we integrate along a vertical line through R and then integrate from left to right to include all the vertical lines in R .

(c) Integration limits of

$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} f(x, y) dx dy.$$

If we integrate first with respect to x , we integrate along a horizontal line through R and then integrate from bottom to top to include all the horizontal lines in R .

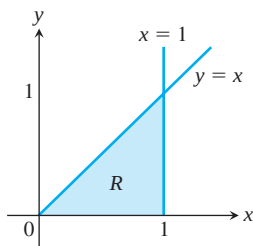


FIGURE 15.12 The region of integration in Example 3.

we run into a problem, because $\int ((\sin x)/x) dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

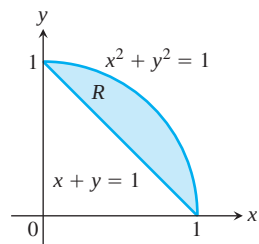
There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you first choose doesn't work, try the other. Sometimes neither order will work, and then we need to use numerical approximations. ■

Finding Limits of Integration

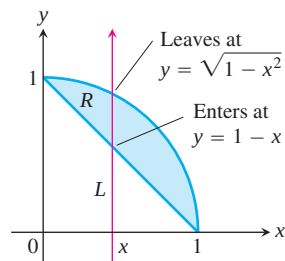
We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating $\iint_R f(x, y) dA$, integrating first with respect to y and then with respect to x , do the following:

1. *Sketch.* Sketch the region of integration and label the bounding curves.

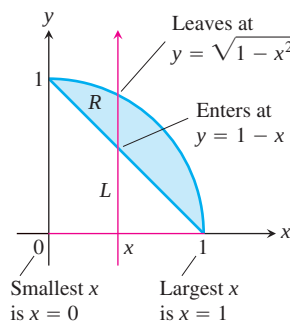


2. *Find the y-limits of integration.* Imagine a vertical line L cutting through R in the direction of increasing y . Mark the y -values where L enters and leaves. These are the y -limits of integration and are usually functions of x (instead of constants).



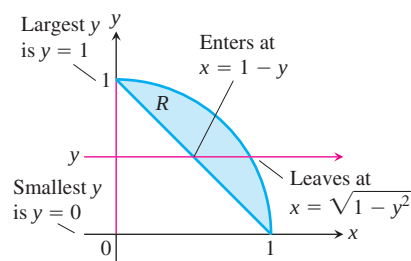
3. *Find the x-limits of integration.* Choose x -limits that include all the vertical lines through R . The integral shown here is

$$\iint_R f(x, y) dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y) dy dx.$$



To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral is

$$\iint_R f(x, y) dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y) dx dy.$$



EXAMPLE 4 Reversing the Order of Integration

Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2) dy dx$$

and write an equivalent integral with the order of integration reversed.

Solution The region of integration is given by the inequalities $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure 15.13a).

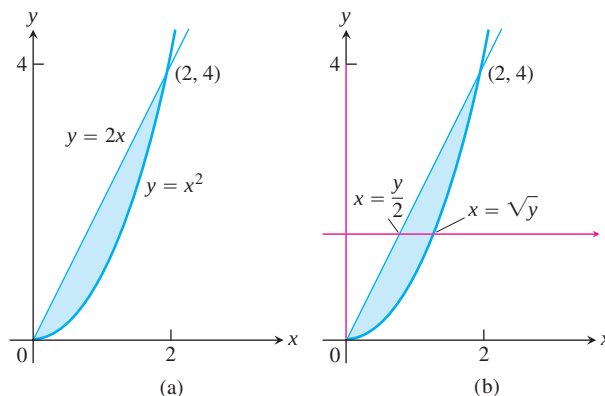


FIGURE 15.13 Region of integration for Example 4.

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$ (Figure 15.13b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2) dx dy.$$

The common value of these integrals is 8. ■

Properties of Double Integrals

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous, then

1. *Constant Multiple:* $\iint_R cf(x, y) dA = c \iint_R f(x, y) dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y)) dA = \iint_R f(x, y) dA \pm \iint_R g(x, y) dA$$

3. *Domination:*

(a) $\iint_R f(x, y) dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\iint_R f(x, y) dA \geq \iint_R g(x, y) dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$

if R is the union of two nonoverlapping regions R_1 and R_2 (Figure 15.7).

The idea behind these properties is that integrals behave like sums. If the function $f(x, y)$ is replaced by its constant multiple $cf(x, y)$, then a Riemann sum for f

$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$

is replaced by a Riemann sum for cf

$$\sum_{k=1}^n cf(x_k, y_k) \Delta A_k = c \sum_{k=1}^n f(x_k, y_k) \Delta A_k = cS_n.$$

Taking limits as $n \rightarrow \infty$ shows that $c \lim_{n \rightarrow \infty} S_n = c \iint_R f dA$ and $\lim_{n \rightarrow \infty} cS_n = \iint_R cf dA$ are equal. It follows that the constant multiple property carries over from sums to double integrals.

The other properties are also easy to verify for Riemann sums, and carry over to double integrals for the same reason. While this discussion gives the idea, an actual proof that these properties hold requires a more careful analysis of how Riemann sums converge.

EXERCISES 15.1

Finding Regions of Integration and Double Integrals

In Exercises 1–10, sketch the region of integration and evaluate the integral.

1. $\int_0^3 \int_0^2 (4 - y^2) dy dx$
2. $\int_0^3 \int_{-2}^0 (x^2 y - 2xy) dy dx$
3. $\int_{-1}^0 \int_{-1}^1 (x + y + 1) dx dy$
4. $\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y) dx dy$
5. $\int_0^{\pi} \int_0^x x \sin y dy dx$
6. $\int_0^{\pi} \int_0^{\sin x} y dy dx$
7. $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} dx dy$
8. $\int_1^2 \int_y^{y^2} dx dy$
9. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dx dy$
10. $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} dy dx$

In Exercises 11–16, integrate f over the given region.

11. **Quadrilateral** $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, $x = 2$
12. **Square** $f(x, y) = 1/(xy)$ over the square $1 \leq x \leq 2$, $1 \leq y \leq 2$
13. **Triangle** $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$
14. **Rectangle** $f(x, y) = y \cos xy$ over the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq 1$
15. **Triangle** $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv -plane by the line $u + v = 1$
16. **Curved region** $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st -plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$

Each of Exercises 17–20 gives an integral over a region in a Cartesian coordinate plane. Sketch the region and evaluate the integral.

17. $\int_{-2}^0 \int_v^{-v} 2 dp dv$ (the pv -plane)
18. $\int_0^1 \int_0^{\sqrt{1-s^2}} 8t dt ds$ (the st -plane)
19. $\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t du dt$ (the tu -plane)
20. $\int_0^3 \int_1^{4-2u} \frac{4-2u}{v^2} dv du$ (the uv -plane)

Reversing the Order of Integration

In Exercises 21–30, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

21. $\int_0^1 \int_2^{4-2x} dy dx$
22. $\int_0^2 \int_{y-2}^0 dx dy$
23. $\int_0^1 \int_y^{\sqrt{y}} dx dy$
24. $\int_0^1 \int_{1-x}^{1-x^2} dy dx$
25. $\int_0^1 \int_1^{e^x} dy dx$
26. $\int_0^{\ln 2} \int_{e^x}^2 dx dy$
27. $\int_0^{3/2} \int_0^{9-4x^2} 16x dy dx$
28. $\int_0^2 \int_0^{4-y^2} y dx dy$
29. $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y dx dy$
30. $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x dy dx$

Evaluating Double Integrals

In Exercises 31–40, sketch the region of integration, reverse the order of integration, and evaluate the integral.

31. $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} dy dx$
32. $\int_0^2 \int_x^2 2y^2 \sin xy dy dx$
33. $\int_0^1 \int_y^1 x^2 e^{xy} dx dy$
34. $\int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y} dy dx$
35. $\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2} dx dy$
36. $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3} dy dx$
37. $\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5) dx dy$
38. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy dx}{y^4 + 1}$
39. **Square region** $\iint_R (y - 2x^2) dA$ where R is the region bounded by the square $|x| + |y| = 1$
40. **Triangular region** $\iint_R xy dA$ where R is the region bounded by the lines $y = x$, $y = 2x$, and $x + y = 2$

Volume Beneath a Surface $z = f(x, y)$

41. Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy -plane.
42. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy -plane.
43. Find the volume of the solid whose base is the region in the xy -plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.
44. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.



45. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.
46. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.
47. Find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.
48. Find the volume of the solid cut from the square column $|x| + |y| \leq 1$ by the planes $z = 0$ and $3x + z = 3$.
49. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$ and $x = 1$, on the sides by the cylinders $y = \pm 1/x$, and above and below by the planes $z = x + 1$ and $z = 0$.
50. Find the volume of the solid bounded on the front and back by the planes $x = \pm \pi/3$, on the sides by the cylinders $y = \pm \sec x$, above by the cylinder $z = 1 + y^2$, and below by the xy -plane.

Integrals over Unbounded Regions

Improper double integrals can often be computed similarly to improper integrals of one variable. The first iteration of the following improper integrals is conducted just as if they were proper integrals. One then evaluates an improper integral of a single variable by taking appropriate limits, as in Section 8.8. Evaluate the improper integrals in Exercises 51–54 as iterated integrals.



51. $\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y} dy dx$
52. $\int_{-1}^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y + 1) dy dx$
53. $\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2 + 1)(y^2 + 1)} dx dy$
54. $\int_0^\infty \int_0^\infty x e^{-(x+2y)} dx dy$

Approximating Double Integrals

In Exercises 55 and 56, approximate the double integral of $f(x, y)$ over the region R partitioned by the given vertical lines $x = a$ and horizontal lines $y = c$. In each subrectangle, use (x_k, y_k) as indicated for your approximation.

$$\iint_R f(x, y) dA \approx \sum_{k=1}^n f(x_k, y_k) \Delta A_k$$



55. $f(x, y) = x + y$ over the region R bounded above by the semicircle $y = \sqrt{1 - x^2}$ and below by the x -axis, using the partition $x = -1, -1/2, 0, 1/4, 1/2, 1$ and $y = 0, 1/2, 1$ with (x_k, y_k) the lower left corner in the k th subrectangle (provided the subrectangle lies within R)
56. $f(x, y) = x + 2y$ over the region R inside the circle $(x - 2)^2 + (y - 3)^2 = 1$ using the partition $x = 1, 3/2, 2, 5/2, 3$ and $y = 2, 5/2, 3, 7/2, 4$ with (x_k, y_k) the center (centroid) in the k th subrectangle (provided the subrectangle lies within R)

Theory and Examples

57. **Circular sector** Integrate $f(x, y) = \sqrt{4 - x^2}$ over the smaller sector cut from the disk $x^2 + y^2 \leq 4$ by the rays $\theta = \pi/6$ and $\theta = \pi/2$.
58. **Unbounded region** Integrate $f(x, y) = 1/[(x^2 - x)(y - 1)^{2/3}]$ over the infinite rectangle $2 \leq x < \infty, 0 \leq y \leq 2$.
59. **Noncircular cylinder** A solid right (noncircular) cylinder has its base R in the xy -plane and is bounded above by the paraboloid $z = x^2 + y^2$. The cylinder's volume is

$$V = \int_0^1 \int_0^y (x^2 + y^2) dx dy + \int_1^2 \int_0^{2-y} (x^2 + y^2) dx dy.$$

Sketch the base region R and express the cylinder's volume as a single iterated integral with the order of integration reversed. Then evaluate the integral to find the volume.

60. **Converting to a double integral** Evaluate the integral

$$\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x) dx.$$

(Hint: Write the integrand as an integral.)

61. **Maximizing a double integral** What region R in the xy -plane maximizes the value of

$$\iint_R (4 - x^2 - 2y^2) dA?$$

Give reasons for your answer.

62. **Minimizing a double integral** What region R in the xy -plane minimizes the value of

$$\iint_R (x^2 + y^2 - 9) dA?$$

Give reasons for your answer.

63. Is it possible to evaluate the integral of a continuous function $f(x, y)$ over a rectangular region in the xy -plane and get different answers depending on the order of integration? Give reasons for your answer.
64. How would you evaluate the double integral of a continuous function $f(x, y)$ over the region R in the xy -plane enclosed by the triangle with vertices $(0, 1)$, $(2, 0)$, and $(1, 2)$? Give reasons for your answer.
65. **Unbounded region** Prove that

$$\begin{aligned} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2} dx dy &= \lim_{b \rightarrow \infty} \int_{-b}^b \int_{-b}^b e^{-x^2-y^2} dx dy \\ &= 4 \left(\int_0^\infty e^{-x^2} dx \right)^2. \end{aligned}$$

66. **Improper double integral** Evaluate the improper integral

$$\int_0^1 \int_0^3 \frac{x^2}{(y - 1)^{2/3}} dy dx.$$

COMPUTER EXPLORATIONS

Evaluating Double Integrals Numerically

Use a CAS double-integral evaluator to estimate the values of the integrals in Exercises 67–70.

$$67. \int_1^3 \int_1^x \frac{1}{xy} dy dx \qquad 68. \int_0^1 \int_0^1 e^{-(x^2+y^2)} dy dx$$

$$69. \int_0^1 \int_0^1 \tan^{-1} xy dy dx$$

$$70. \int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2} dy dx$$

Use a CAS double-integral evaluator to find the integrals in Exercises 71–76. Then reverse the order of integration and evaluate, again with a CAS.

$$71. \int_0^1 \int_{2y}^4 e^{x^2} dx dy \qquad 72. \int_0^3 \int_{x^2}^9 x \cos(y^2) dy dx$$

$$73. \int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2) dx dy$$

$$74. \int_0^2 \int_0^{4-y^2} e^{xy} dx dy$$

$$75. \int_1^2 \int_0^{x^2} \frac{1}{x+y} dy dx \qquad 76. \int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}} dx dy$$

15.2

Area, Moments, and Centers of Mass

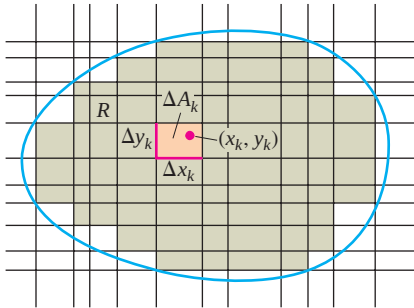


FIGURE 15.14 As the norm of a partition of the region R approaches zero, the sum of the areas ΔA_k gives the area of R defined by the double integral $\iint_R dA$.

In this section, we show how to use double integrals to calculate the areas of bounded regions in the plane and to find the average value of a function of two variables. Then we study the physical problem of finding the center of mass of a thin plate covering a region in the plane.

Areas of Bounded Regions in the Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^n f(x_k, y_k) \Delta A_k = \sum_{k=1}^n \Delta A_k. \quad (1)$$

This is simply the sum of the areas of the small rectangles in the partition of R , and approximates what we would like to call the area of R . As the norm of a partition of R approaches zero, the height and width of all rectangles in the partition approach zero, and the coverage of R becomes increasingly complete (Figure 15.14). We define the area of R to be the limit

$$\text{Area} = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \Delta A_k = \iint_R dA \quad (2)$$

DEFINITION Area

The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

As with the other definitions in this chapter, the definition here applies to a greater variety of regions than does the earlier single-variable definition of area, but it agrees with the earlier definition on regions to which they both apply. To evaluate the integral in the definition of area, we integrate the constant function $f(x, y) = 1$ over R .

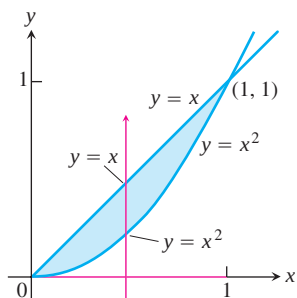


FIGURE 15.15 The region in Example 1.

**EXAMPLE 1** Finding Area

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Figure 15.15), noting where the two curves intersect, and calculate the area as

$$\begin{aligned} A &= \int_0^1 \int_{x^2}^x dy \, dx = \int_0^1 \left[y \right]_{x^2}^x dx \\ &= \int_0^1 (x - x^2) \, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}. \end{aligned}$$

Notice that the single integral $\int_0^1 (x - x^2) \, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.5. ■

**EXAMPLE 2** Finding Area

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution If we divide R into the regions R_1 and R_2 shown in Figure 15.16a, we may calculate the area as

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx \, dy.$$

On the other hand, reversing the order of integration (Figure 15.16b) gives

$$A = \int_{-1}^2 \int_{x^2}^{x+2} dy \, dx.$$

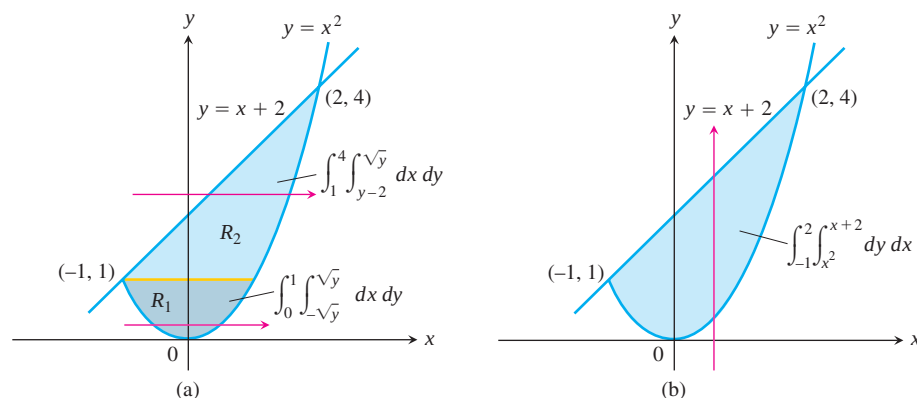


FIGURE 15.16 Calculating this area takes (a) two double integrals if the first integration is with respect to x , but (b) only one if the first integration is with respect to y (Example 2).

This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. The area is

$$A = \int_{-1}^2 \left[y \right]_{x^2}^{x+2} dx = \int_{-1}^2 (x + 2 - x^2) dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 = \frac{9}{2}. \quad \blacksquare$$

Average Value

The average value of an integrable function of one variable on a closed interval is the integral of the function over the interval divided by the length of the interval. For an integrable function of two variables defined on a bounded region in the plane, the average value is the integral over the region divided by the area of the region. This can be visualized by thinking of the function as giving the height at one instant of some water sloshing around in a tank whose vertical walls lie over the boundary of the region. The average height of the water in the tank can be found by letting the water settle down to a constant height. The height is then equal to the volume of water in the tank divided by the area of R . We are led to define the average value of an integrable function f over a region R to be

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f \, dA. \quad (3)$$

If f is the temperature of a thin plate covering R , then the double integral of f over R divided by the area of R is the plate's average temperature. If $f(x, y)$ is the distance from the point (x, y) to a fixed point P , then the average value of f over R is the average distance of points in R from P .



EXAMPLE 3 Finding Average Value

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R: 0 \leq x \leq \pi, 0 \leq y \leq 1$.

Solution The value of the integral of f over R is

$$\begin{aligned} \int_0^\pi \int_0^1 x \cos xy \, dy \, dx &= \int_0^\pi \left[\sin xy \right]_{y=0}^{y=1} dx && \int x \cos xy \, dy = \sin xy + C \\ &= \int_0^\pi (\sin x - 0) \, dx = -\cos x \Big|_0^\pi = 1 + 1 = 2. \end{aligned}$$

The area of R is π . The average value of f over R is $2/\pi$. ■

Moments and Centers of Mass for Thin Flat Plates

In Section 6.4 we introduced the concepts of moments and centers of mass, and we saw how to compute these quantities for thin rods or strips and for plates of constant density. Using multiple integrals we can extend these calculations to a great variety of shapes with varying density. We first consider the problem of finding the center of mass of a thin flat plate: a disk of aluminum, say, or a triangular sheet of metal. We assume the distribution of

mass in such a plate to be continuous. A material's *density* function, denoted by $\delta(x, y)$, is the mass per unit area. The *mass* of a plate is obtained by integrating the density function over the region R forming the plate. The first moment about an axis is calculated by integrating over R the distance from the axis times the density. The center of mass is found from the first moments. Table 15.1 gives the double integral formulas for mass, first moments, and center of mass.

TABLE 15.1 Mass and first moment formulas for thin plates covering a region R in the xy -plane

Mass:	$M = \iint_R \delta(x, y) dA$	$\delta(x, y)$ is the density at (x, y)
First moments:	$M_x = \iint_R y\delta(x, y) dA, \quad M_y = \iint_R x\delta(x, y) dA$	
Center of mass:	$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$	

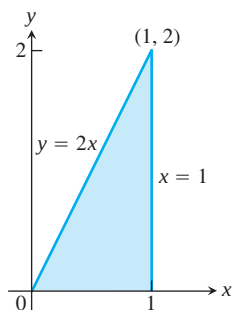


FIGURE 15.17 The triangular region covered by the plate in Example 4.



EXAMPLE 4 Finding the Center of Mass of a Thin Plate of Variable Density

A thin plate covers the triangular region bounded by the x -axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\delta(x, y) = 6x + 6y + 6$. Find the plate's mass, first moments, and center of mass about the coordinate axes.

Solution We sketch the plate and put in enough detail to determine the limits of integration for the integrals we have to evaluate (Figure 15.17).

The plate's mass is

$$\begin{aligned}
 M &= \int_0^1 \int_0^{2x} \delta(x, y) dy dx = \int_0^1 \int_0^{2x} (6x + 6y + 6) dy dx \\
 &= \int_0^1 \left[6xy + 3y^2 + 6y \right]_{y=0}^{y=2x} dx \\
 &= \int_0^1 (24x^2 + 12x) dx = \left[8x^3 + 6x^2 \right]_0^1 = 14.
 \end{aligned}$$

The first moment about the x -axis is

$$\begin{aligned}
 M_x &= \int_0^1 \int_0^{2x} y\delta(x, y) dy dx = \int_0^1 \int_0^{2x} (6xy + 6y^2 + 6y) dy dx \\
 &= \int_0^1 \left[3xy^2 + 2y^3 + 3y^2 \right]_{y=0}^{y=2x} dx = \int_0^1 (28x^3 + 12x^2) dx \\
 &= \left[7x^4 + 4x^3 \right]_0^1 = 11.
 \end{aligned}$$

A similar calculation gives the moment about the y -axis:

$$M_y = \int_0^1 \int_0^{2x} x \delta(x, y) dy dx = 10.$$

The coordinates of the center of mass are therefore

$$\bar{x} = \frac{M_y}{M} = \frac{10}{14} = \frac{5}{7}, \quad \bar{y} = \frac{M_x}{M} = \frac{11}{14}.$$

Moments of Inertia

A body's first moments (Table 15.1) tell us about balance and about the torque the body exerts about different axes in a gravitational field. If the body is a rotating shaft, however, we are more likely to be interested in how much energy is stored in the shaft or about how much energy it will take to accelerate the shaft to a particular angular velocity. This is where the second moment or moment of inertia comes in.

Think of partitioning the shaft into small blocks of mass Δm_k and let r_k denote the distance from the k th block's center of mass to the axis of rotation (Figure 15.18). If the shaft rotates at an angular velocity of $\omega = d\theta/dt$ radians per second, the block's center of mass will trace its orbit at a linear speed of

$$v_k = \frac{d}{dt}(r_k \theta) = r_k \frac{d\theta}{dt} = r_k \omega.$$

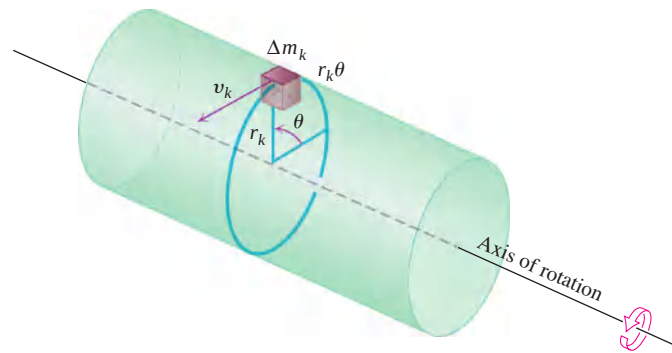


FIGURE 15.18 To find an integral for the amount of energy stored in a rotating shaft, we first imagine the shaft to be partitioned into small blocks. Each block has its own kinetic energy. We add the contributions of the individual blocks to find the kinetic energy of the shaft.

The block's kinetic energy will be approximately

$$\frac{1}{2} \Delta m_k v_k^2 = \frac{1}{2} \Delta m_k (r_k \omega)^2 = \frac{1}{2} \omega^2 r_k^2 \Delta m_k.$$

The kinetic energy of the shaft will be approximately

$$\sum \frac{1}{2} \omega^2 r_k^2 \Delta m_k.$$

The integral approached by these sums as the shaft is partitioned into smaller and smaller blocks gives the shaft's kinetic energy:

$$KE_{\text{shaft}} = \int \frac{1}{2} \omega^2 r^2 dm = \frac{1}{2} \omega^2 \int r^2 dm. \quad (4)$$

The factor

$$I = \int r^2 dm$$

is the *moment of inertia* of the shaft about its axis of rotation, and we see from Equation (4) that the shaft's kinetic energy is

$$KE_{\text{shaft}} = \frac{1}{2} I \omega^2.$$

The moment of inertia of a shaft resembles in some ways the inertia of a locomotive. To start a locomotive with mass m moving at a linear velocity v , we need to provide a kinetic energy of $KE = (1/2)mv^2$. To stop the locomotive we have to remove this amount of energy. To start a shaft with moment of inertia I rotating at an angular velocity ω , we need to provide a kinetic energy of $KE = (1/2)I\omega^2$. To stop the shaft we have to take this amount of energy back out. The shaft's moment of inertia is analogous to the locomotive's mass. What makes the locomotive hard to start or stop is its mass. What makes the shaft hard to start or stop is its moment of inertia. The moment of inertia depends not only on the mass of the shaft, but also its distribution.

The moment of inertia also plays a role in determining how much a horizontal metal beam will bend under a load. The stiffness of the beam is a constant times I , the moment of inertia of a typical cross-section of the beam about the beam's longitudinal axis. The greater the value of I , the stiffer the beam and the less it will bend under a given load. That is why we use I-beams instead of beams whose cross-sections are square. The flanges at the top and bottom of the beam hold most of the beam's mass away from the longitudinal axis to maximize the value of I (Figure 15.19).

To see the moment of inertia at work, try the following experiment. Tape two coins to the ends of a pencil and twiddle the pencil about the center of mass. The moment of inertia accounts for the resistance you feel each time you change the direction of motion. Now move the coins an equal distance toward the center of mass and twiddle the pencil again. The system has the same mass and the same center of mass but now offers less resistance to the changes in motion. The moment of inertia has been reduced. The moment of inertia is what gives a baseball bat, golf club, or tennis racket its "feel." Tennis rackets that weigh the same, look the same, and have identical centers of mass will feel different and behave differently if their masses are not distributed the same way.

Computations of moments of inertia for thin plates in the plane lead to double integral formulas, which are summarized in Table 15.2. A small thin piece of mass Δm is equal to its small area ΔA multiplied by the density of a point in the piece. Computations of moments of inertia for objects occupying a region in space are discussed in Section 15.5.

The mathematical difference between the **first moments** M_x and M_y and the **moments of inertia**, or **second moments**, I_x and I_y is that the second moments use the *squares* of the "lever-arm" distances x and y .

The moment I_0 is also called the **polar moment** of inertia about the origin. It is calculated by integrating the density $\delta(x, y)$ (mass per unit area) times $r^2 = x^2 + y^2$, the square of the distance from a representative point (x, y) to the origin. Notice that $I_0 = I_x + I_y$; once we find two, we get the third automatically. (The moment I_0 is sometimes called I_z , for

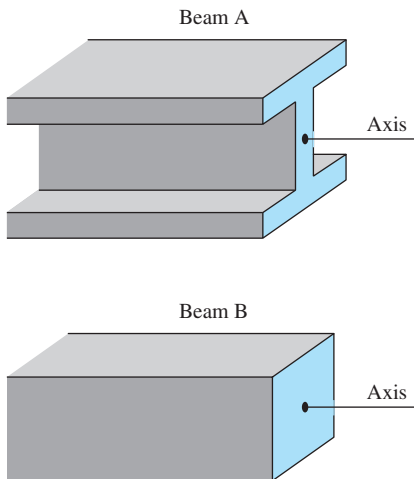


FIGURE 15.19 The greater the polar moment of inertia of the cross-section of a beam about the beam's longitudinal axis, the stiffer the beam. Beams A and B have the same cross-sectional area, but A is stiffer.

moment of inertia about the z -axis. The identity $I_z = I_x + I_y$ is then called the **Perpendicular Axis Theorem**.)

The **radius of gyration** R_x is defined by the equation

$$I_x = MR_x^2.$$

It tells how far from the x -axis the entire mass of the plate might be concentrated to give the same I_x . The radius of gyration gives a convenient way to express the moment of inertia in terms of a mass and a length. The radii R_y and R_0 are defined in a similar way, with

$$I_y = MR_y^2 \quad \text{and} \quad I_0 = MR_0^2.$$

We take square roots to get the formulas in Table 15.2, which gives the formulas for moments of inertia (second moments) as well as for radii of gyration.

TABLE 15.2 Second moment formulas for thin plates in the xy -plane

Moments of inertia (second moments):

About the x -axis: $I_x = \iint y^2 \delta(x, y) \, dA$

About the y -axis: $I_y = \iint x^2 \delta(x, y) \, dA$

About a line L : $I_L = \iint r^2(x, y) \delta(x, y) \, dA,$
where $r(x, y)$ = distance from (x, y) to L

About the origin
(polar moment): $I_0 = \iint (x^2 + y^2) \delta(x, y) \, dA = I_x + I_y$

Radii of gyration:

About the x -axis:	$R_x = \sqrt{I_x/M}$
About the y -axis:	$R_y = \sqrt{I_y/M}$
About the origin:	$R_0 = \sqrt{I_0/M}$



EXAMPLE 5 Finding Moments of Inertia and Radii of Gyration

For the thin plate in Example 4 (Figure 15.17), find the moments of inertia and radii of gyration about the coordinate axes and the origin.

Solution Using the density function $\delta(x, y) = 6x + 6y + 6$ given in Example 4, the moment of inertia about the x -axis is

$$\begin{aligned} I_x &= \int_0^1 \int_0^{2x} y^2 \delta(x, y) \, dy \, dx = \int_0^1 \int_0^{2x} (6xy^2 + 6y^3 + 6y^2) \, dy \, dx \\ &= \int_0^1 \left[2xy^3 + \frac{3}{2}y^4 + 2y^3 \right]_{y=0}^{y=2x} dx = \int_0^1 (40x^4 + 16x^3) \, dx \\ &= \left[8x^5 + 4x^4 \right]_0^1 = 12. \end{aligned}$$

Similarly, the moment of inertia about the y -axis is

$$I_y = \int_0^1 \int_0^{2x} x^2 \delta(x, y) dy dx = \frac{39}{5}.$$

Notice that we integrate y^2 times density in calculating I_x and x^2 times density to find I_y .

Since we know I_x and I_y , we do not need to evaluate an integral to find I_0 ; we can use the equation $I_0 = I_x + I_y$ instead:

$$I_0 = 12 + \frac{39}{5} = \frac{60 + 39}{5} = \frac{99}{5}.$$

The three radii of gyration are

$$R_x = \sqrt{I_x/M} = \sqrt{12/14} = \sqrt{6/7} \approx 0.93$$

$$R_y = \sqrt{I_y/M} = \sqrt{\left(\frac{39}{5}\right)/14} = \sqrt{39/70} \approx 0.75$$

$$R_0 = \sqrt{I_0/M} = \sqrt{\left(\frac{99}{5}\right)/14} = \sqrt{99/70} \approx 1.19. \quad \blacksquare$$

Moments are also of importance in statistics. The first moment is used in computing the mean μ of a set of data, and the second moment is used in computing the variance (Σ^2) and the standard deviation (Σ). Third and fourth moments are used for computing statistical quantities known as skewness and kurtosis.

Centroids of Geometric Figures

When the density of an object is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$ in Table 15.1. As far as $\bar{x}$ and $\bar{y}$ are concerned, δ might as well be 1. Thus, when δ is constant, the location of the center of mass becomes a feature of the object's shape and not of the material of which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape. To find a centroid, we set δ equal to 1 and proceed to find $\bar{x}$ and $\bar{y}$ as before, by dividing first moments by masses.



EXAMPLE 6 Finding the Centroid of a Region

Find the centroid of the region in the first quadrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$.

Solution We sketch the region and include enough detail to determine the limits of integration (Figure 15.20). We then set δ equal to 1 and evaluate the appropriate formulas from Table 15.1:

$$M = \int_0^1 \int_{x^2}^x 1 dy dx = \int_0^1 \left[y \right]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2) dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$\begin{aligned} M_x &= \int_0^1 \int_{x^2}^x y dy dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=x} dx \\ &= \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx = \left[\frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{15} \end{aligned}$$

$$M_y = \int_0^1 \int_{x^2}^x x dy dx = \int_0^1 \left[xy \right]_{y=x^2}^{y=x} dx = \int_0^1 (x^2 - x^3) dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}.$$

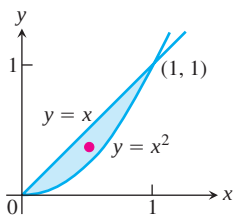


FIGURE 15.20 The centroid of this region is found in Example 6.

From these values of M , M_x , and M_y , we find

$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}.$$

The centroid is the point $(1/2, 2/5)$. ■

EXERCISES 15.2

Area by Double Integration

In Exercises 1–8, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

1. The coordinate axes and the line $x + y = 2$
2. The lines $x = 0$, $y = 2x$, and $y = 4$
3. The parabola $x = -y^2$ and the line $y = x + 2$
4. The parabola $x = y - y^2$ and the line $y = -x$
5. The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$
6. The curves $y = \ln x$ and $y = 2 \ln x$ and the line $x = e$, in the first quadrant
7. The parabolas $x = y^2$ and $x = 2y - y^2$
8. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$

Identifying the Region of Integration

The integrals and sums of integrals in Exercises 9–14 give the areas of regions in the xy -plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

9. $\int_0^6 \int_{y^2/3}^{2y} dx dy$
10. $\int_0^3 \int_{-x}^{x(2-x)} dy dx$
11. $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy dx$
12. $\int_{-1}^2 \int_{y^2}^{y+2} dx dy$
13. $\int_{-1}^0 \int_{-2x}^{1-x} dy dx + \int_0^2 \int_{-x/2}^{1-x} dy dx$
14. $\int_0^2 \int_{x^2-4}^0 dy dx + \int_0^4 \int_0^{\sqrt{x}} dy dx$

Average Values

15. Find the average value of $f(x, y) = \sin(x + y)$ over
 - a. the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi$
 - b. the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi/2$
16. Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, or the average value of f over the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant? Calculate them to find out.

17. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \leq x \leq 2$, $0 \leq y \leq 2$.
18. Find the average value of $f(x, y) = 1/(xy)$ over the square $\ln 2 \leq x \leq 2 \ln 2$, $\ln 2 \leq y \leq 2 \ln 2$.

Constant Density

19. **Finding center of mass** Find a center of mass of a thin plate of density $\delta = 3$ bounded by the lines $x = 0$, $y = x$, and the parabola $y = 2 - x^2$ in the first quadrant.
20. **Finding moments of inertia and radii of gyration** Find the moments of inertia and radii of gyration about the coordinate axes of a thin rectangular plate of constant density δ bounded by the lines $x = 3$ and $y = 3$ in the first quadrant.
21. **Finding a centroid** Find the centroid of the region in the first quadrant bounded by the x -axis, the parabola $y^2 = 2x$, and the line $x + y = 4$.
22. **Finding a centroid** Find the centroid of the triangular region cut from the first quadrant by the line $x + y = 3$.
23. **Finding a centroid** Find the centroid of the semicircular region bounded by the x -axis and the curve $y = \sqrt{1 - x^2}$.
24. **Finding a centroid** The area of the region in the first quadrant bounded by the parabola $y = 6x - x^2$ and the line $y = x$ is $125/6$ square units. Find the centroid.
25. **Finding a centroid** Find the centroid of the region cut from the first quadrant by the circle $x^2 + y^2 = a^2$.
26. **Finding a centroid** Find the centroid of the region between the x -axis and the arch $y = \sin x$, $0 \leq x \leq \pi$.
27. **Finding moments of inertia** Find the moment of inertia about the x -axis of a thin plate of density $\delta = 1$ bounded by the circle $x^2 + y^2 = 4$. Then use your result to find I_y and I_0 for the plate.
28. **Finding a moment of inertia** Find the moment of inertia with respect to the y -axis of a thin sheet of constant density $\delta = 1$ bounded by the curve $y = (\sin^2 x)/x^2$ and the interval $\pi \leq x \leq 2\pi$ of the x -axis.
29. **The centroid of an infinite region** Find the centroid of the infinite region in the second quadrant enclosed by the coordinate axes and the curve $y = e^x$. (Use improper integrals in the mass-moment formulas.)





- 30. The first moment of an infinite plate** Find the first moment about the y -axis of a thin plate of density $\delta(x, y) = 1$ covering the infinite region under the curve $y = e^{-x^2/2}$ in the first quadrant.

Variable Density



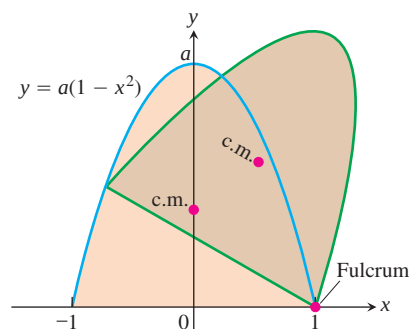
- 31. Finding a moment of inertia and radius of gyration** Find the moment of inertia and radius of gyration about the x -axis of a thin plate bounded by the parabola $x = y - y^2$ and the line $x + y = 0$ if $\delta(x, y) = x + y$.
- 32. Finding mass** Find the mass of a thin plate occupying the smaller region cut from the ellipse $x^2 + 4y^2 = 12$ by the parabola $x = 4y^2$ if $\delta(x, y) = 5x$.
- 33. Finding a center of mass** Find the center of mass of a thin triangular plate bounded by the y -axis and the lines $y = x$ and $y = 2 - x$ if $\delta(x, y) = 6x + 3y + 3$.
- 34. Finding a center of mass and moment of inertia** Find the center of mass and moment of inertia about the x -axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is $\delta(x, y) = y + 1$.
- 35. Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y -axis of a thin rectangular plate cut from the first quadrant by the lines $x = 6$ and $y = 1$ if $\delta(x, y) = x + y + 1$.
- 36. Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y -axis of a thin plate bounded by the line $y = 1$ and the parabola $y = x^2$ if the density is $\delta(x, y) = y + 1$.
- 37. Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the y -axis of a thin plate bounded by the x -axis, the lines $x = \pm 1$, and the parabola $y = x^2$ if $\delta(x, y) = 7y + 1$.
- 38. Center of mass, moment of inertia, and radius of gyration** Find the center of mass and the moment of inertia and radius of gyration about the x -axis of a thin rectangular plate bounded by the lines $x = 0$, $x = 20$, $y = -1$, and $y = 1$ if $\delta(x, y) = 1 + (x/20)$.
- 39. Center of mass, moments of inertia, and radii of gyration** Find the center of mass, the moment of inertia and radii of gyration about the coordinate axes, and the polar moment of inertia and radius of gyration of a thin triangular plate bounded by the lines $y = x$, $y = -x$, and $y = 1$ if $\delta(x, y) = y + 1$.
- 40. Center of mass, moments of inertia, and radii of gyration** Repeat Exercise 39 for $\delta(x, y) = 3x^2 + 1$.

Theory and Examples

- 41. Bacterium population** If $f(x, y) = (10,000e^y)/(1 + |x|/2)$ represents the "population density" of a certain bacterium on the xy -plane, where x and y are measured in centimeters, find the total population of bacteria within the rectangle $-5 \leq x \leq 5$ and $-2 \leq y \leq 0$.

- 42. Regional population** If $f(x, y) = 100(y + 1)$ represents the population density of a planar region on Earth, where x and y are measured in miles, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.

- 43. Appliance design** When we design an appliance, one of the concerns is how hard the appliance will be to tip over. When tipped, it will right itself as long as its center of mass lies on the correct side of the *fulcrum*, the point on which the appliance is riding as it tips. Suppose that the profile of an appliance of approximately constant density is parabolic, like an old-fashioned radio. It fills the region $0 \leq y \leq a(1 - x^2)$, $-1 \leq x \leq 1$, in the xy -plane (see accompanying figure). What values of a will guarantee that the appliance will have to be tipped more than 45° to fall over?



- 44. Minimizing a moment of inertia** A rectangular plate of constant density $\delta(x, y) = 1$ occupies the region bounded by the lines $x = 4$ and $y = 2$ in the first quadrant. The moment of inertia I_a of the rectangle about the line $y = a$ is given by the integral

$$I_a = \int_0^4 \int_0^2 (y - a)^2 dy dx.$$

Find the value of a that minimizes I_a .

- 45. Centroid of unbounded region** Find the centroid of the infinite region in the xy -plane bounded by the curves $y = 1/\sqrt{1 - x^2}$, $y = -1/\sqrt{1 - x^2}$, and the lines $x = 0$, $x = 1$.
- 46. Radius of gyration of slender rod** Find the radius of gyration of a slender rod of constant linear density δ gm/cm and length L cm with respect to an axis
- through the rod's center of mass perpendicular to the rod's axis.
 - perpendicular to the rod's axis at one end of the rod.
- 47. (Continuation of Exercise 34.)** A thin plate of now constant density δ occupies the region R in the xy -plane bounded by the curves $x = y^2$ and $x = 2y - y^2$.
- Constant density** Find δ such that the plate has the same mass as the plate in Exercise 34.
 - Average value** Compare the value of δ found in part (a) with the average value of $\delta(x, y) = y + 1$ over R .

- 48. Average temperature in Texas** According to the *Texas Almanac*, Texas has 254 counties and a National Weather Service station in each county. Assume that at time t_0 , each of the 254 weather stations recorded the local temperature. Find a formula that would give a reasonable approximation to the average temperature in Texas at time t_0 . Your answer should involve information that you would expect to be readily available in the *Texas Almanac*.

The Parallel Axis Theorem

Let $L_{c.m.}$ be a line in the xy -plane that runs through the center of mass of a thin plate of mass m covering a region in the plane. Let L be a line in the plane parallel to and h units away from $L_{c.m.}$. The **Parallel Axis Theorem** says that under these conditions the moments of inertia I_L and $I_{c.m.}$ of the plate about L and $L_{c.m.}$ satisfy the equation

$$I_L = I_{c.m.} + mh^2.$$

This equation gives a quick way to calculate one moment when the other moment and the mass are known.

49. Proof of the Parallel Axis Theorem

- Show that the first moment of a thin flat plate about any line in the plane of the plate through the plate's center of mass is zero. (*Hint*: Place the center of mass at the origin with the line along the y -axis. What does the formula $\bar{x} = M_y/M$ then tell you?)
- Use the result in part (a) to derive the Parallel Axis Theorem. Assume that the plane is coordinatized in a way that makes $L_{c.m.}$ the y -axis and L the line $x = h$. Then expand the integrand of the integral for I_L to rewrite the integral as the sum of integrals whose values you recognize.

50. Finding moments of inertia

- Use the Parallel Axis Theorem and the results of Example 4 to find the moments of inertia of the plate in Example 4 about the vertical and horizontal lines through the plate's center of mass.
- Use the results in part (a) to find the plate's moments of inertia about the lines $x = 1$ and $y = 2$.

Pappus's Formula

Pappus knew that the centroid of the union of two nonoverlapping plane regions lies on the line segment joining their individual centroids. More specifically, suppose that m_1 and m_2 are the masses of thin plates P_1 and P_2 that cover nonoverlapping regions in the xy -plane. Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be the vectors from the origin to the respective centers of mass of P_1 and P_2 . Then the center of mass of the union $P_1 \cup P_2$ of the two plates is determined by the vector

$$\mathbf{c} = \frac{m_1 \mathbf{c}_1 + m_2 \mathbf{c}_2}{m_1 + m_2}. \quad (5)$$

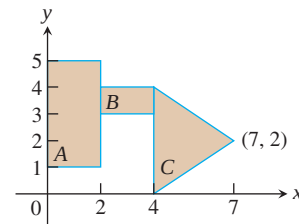
Equation (5) is known as **Pappus's formula**. For more than two nonoverlapping plates, as long as their number is finite, the formula

generalizes to

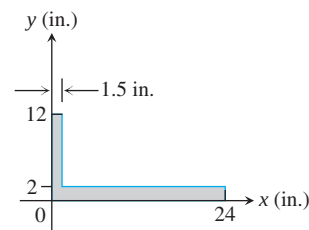
$$\mathbf{c} = \frac{m_1 \mathbf{c}_1 + m_2 \mathbf{c}_2 + \cdots + m_n \mathbf{c}_n}{m_1 + m_2 + \cdots + m_n}. \quad (6)$$

This formula is especially useful for finding the centroid of a plate of irregular shape that is made up of pieces of constant density whose centroids we know from geometry. We find the centroid of each piece and apply Equation (6) to find the centroid of the plate.

- Derive Pappus's formula (Equation (5)). (*Hint*: Sketch the plates as regions in the first quadrant and label their centers of mass as $(\bar{x}_1, \bar{y}_1)$ and $(\bar{x}_2, \bar{y}_2)$. What are the moments of $P_1 \cup P_2$ about the coordinate axes?)
- Use Equation (5) and mathematical induction to show that Equation (6) holds for any positive integer $n > 2$.
- Let A , B , and C be the shapes indicated in the accompanying figure. Use Pappus's formula to find the centroid of
 - $A \cup B$
 - $A \cup C$
 - $B \cup C$
 - $A \cup B \cup C$.



- 54. Locating center of mass** Locate the center of mass of the carpenter's square, shown here.



- An isosceles triangle T has base $2a$ and altitude h . The base lies along the diameter of a semicircular disk D of radius a so that the two together make a shape resembling an ice cream cone. What relation must hold between a and h to place the centroid of $T \cup D$ on the common boundary of T and D ? Inside T ?
- An isosceles triangle T of altitude h has as its base one side of a square Q whose edges have length s . (The square and triangle do not overlap.) What relation must hold between h and s to place the centroid of $T \cup Q$ on the base of the triangle? Compare your answer with the answer to Exercise 55.

15.3 Double Integrals in Polar Form

Integrals are sometimes easier to evaluate if we change to polar coordinates. This section shows how to accomplish the change and how to evaluate integrals over regions whose boundaries are given by polar equations.

Integrals in Polar Coordinates

When we defined the double integral of a function over a region R in the xy -plane, we began by cutting R into rectangles whose sides were parallel to the coordinate axes. These were the natural shapes to use because their sides have either constant x -values or constant y -values. In polar coordinates, the natural shape is a “polar rectangle” whose sides have constant r - and θ -values.

Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \leq g_1(\theta) \leq g_2(\theta) \leq a$ for every value of θ between α and β . Then R lies in a fan-shaped region Q defined by the inequalities $0 \leq r \leq a$ and $\alpha \leq \theta \leq \beta$. See Figure 15.21.

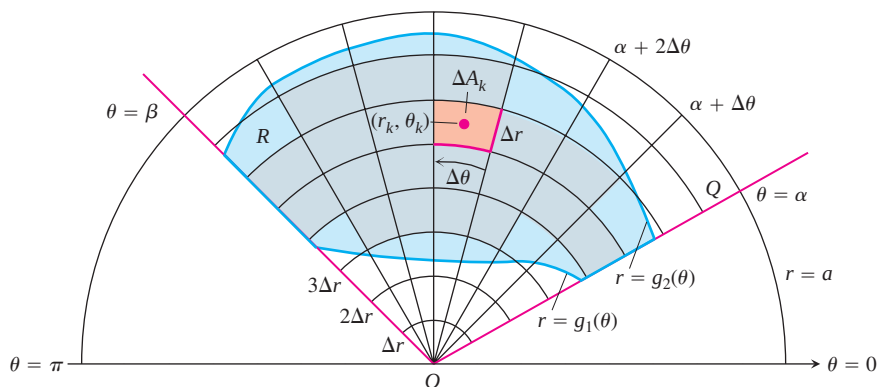


FIGURE 15.21 The region $R: g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta$, is contained in the fan-shaped region $Q: 0 \leq r \leq a, \alpha \leq \theta \leq \beta$. The partition of Q by circular arcs and rays induces a partition of R .

We cover Q by a grid of circular arcs and rays. The arcs are cut from circles centered at the origin, with radii $\Delta r, 2\Delta r, \dots, m\Delta r$, where $\Delta r = a/m$. The rays are given by

$$\theta = \alpha, \quad \theta = \alpha + \Delta\theta, \quad \theta = \alpha + 2\Delta\theta, \quad \dots, \quad \theta = \alpha + m'\Delta\theta = \beta,$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition Q into small patches called “polar rectangles.”

We number the polar rectangles that lie inside R (the order does not matter), calling their areas $\Delta A_1, \Delta A_2, \dots, \Delta A_n$. We let (r_k, θ_k) be any point in the polar rectangle whose area is ΔA_k . We then form the sum

$$S_n = \sum_{k=1}^n f(r_k, \theta_k) \Delta A_k.$$

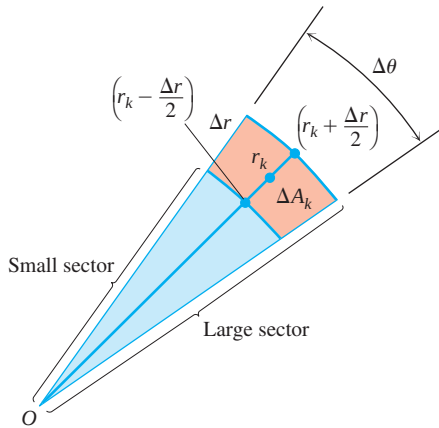


FIGURE 15.22 The observation that

$$\Delta A_k = \left(\begin{array}{c} \text{area of} \\ \text{large sector} \end{array} \right) - \left(\begin{array}{c} \text{area of} \\ \text{small sector} \end{array} \right)$$

leads to the formula $\Delta A_k = r_k \Delta r \Delta \theta$.

If f is continuous throughout R , this sum will approach a limit as we refine the grid to make Δr and $\Delta \theta$ go to zero. The limit is called the double integral of f over R . In symbols,

$$\lim_{n \rightarrow \infty} S_n = \iint_R f(r, \theta) dA.$$

To evaluate this limit, we first have to write the sum S_n in a way that expresses ΔA_k in terms of Δr and $\Delta \theta$. For convenience we choose r_k to be the average of the radii of the inner and outer arcs bounding the k th polar rectangle ΔA_k . The radius of the inner arc bounding ΔA_k is then $r_k - (\Delta r/2)$ (Figure 15.22). The radius of the outer arc is $r_k + (\Delta r/2)$.

The area of a wedge-shaped sector of a circle having radius r and angle θ is

$$A = \frac{1}{2} \theta \cdot r^2,$$

as can be seen by multiplying πr^2 , the area of the circle, by $\theta/2\pi$, the fraction of the circle's area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are

$$\text{Inner radius:} \quad \frac{1}{2} \left(r_k - \frac{\Delta r}{2} \right)^2 \Delta \theta$$

$$\text{Outer radius:} \quad \frac{1}{2} \left(r_k + \frac{\Delta r}{2} \right)^2 \Delta \theta.$$

Therefore,

$$\begin{aligned} \Delta A_k &= \text{area of large sector} - \text{area of small sector} \\ &= \frac{\Delta \theta}{2} \left[\left(r_k + \frac{\Delta r}{2} \right)^2 - \left(r_k - \frac{\Delta r}{2} \right)^2 \right] = \frac{\Delta \theta}{2} (2r_k \Delta r) = r_k \Delta r \Delta \theta. \end{aligned}$$

Combining this result with the sum defining S_n gives

$$S_n = \sum_{k=1}^n f(r_k, \theta_k) r_k \Delta r \Delta \theta.$$

As $n \rightarrow \infty$ and the values of Δr and $\Delta \theta$ approach zero, these sums converge to the double integral

$$\lim_{n \rightarrow \infty} S_n = \iint_R f(r, \theta) r dr d\theta.$$

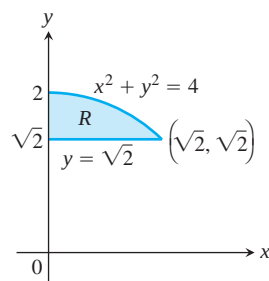
A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint_R f(r, \theta) dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta) r dr d\theta.$$

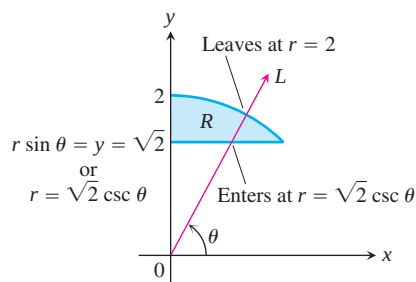
Finding Limits of Integration

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r, \theta) dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ , take the following steps.

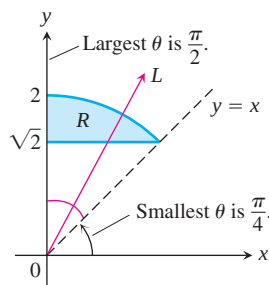
1. *Sketch:* Sketch the region and label the bounding curves.



2. *Find the r -limits of integration:* Imagine a ray L from the origin cutting through R in the direction of increasing r . Mark the r -values where L enters and leaves R . These are the r -limits of integration. They usually depend on the angle θ that L makes with the positive x -axis.



3. *Find the θ -limits of integration:* Find the smallest and largest θ -values that bound R . These are the θ -limits of integration.



The integral is

$$\iint_R f(r, \theta) dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2} \csc \theta}^{r=2} f(r, \theta) r dr d\theta.$$

EXAMPLE 1 Finding Limits of Integration

Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves (Figure 15.23).
2. Next we find the r -limits of integration. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos \theta$.

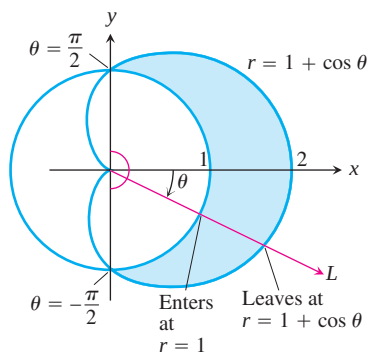


FIGURE 15.23 Finding the limits of integration in polar coordinates for the region in Example 1.

3. Finally we find the θ -limits of integration. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} f(r, \theta) r \, dr \, d\theta.$$

If $f(r, \theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R .

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r \, dr \, d\theta.$$

This formula for area is consistent with all earlier formulas, although we do not prove this fact.

EXAMPLE 2 Finding Area in Polar Coordinates

Find the area enclosed by the lemniscate $r^2 = 4 \cos 2\theta$.

Solution We graph the lemniscate to determine the limits of integration (Figure 15.24) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$\begin{aligned} A &= 4 \int_0^{\pi/4} \int_0^{\sqrt{4 \cos 2\theta}} r \, dr \, d\theta = 4 \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_{r=0}^{r=\sqrt{4 \cos 2\theta}} d\theta \\ &= 4 \int_0^{\pi/4} 2 \cos 2\theta \, d\theta = 4 \sin 2\theta \Big|_0^{\pi/4} = 4. \end{aligned}$$

Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint_R f(x, y) \, dx \, dy$ into a polar integral has two steps. First substitute $x = r \cos \theta$ and $y = r \sin \theta$, and replace $dx \, dy$ by $r \, dr \, d\theta$ in the Cartesian integral. Then supply polar limits of integration for the boundary of R .

The Cartesian integral then becomes

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(r \cos \theta, r \sin \theta) r \, dr \, d\theta,$$

where G denotes the region of integration in polar coordinates. This is like the substitution method in Chapter 5 except that there are now two variables to substitute for instead of one. Notice that $dx \, dy$ is not replaced by $dr \, d\theta$ but by $r \, dr \, d\theta$. A more general discussion of changes of variables (substitutions) in multiple integrals is given in Section 15.7.

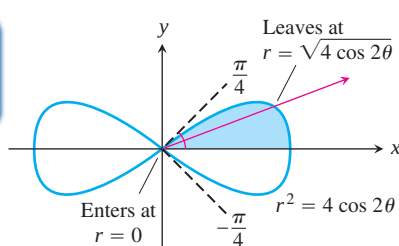


FIGURE 15.24 To integrate over the shaded region, we run r from 0 to $\sqrt{4 \cos 2\theta}$ and θ from 0 to $\pi/4$ (Example 2).

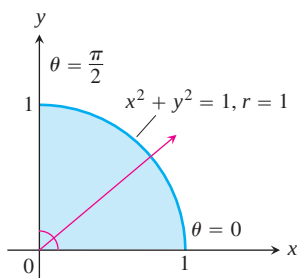


FIGURE 15.25 In polar coordinates, this region is described by simple inequalities:

$$0 \leq r \leq 1 \quad \text{and} \quad 0 \leq \theta \leq \pi/2$$

(Example 3).



EXAMPLE 3 Changing Cartesian Integrals to Polar Integrals

Find the polar moment of inertia about the origin of a thin plate of density $\delta(x, y) = 1$ bounded by the quarter circle $x^2 + y^2 = 1$ in the first quadrant.

Solution We sketch the plate to determine the limits of integration (Figure 15.25). In Cartesian coordinates, the polar moment is the value of the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx.$$

Integration with respect to y gives

$$\int_0^1 \left(x^2 \sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables.

Things go better if we change the original integral to polar coordinates. Substituting $x = r \cos \theta$, $y = r \sin \theta$ and replacing $dx dy$ by $r dr d\theta$, we get

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx &= \int_0^{\pi/2} \int_0^1 (r^2) r dr d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4} d\theta = \frac{\pi}{8}. \end{aligned}$$

Why is the polar coordinate transformation so effective here? One reason is that $x^2 + y^2$ simplifies to r^2 . Another is that the limits of integration become constants. ■

EXAMPLE 4 Evaluating Integrals Using Polar Coordinates

Evaluate

$$\iint_R e^{x^2+y^2} dy dx,$$

where R is the semicircular region bounded by the x -axis and the curve $y = \sqrt{1-x^2}$ (Figure 15.26).

Solution In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y . Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to find a way to evaluate it. Polar coordinates save the day. Substituting $x = r \cos \theta$, $y = r \sin \theta$ and replacing $dy dx$ by $r dr d\theta$ enables us to evaluate the integral as

$$\begin{aligned} \iint_R e^{x^2+y^2} dy dx &= \int_0^\pi \int_0^1 e^{r^2} r dr d\theta = \int_0^\pi \left[\frac{1}{2} e^{r^2} \right]_0^1 d\theta \\ &= \int_0^\pi \frac{1}{2} (e - 1) d\theta = \frac{\pi}{2} (e - 1). \end{aligned}$$

The r in the $r dr d\theta$ was just what we needed to integrate e^{r^2} . Without it, we would have been unable to proceed. ■

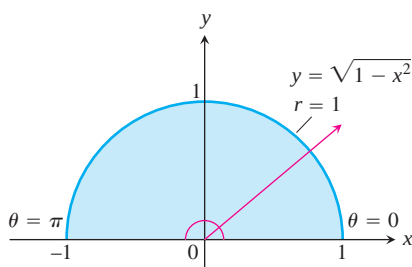


FIGURE 15.26 The semicircular region in Example 4 is the region

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi.$$



EXERCISES 15.3

Evaluating Polar Integrals

In Exercises 1–16, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

1. $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} dy \, dx$
2. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx$
3. $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) \, dx \, dy$
4. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2) \, dy \, dx$
5. $\int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \, dx$
6. $\int_0^2 \int_0^{\sqrt{4-y^2}} (x^2 + y^2) \, dx \, dy$
7. $\int_0^6 \int_0^y x \, dx \, dy$
8. $\int_0^2 \int_0^x y \, dy \, dx$
9. $\int_{-1}^0 \int_{-\sqrt{1-x^2}}^0 \frac{2}{1 + \sqrt{x^2 + y^2}} \, dy \, dx$
10. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^0 \frac{4\sqrt{x^2 + y^2}}{1 + x^2 + y^2} \, dx \, dy$
11. $\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2 + y^2}} \, dx \, dy$
12. $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{-(x^2 + y^2)} \, dy \, dx$
13. $\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} \frac{x + y}{x^2 + y^2} \, dy \, dx$
14. $\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 xy^2 \, dx \, dy$
15. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) \, dx \, dy$
16. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1 + x^2 + y^2)^2} \, dy \, dx$

Finding Area in Polar Coordinates

17. Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$.
18. **Cardioid overlapping a circle** Find the area of the region that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.
19. **One leaf of a rose** Find the area enclosed by one leaf of the rose $r = 12 \cos 3\theta$.
20. **Snail shell** Find the area of the region enclosed by the positive x -axis and spiral $r = 4\theta/3$, $0 \leq \theta \leq 2\pi$. The region looks like a snail shell.
21. **Cardioid in the first quadrant** Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin \theta$.
22. **Overlapping cardioids** Find the area of the region common to the interiors of the cardioids $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$.

Masses and Moments

23. **First moment of a plate** Find the first moment about the x -axis of a thin plate of constant density $\delta(x, y) = 3$, bounded below by the x -axis and above by the cardioid $r = 1 - \cos \theta$.
24. **Inertial and polar moments of a disk** Find the moment of inertia about the x -axis and the polar moment of inertia about the origin of a thin disk bounded by the circle $x^2 + y^2 = a^2$ if the disk's density at the point (x, y) is $\delta(x, y) = k(x^2 + y^2)$, k a constant.
25. **Mass of a plate** Find the mass of a thin plate covering the region outside the circle $r = 3$ and inside the circle $r = 6 \sin \theta$ if the plate's density function is $\delta(x, y) = 1/r$.
26. **Polar moment of a cardioid overlapping circle** Find the polar moment of inertia about the origin of a thin plate covering the region that lies inside the cardioid $r = 1 - \cos \theta$ and outside the circle $r = 1$ if the plate's density function is $\delta(x, y) = 1/r^2$.
27. **Centroid of a cardioid region** Find the centroid of the region enclosed by the cardioid $r = 1 + \cos \theta$.
28. **Polar moment of a cardioid region** Find the polar moment of inertia about the origin of a thin plate enclosed by the cardioid $r = 1 + \cos \theta$ if the plate's density function is $\delta(x, y) = 1$.

Average Values

29. **Average height of a hemisphere** Find the average height of the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the xy -plane.
30. **Average height of a cone** Find the average height of the (single) cone $z = \sqrt{x^2 + y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the xy -plane.
31. **Average distance from interior of disk to center** Find the average distance from a point $P(x, y)$ in the disk $x^2 + y^2 \leq a^2$ to the origin.
32. **Average distance squared from a point in a disk to a point in its boundary** Find the average value of the *square* of the distance from the point $P(x, y)$ in the disk $x^2 + y^2 \leq 1$ to the boundary point $A(1, 0)$.

Theory and Examples

33. **Converting to a polar integral** Integrate $f(x, y) = [\ln(x^2 + y^2)]/\sqrt{x^2 + y^2}$ over the region $1 \leq x^2 + y^2 \leq e$.
34. **Converting to a polar integral** Integrate $f(x, y) = [\ln(x^2 + y^2)]/(x^2 + y^2)$ over the region $1 \leq x^2 + y^2 \leq e^2$.
35. **Volume of noncircular right cylinder** The region that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.

- 36. Volume of noncircular right cylinder** The region enclosed by the lemniscate $r^2 = 2 \cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

37. Converting to polar integrals

- a. The usual way to evaluate the improper integral

$I = \int_0^\infty e^{-x^2} dx$ is first to calculate its square:

$$I^2 = \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy.$$

Evaluate the last integral using polar coordinates and solve the resulting equation for I .

- b. Evaluate

$$\lim_{x \rightarrow \infty} \operatorname{erf}(x) = \lim_{x \rightarrow \infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} dt.$$

- 38. Converting to a polar integral** Evaluate the integral

$$\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2} dx dy.$$

- 39. Existence** Integrate the function $f(x, y) = 1/(1 - x^2 - y^2)$ over the disk $x^2 + y^2 \leq 3/4$. Does the integral of $f(x, y)$ over the disk $x^2 + y^2 \leq 1$ exist? Give reasons for your answer.

- 40. Area formula in polar coordinates** Use the double integral in polar coordinates to derive the formula

$$A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta$$

for the area of the fan-shaped region between the origin and polar curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$.

- 41. Average distance to a given point inside a disk** Let P_0 be a point inside a circle of radius a and let h denote the distance from

P_0 to the center of the circle. Let d denote the distance from an arbitrary point P to P_0 . Find the average value of d^2 over the region enclosed by the circle. (*Hint:* Simplify your work by placing the center of the circle at the origin and P_0 on the x -axis.)

- 42. Area** Suppose that the area of a region in the polar coordinate plane is

$$A = \int_{\pi/4}^{3\pi/4} \int_{\csc \theta}^{2 \sin \theta} r dr d\theta.$$

Sketch the region and find its area.

COMPUTER EXPLORATIONS

Coordinate Conversions

In Exercises 43–46, use a CAS to change the Cartesian integrals into an equivalent polar integral and evaluate the polar integral. Perform the following steps in each exercise.

- Plot the Cartesian region of integration in the xy -plane.
- Change each boundary curve of the Cartesian region in part (a) to its polar representation by solving its Cartesian equation for r and θ .
- Using the results in part (b), plot the polar region of integration in the $r\theta$ -plane.
- Change the integrand from Cartesian to polar coordinates. Determine the limits of integration from your plot in part (c) and evaluate the polar integral using the CAS integration utility.

43. $\int_0^1 \int_x^1 \frac{y}{x^2 + y^2} dy dx$

44. $\int_0^1 \int_0^{x/2} \frac{x}{x^2 + y^2} dy dx$

45. $\int_0^1 \int_{-y/3}^{y/3} \frac{y}{\sqrt{x^2 + y^2}} dx dy$

46. $\int_0^1 \int_y^{2-y} \sqrt{x+y} dx dy$

15.4

Triple Integrals in Rectangular Coordinates

Just as double integrals allow us to deal with more general situations than could be handled by single integrals, triple integrals enable us to solve still more general problems. We use triple integrals to calculate the volumes of three-dimensional shapes, the masses and moments of solids of varying density, and the average value of a function over a three-dimensional region. Triple integrals also arise in the study of vector fields and fluid flow in three dimensions, as we will see in Chapter 16.

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed bounded region D in space, such as the region occupied by a solid ball or a lump of clay, then the integral of F over D may be defined in

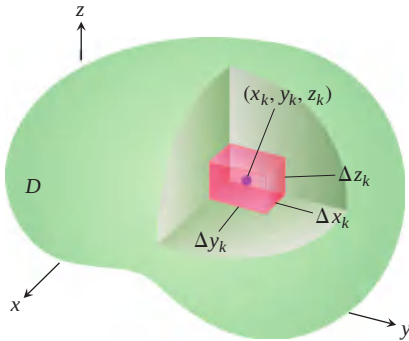


FIGURE 15.27 Partitioning a solid with rectangular cells of volume ΔV_k .

the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axis (Figure 15.27). We number the cells that lie inside D from 1 to n in some order, the k th cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^n F(x_k, y_k, z_k) \Delta V_k. \quad (1)$$

We are interested in what happens as D is partitioned by smaller and smaller cells, so that Δx_k , Δy_k , Δz_k and the norm of the partition $\|P\|$, the largest value among Δx_k , Δy_k , Δz_k , all approach zero. When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is **integrable** over D . As before, it can be shown that when F is continuous and the bounding surface of D is formed from finitely many smooth surfaces joined together along finitely many smooth curves, then F is integrable. As $\|P\| \rightarrow 0$ and the number of cells n goes to ∞ , the sums S_n approach a limit. We call this limit the **triple integral of F over D** and write

$$\lim_{n \rightarrow \infty} S_n = \iiint_D F(x, y, z) \, dV \quad \text{or} \quad \lim_{\|P\| \rightarrow 0} S_n = \iiint_D F(x, y, z) \, dx \, dy \, dz.$$

The regions D over which continuous functions are integrable are those that can be closely approximated by small rectangular cells. Such regions include those encountered in applications.

Volume of a Region in Space

If F is the constant function whose value is 1, then the sums in Equation (1) reduce to

$$S_n = \sum F(x_k, y_k, z_k) \Delta V_k = \sum 1 \cdot \Delta V_k = \sum \Delta V_k.$$

As Δx_k , Δy_k , and Δz_k approach zero, the cells ΔV_k become smaller and more numerous and fill up more and more of D . We therefore define the volume of D to be the triple integral

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta V_k = \iiint_D dV.$$

DEFINITION Volume

The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

This definition is in agreement with our previous definitions of volume, though we omit the verification of this fact. As we see in a moment, this integral enables us to calculate the volumes of solids enclosed by curved surfaces.

Finding Limits of Integration

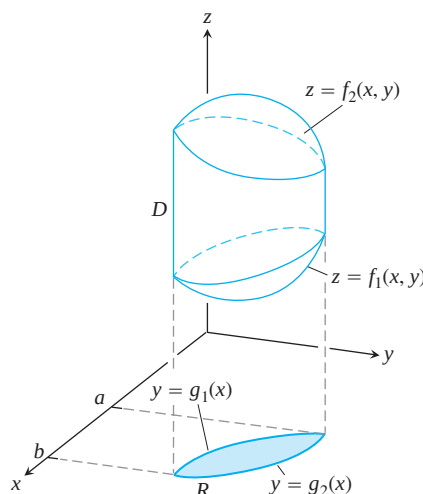
We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem (Section 15.1) to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these single integrals.

To evaluate

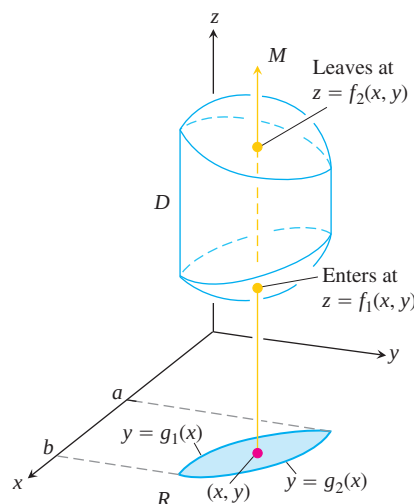
$$\iiint_D F(x, y, z) \, dV$$

over a region D , integrate first with respect to z , then with respect to y , finally with x .

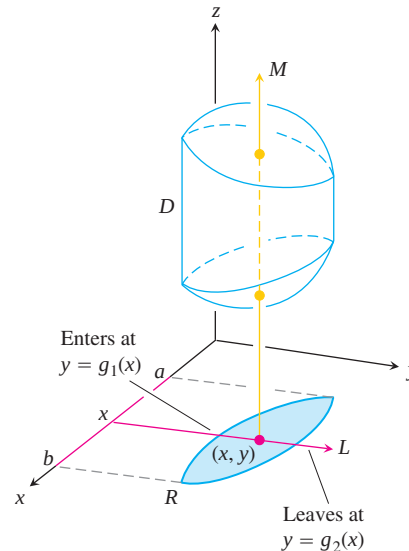
1. *Sketch:* Sketch the region D along with its “shadow” R (vertical projection) in the xy -plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R .



2. *Find the z-limits of integration:* Draw a line M passing through a typical point (x, y) in R parallel to the z -axis. As z increases, M enters D at $z = f1(x, y)$ and leaves at $z = f2(x, y)$. These are the z -limits of integration.



3. *Find the y -limits of integration:* Draw a line L through (x, y) parallel to the y -axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y -limits of integration.



4. *Find the x -limits of integration:* Choose x -limits that include all lines through R parallel to the y -axis ($x = a$ and $x = b$ in the preceding figure). These are the x -limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z) \, dz \, dy \, dx.$$

Follow similar procedures if you change the order of integration. The “shadow” of region D lies in the plane of the last two variables with respect to which the iterated integration takes place.

The above procedure applies whenever a solid region D is bounded above and below by a surface, and when the “shadow” region R is bounded by a lower and upper curve. It does not apply to regions with complicated holes through them, although sometimes such regions can be subdivided into simpler regions for which the procedure does apply.



EXAMPLE 1 Finding a Volume

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution The volume is

$$V = \iiint_D dz \, dy \, dx,$$

the integral of $F(x, y, z) = 1$ over D . To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure 15.28) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region R , the projection of D onto the xy -plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The “upper” boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

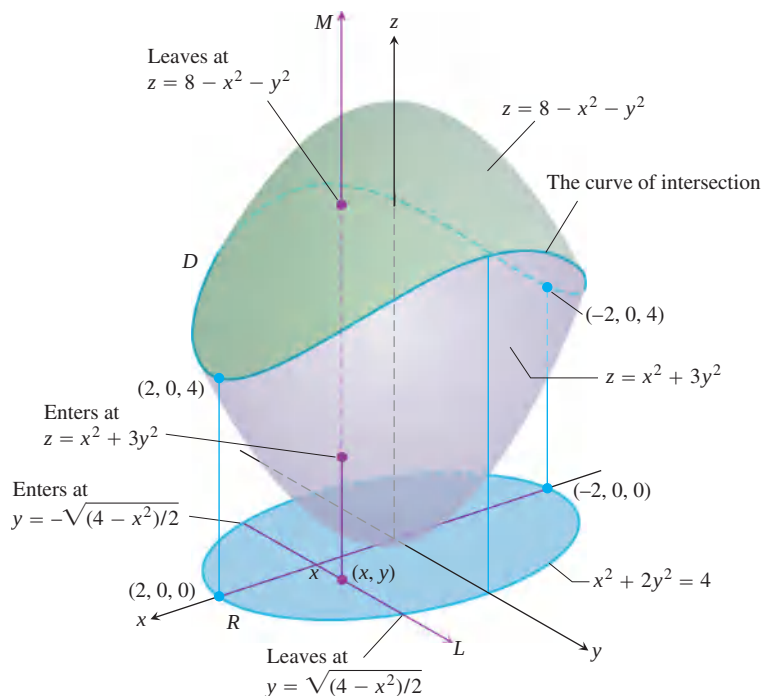


FIGURE 15.28 The volume of the region enclosed by two paraboloids, calculated in Example 1.

Now we find the z -limits of integration. The line M passing through a typical point (x, y) in R parallel to the z -axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

Next we find the y -limits of integration. The line L through (x, y) parallel to the y -axis enters R at $y = -\sqrt{(4 - x^2)}/2$ and leaves at $y = \sqrt{(4 - x^2)}/2$.

Finally we find the x -limits of integration. As L sweeps across R , the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$\begin{aligned}
 V &= \iiint_D dz \, dy \, dx \\
 &= \int_{-2}^2 \int_{-\sqrt{(4-x^2)}/2}^{\sqrt{(4-x^2)}/2} \int_{x^2+3y^2}^{8-x^2-y^2} dz \, dy \, dx \\
 &= \int_{-2}^2 \int_{-\sqrt{(4-x^2)}/2}^{\sqrt{(4-x^2)}/2} (8 - 2x^2 - 4y^2) \, dy \, dx \\
 &= \int_{-2}^2 \left[(8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)}/2}^{y=\sqrt{(4-x^2)}/2} dx \\
 &= \int_{-2}^2 \left(2(8 - 2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3} \left(\frac{4-x^2}{2} \right)^{3/2} \right) dx \\
 &= \int_{-2}^2 \left[8 \left(\frac{4-x^2}{2} \right)^{3/2} - \frac{8}{3} \left(\frac{4-x^2}{2} \right)^{3/2} \right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^2 (4 - x^2)^{3/2} dx \\
 &= 8\pi\sqrt{2}. \quad \text{After integration with the substitution } x = 2 \sin u.
 \end{aligned}$$

In the next example, we project D onto the xz -plane instead of the xy -plane, to show how to use a different order of integration.

EXAMPLE 2 Finding the Limits of Integration in the Order $dy \, dz \, dx$

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$.

Solution We sketch D along with its “shadow” R in the xz -plane (Figure 15.29). The upper (right-hand) bounding surface of D lies in the plane $y = 1$. The lower (left-hand) bounding surface lies in the plane $y = x + z$. The upper boundary of R is the line $z = 1 - x$. The lower boundary is the line $z = 0$.

First we find the y -limits of integration. The line through a typical point (x, z) in R parallel to the y -axis enters D at $y = x + z$ and leaves at $y = 1$.

Next we find the z -limits of integration. The line L through (x, z) parallel to the z -axis enters R at $z = 0$ and leaves at $z = 1 - x$.

Finally we find the x -limits of integration. As L sweeps across R , the value of x varies from $x = 0$ to $x = 1$. The integral is

$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x, y, z) \, dy \, dz \, dx.$$

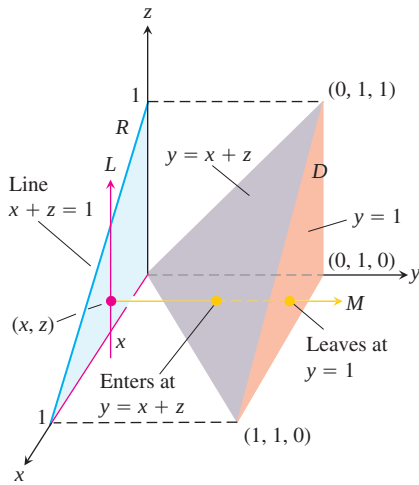


FIGURE 15.29 Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron D (Example 2).

EXAMPLE 3 Revisiting Example 2 Using the Order $dz \, dy \, dx$

To integrate $F(x, y, z)$ over the tetrahedron D in the order $dz \, dy \, dx$, we perform the steps in the following way.

First we find the z -limits of integration. A line parallel to the z -axis through a typical point (x, y) in the xy -plane “shadow” enters the tetrahedron at $z = 0$ and exits through the upper plane where $z = y - x$ (Figure 15.29).

Next we find the y -limits of integration. On the xy -plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line through (x, y) parallel to the y -axis enters the shadow in the xy -plane at $y = x$ and exits at $y = 1$.

Finally we find the x -limits of integration. As the line parallel to the y -axis in the previous step sweeps out the shadow, the value of x varies from $x = 0$ to $x = 1$ at the point $(1, 1, 0)$. The integral is

$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z) \, dz \, dy \, dx.$$

For example, if $F(x, y, z) = 1$, we would find the volume of the tetrahedron to be

$$\begin{aligned} V &= \int_0^1 \int_x^1 \int_0^{y-x} dz \, dy \, dx \\ &= \int_0^1 \int_x^1 (y - x) \, dy \, dx \\ &= \int_0^1 \left[\frac{1}{2} y^2 - xy \right]_{y=x}^{y=1} dx \\ &= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2} x^2 \right) dx \\ &= \left[\frac{1}{2} x - \frac{1}{2} x^2 + \frac{1}{6} x^3 \right]_0^1 \\ &= \frac{1}{6}. \end{aligned}$$

We get the same result by integrating with the order $dy \, dz \, dx$,

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy \, dz \, dx = \frac{1}{6}.$$

As we have seen, there are sometimes (but not always) two different orders in which the iterated single integrations for evaluating a double integral may be worked. For triple integrals, there can be as many as six, since there are six ways of ordering dx , dy , and dz . Each ordering leads to a different description of the region of integration in space, and to different limits of integration.

EXAMPLE 4 Using Different Orders of Integration

Each of the following integrals gives the volume of the solid shown in Figure 15.30.

- (a) $\int_0^1 \int_0^{1-z} \int_0^2 dx \, dy \, dz$ (b) $\int_0^1 \int_0^{1-y} \int_0^2 dx \, dz \, dy$
- (c) $\int_0^1 \int_0^2 \int_0^{1-z} dy \, dx \, dz$ (d) $\int_0^2 \int_0^1 \int_0^{1-z} dy \, dz \, dx$
- (e) $\int_0^1 \int_0^2 \int_0^{1-y} dz \, dx \, dy$ (f) $\int_0^2 \int_0^1 \int_0^{1-y} dz \, dy \, dx$

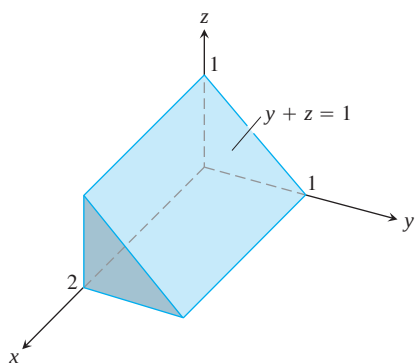


FIGURE 15.30 Example 4 gives six different iterated triple integrals for the volume of this prism.

We work out the integrals in parts (b) and (c):

$$\begin{aligned} V &= \int_0^1 \int_0^{1-y} \int_0^2 dx \, dz \, dy && \text{Integral in part (b)} \\ &= \int_0^1 \int_0^{1-y} 2 \, dz \, dy \\ &= \int_0^1 \left[2z \right]_{z=0}^{z=1-y} dy \\ &= \int_0^1 2(1-y) \, dy \\ &= 1. \end{aligned}$$

Also,

$$\begin{aligned} V &= \int_0^1 \int_0^2 \int_0^{1-z} dy \, dx \, dz && \text{Integral in part (c)} \\ &= \int_0^1 \int_0^2 (1-z) \, dx \, dz \\ &= \int_0^1 \left[x - zx \right]_{x=0}^{x=2} dz \end{aligned}$$



$$\begin{aligned}
 &= \int_0^1 (2 - 2z) \, dz \\
 &= 1.
 \end{aligned}$$

The integrals in parts (a), (d), (e), and (f) also give $V = 1$. ■

Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\text{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F \, dV. \quad (2)$$

For example, if $F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, then the average value of F over D is the average distance of points in D from the origin. If $F(x, y, z)$ is the temperature at (x, y, z) on a solid that occupies a region D in space, then the average value of F over D is the average temperature of the solid.

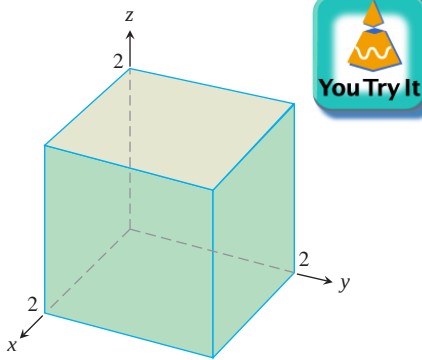


FIGURE 15.31 The region of integration in Example 5.

EXAMPLE 5 Finding an Average Value

Find the average value of $F(x, y, z) = xyz$ over the cube bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution We sketch the cube with enough detail to show the limits of integration (Figure 15.31). We then use Equation (2) to calculate the average value of F over the cube.

The volume of the cube is $(2)(2)(2) = 8$. The value of the integral of F over the cube is

$$\begin{aligned}
 \int_0^2 \int_0^2 \int_0^2 xyz \, dx \, dy \, dz &= \int_0^2 \int_0^2 \left[\frac{x^2}{2} yz \right]_{x=0}^{x=2} dy \, dz = \int_0^2 \int_0^2 2yz \, dy \, dz \\
 &= \int_0^2 \left[y^2 z \right]_{y=0}^{y=2} dz = \int_0^2 4z \, dz = \left[2z^2 \right]_0^2 = 8.
 \end{aligned}$$

With these values, Equation (2) gives

$$\text{Average value of } xyz \text{ over the cube} = \frac{1}{\text{volume}} \iiint_{\text{cube}} xyz \, dV = \left(\frac{1}{8} \right) (8) = 1.$$

In evaluating the integral, we chose the order $dx \, dy \, dz$, but any of the other five possible orders would have done as well. ■

Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals.

Properties of Triple Integrals

If $F = F(x, y, z)$ and $G = G(x, y, z)$ are continuous, then

$$1. \text{ Constant Multiple: } \iiint_D kF \, dV = k \iiint_D F \, dV \quad (\text{any number } k)$$

$$2. \text{ Sum and Difference: } \iiint_D (F \pm G) \, dV = \iiint_D F \, dV \pm \iiint_D G \, dV$$

3. Domination:

$$(a) \quad \iiint_D F \, dV \geq 0 \quad \text{if } F \geq 0 \text{ on } D$$

$$(b) \quad \iiint_D F \, dV \geq \iiint_D G \, dV \quad \text{if } F \geq G \text{ on } D$$

$$4. \text{ Additivity: } \iiint_D F \, dV = \iiint_{D_1} F \, dV + \iiint_{D_2} F \, dV$$

if D is the union of two nonoverlapping regions D_1 and D_2 .

EXERCISES 15.4

Evaluating Triple Integrals in Different Iterations



1. Evaluate the integral in Example 2 taking $F(x, y, z) = 1$ to find the volume of the tetrahedron.
2. **Volume of rectangular solid** Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 2$, and $z = 3$. Evaluate one of the integrals.
3. **Volume of tetrahedron** Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.
4. **Volume of solid** Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$. Evaluate one of the integrals.
5. **Volume enclosed by paraboloids** Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D . Evaluate one of the integrals.
6. **Volume inside paraboloid beneath a plane** Let D be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz dx dy$ and $dz dy dx$ that give the volume of D . Do not evaluate either integral.

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dz dy dx$
8. $\int_0^{\sqrt{2}} \int_0^{3y} \int_{x^2+3y^2}^{8-x^2-y^2} dz dx dy$
9. $\int_1^e \int_1^e \int_1^e \frac{1}{xyz} dx dy dz$
10. $\int_0^1 \int_0^{3-3x} \int_0^{3-3x-y} dz dy dx$
11. $\int_0^1 \int_0^\pi \int_0^\pi y \sin z dx dy dz$
12. $\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (x + y + z) dy dx dz$
13. $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2}} dz dy dx$
14. $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz dx dy$
15. $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} dz dy dx$
16. $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x dz dy dx$
17. $\int_0^\pi \int_0^\pi \int_0^\pi \cos(u + v + w) du dv dw \quad (uvw\text{-space})$
18. $\int_1^e \int_1^e \int_1^e \ln r \ln s \ln t dt dr ds \quad (rst\text{-space})$
19. $\int_0^{\pi/4} \int_0^{\ln \sec v} \int_{-\infty}^{2t} e^x dx dt dv \quad (tvx\text{-space})$





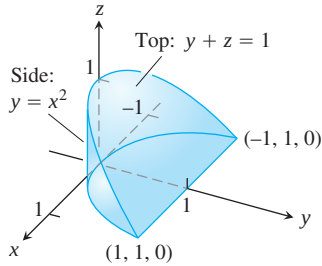
20. $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} dp dq dr$ (pqr -space)

Volumes Using Triple Integrals



21. Here is the region of integration of the integral

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} dz dy dx.$$

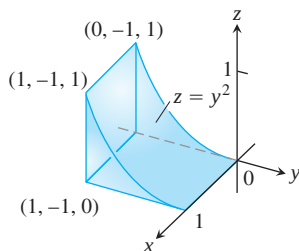


Rewrite the integral as an equivalent iterated integral in the order

- $dy dz dx$
- $dy dx dz$
- $dx dy dz$
- $dx dz dy$
- $dz dx dy$.

22. Here is the region of integration of the integral

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz dy dx.$$

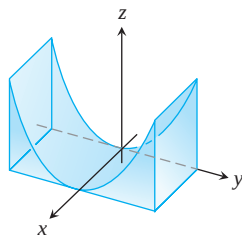


Rewrite the integral as an equivalent iterated integral in the order

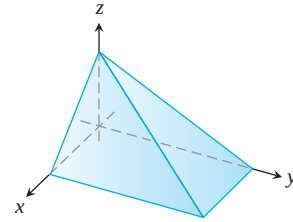
- $dy dz dx$
- $dy dx dz$
- $dx dy dz$
- $dx dz dy$
- $dz dx dy$.

Find the volumes of the regions in Exercises 23–36.

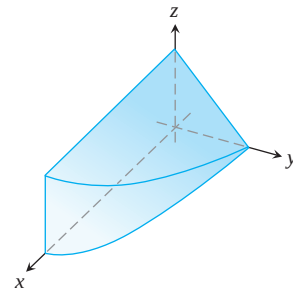
23. The region between the cylinder $z = y^2$ and the xy -plane that is bounded by the planes $x = 0$, $x = 1$, $y = -1$, $y = 1$



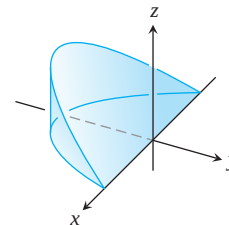
24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$



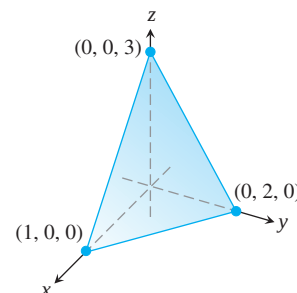
25. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



26. The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$

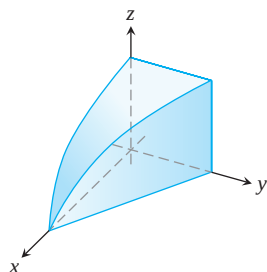


27. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

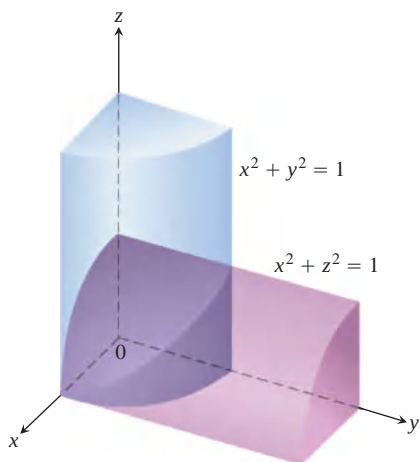




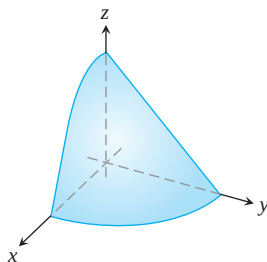
28. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$, and the surface $z = \cos(\pi x/2)$, $0 \leq x \leq 1$



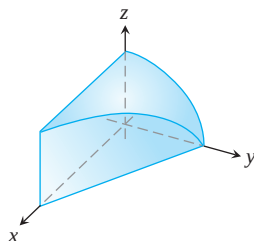
29. The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown in the accompanying figure.



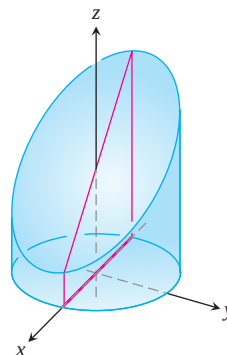
30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y^2$



31. The region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$



32. The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



33. The region between the planes $x + y + 2z = 2$ and $2x + 2y + z = 4$ in the first octant
34. The finite region bounded by the planes $z = x$, $x + z = 8$, $z = y$, $y = 8$, and $z = 0$.
35. The region cut from the solid elliptical cylinder $x^2 + 4y^2 \leq 4$ by the xy -plane and the plane $z = x + 2$
36. The region bounded in back by the plane $x = 0$, on the front and sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$, and on the bottom by the xy -plane

Average Values

In Exercises 37–40, find the average value of $F(x, y, z)$ over the given region.

37. $F(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$
38. $F(x, y, z) = x + y - z$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 2$
39. $F(x, y, z) = x^2 + y^2 + z^2$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$
40. $F(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$



Changing the Order of Integration

Evaluate the integrals in Exercises 41–44 by changing the order of integration in an appropriate way.

41. $\int_0^4 \int_0^1 \int_{2y}^2 \frac{4 \cos(x^2)}{2\sqrt{z}} dx dy dz$
42. $\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2} dy dx dz$
43. $\int_0^1 \int_{\sqrt{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin \pi y^2}{y^2} dx dy dz$
44. $\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z} dy dz dx$



Theory and Examples

- 45. Finding upper limit of iterated integral** Solve for a :

$$\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz \, dy \, dx = \frac{4}{15}.$$

- 46. Ellipsoid** For what value of c is the volume of the ellipsoid $x^2 + (y/2)^2 + (z/c)^2 = 1$ equal to 8π ?
- 47. Minimizing a triple integral** What domain D in space minimizes the value of the integral

$$\iiint_D (4x^2 + 4y^2 + z^2 - 4) \, dV?$$

Give reasons for your answer.

- 48. Maximizing a triple integral** What domain D in space maximizes the value of the integral

$$\iiint_D (1 - x^2 - y^2 - z^2) \, dV?$$

Give reasons for your answer.

COMPUTER EXPLORATIONS

Numerical Evaluations

In Exercises 49–52, use a CAS integration utility to evaluate the triple integral of the given function over the specified solid region.

- 49.** $F(x, y, z) = x^2 y^2 z$ over the solid cylinder bounded by $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$
- 50.** $F(x, y, z) = |xyz|$ over the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 1$
- 51.** $F(x, y, z) = \frac{z}{(x^2 + y^2 + z^2)^{3/2}}$ over the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$
- 52.** $F(x, y, z) = x^4 + y^2 + z^2$ over the solid sphere $x^2 + y^2 + z^2 \leq 1$

15.5

Masses and Moments in Three Dimensions



Project



Project

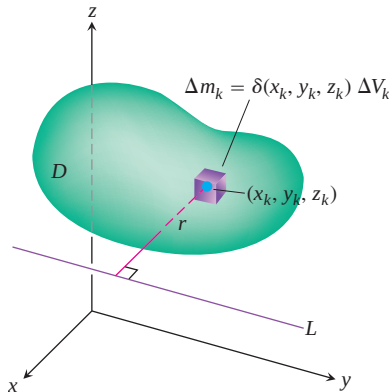


FIGURE 15.32 To define an object's mass and moment of inertia about a line, we first imagine it to be partitioned into a finite number of mass elements Δm_k .

This section shows how to calculate the masses and moments of three-dimensional objects in Cartesian coordinates. The formulas are similar to those for two-dimensional objects. For calculations in spherical and cylindrical coordinates, see Section 15.6.

Masses and Moments

If $\delta(x, y, z)$ is the density of an object occupying a region D in space (mass per unit volume), the integral of δ over D gives the **mass** of the object. To see why, imagine partitioning the object into n mass elements like the one in Figure 15.32. The object's mass is the limit

$$M = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta m_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \delta(x_k, y_k, z_k) \Delta V_k = \iiint_D \delta(x, y, z) dV.$$

We now derive a formula for the moment of inertia. If $r(x, y, z)$ is the distance from the point (x, y, z) in D to a line L , then the moment of inertia of the mass $\Delta m_k = \delta(x_k, y_k, z_k) \Delta V_k$ about the line L (shown in Figure 15.32) is approximately $\Delta I_k = r^2(x_k, y_k, z_k) \Delta m_k$. **The moment of inertia about L** of the entire object is

$$I_L = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Delta I_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n r^2(x_k, y_k, z_k) \delta(x_k, y_k, z_k) \Delta V_k = \iiint_D r^2 \delta dV.$$

If L is the x -axis, then $r^2 = y^2 + z^2$ (Figure 15.33) and

$$I_x = \iiint_D (y^2 + z^2) \delta dV.$$

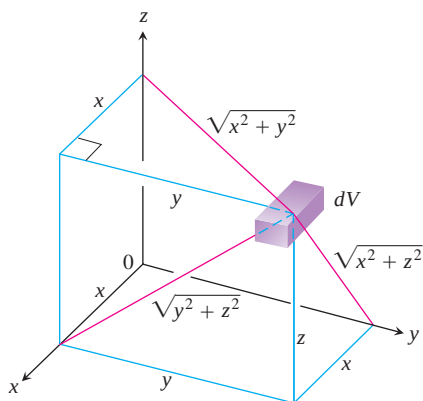


FIGURE 15.33 Distances from dV to the coordinate planes and axes.

Similarly, if L is the y -axis or z -axis we have

$$I_y = \iiint_D (x^2 + z^2) \delta \, dV \quad \text{and} \quad I_z = \iiint_D (x^2 + y^2) \delta \, dV.$$

Likewise, we can obtain the **first moments about the coordinate planes**. For example,

$$M_{yz} = \iiint_D x \delta(x, y, z) \, dV$$

gives the first moment about the yz -plane.

The mass and moment formulas in space analogous to those discussed for planar regions in Section 15.2 are summarized in Table 15.3.

TABLE 15.3 Mass and moment formulas for solid objects in space

Mass: $M = \iiint_D \delta \, dV$ ($\delta = \delta(x, y, z) = \text{density}$)

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \delta \, dV, \quad M_{xz} = \iiint_D y \delta \, dV, \quad M_{xy} = \iiint_D z \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{xz}}{M}, \quad \bar{z} = \frac{M_{xy}}{M}$$

Moments of inertia (second moments) about the coordinate axes:

$$I_x = \iiint_D (y^2 + z^2) \delta \, dV$$

$$I_y = \iiint_D (x^2 + z^2) \delta \, dV$$

$$I_z = \iiint_D (x^2 + y^2) \delta \, dV$$

Moments of inertia about a line L :

$$I_L = \iiint_D r^2 \delta \, dV \quad (r(x, y, z) = \text{distance from the point } (x, y, z) \text{ to line } L)$$

Radius of gyration about a line L :

$$R_L = \sqrt{I_L/M}$$



EXAMPLE 1 Finding the Center of Mass of a Solid in Space

Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Figure 15.34).

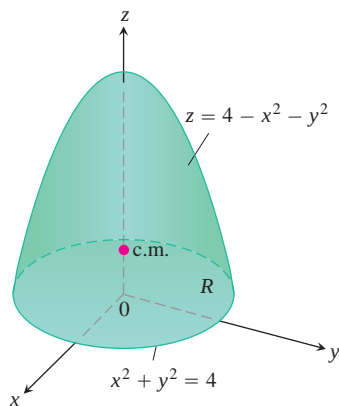


FIGURE 15.34 Finding the center of mass of a solid (Example 1).

Solution By symmetry $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we first calculate

$$\begin{aligned} M_{xy} &= \iiint_R \int_{z=0}^{z=4-x^2-y^2} z \, \delta \, dz \, dy \, dx = \iint_R \left[\frac{z^2}{2} \right]_{z=0}^{z=4-x^2-y^2} \delta \, dy \, dx \\ &= \frac{\delta}{2} \iint_R (4 - x^2 - y^2)^2 \, dy \, dx \\ &= \frac{\delta}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2 r \, dr \, d\theta \quad \text{Polar coordinates} \\ &= \frac{\delta}{2} \int_0^{2\pi} \left[-\frac{1}{6} (4 - r^2)^3 \right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3} \int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}. \end{aligned}$$

A similar calculation gives

$$M = \iiint_R \int_0^{4-x^2-y^2} \delta \, dz \, dy \, dx = 8\pi\delta.$$

Therefore $\bar{z} = (M_{xy}/M) = 4/3$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$. ■

When the density of a solid object is constant (as in Example 1), the center of mass is called the **centroid** of the object (as was the case for two-dimensional shapes in Section 15.2).

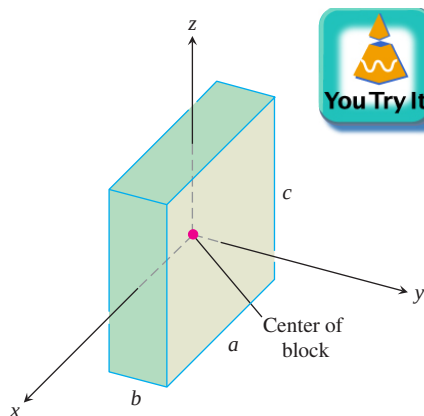


FIGURE 15.35 Finding I_x , I_y , and I_z for the block shown here. The origin lies at the center of the block (Example 2).

EXAMPLE 2 Finding the Moments of Inertia About the Coordinate Axes

Find I_x , I_y , I_z for the rectangular solid of constant density δ shown in Figure 15.35.

Solution The formula for I_x gives

$$I_x = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2) \delta \, dx \, dy \, dz.$$

We can avoid some of the work of integration by observing that $(y^2 + z^2)\delta$ is an even function of x , y , and z . The rectangular solid consists of eight symmetric pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value.

$$\begin{aligned} I_x &= 8 \int_0^{c/2} \int_0^{b/2} \int_0^{a/2} (y^2 + z^2) \delta \, dx \, dy \, dz = 4a\delta \int_0^{c/2} \int_0^{b/2} (y^2 + z^2) \, dy \, dz \\ &= 4a\delta \int_0^{c/2} \left[\frac{y^3}{3} + z^2 y \right]_{y=0}^{y=b/2} dz \\ &= 4a\delta \int_0^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2} \right) dz \\ &= 4a\delta \left(\frac{b^3 c}{48} + \frac{c^3 b}{48} \right) = \frac{abc\delta}{12} (b^2 + c^2) = \frac{M}{12} (b^2 + c^2). \end{aligned}$$

Similarly,

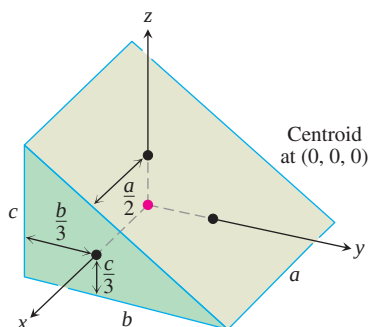
$$I_y = \frac{M}{12} (a^2 + c^2) \quad \text{and} \quad I_z = \frac{M}{12} (a^2 + b^2). \quad \blacksquare$$

EXERCISES 15.5

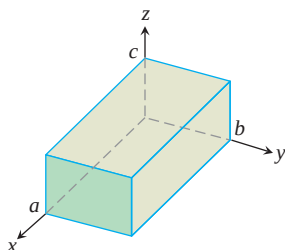
Constant Density

The solids in Exercises 1–12 all have constant density $\delta = 1$.

- (Example 1 Revisited.) Evaluate the integral for I_x in Table 15.3 directly to show that the shortcut in Example 2 gives the same answer. Use the results in Example 2 to find the radius of gyration of the rectangular solid about each coordinate axis.
- Moments of inertia** The coordinate axes in the figure run through the centroid of a solid wedge parallel to the labeled edges. Find I_x , I_y , and I_z if $a = b = 6$ and $c = 4$.



- Moments of inertia** Find the moments of inertia of the rectangular solid shown here with respect to its edges by calculating I_x , I_y , and I_z .



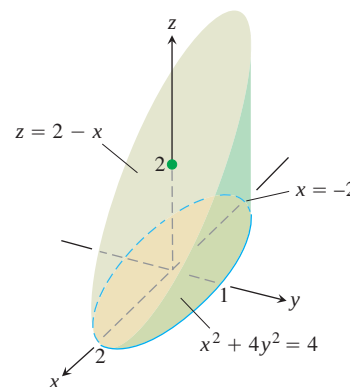
- a. Centroid and moments of inertia** Find the centroid and the moments of inertia I_x , I_y , and I_z of the tetrahedron whose vertices are the points $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.
- b. Radius of gyration** Find the radius of gyration of the tetrahedron about the x -axis. Compare it with the distance from the centroid to the x -axis.
- Center of mass and moments of inertia** A solid “trough” of constant density is bounded below by the surface $z = 4y^2$, above by the plane $z = 4$, and on the ends by the planes $x = 1$ and $x = -1$. Find the center of mass and the moments of inertia with respect to the three axes.
- Center of mass** A solid of constant density is bounded below by the plane $z = 0$, on the sides by the elliptical cylinder $x^2 + 4y^2 = 4$, and above by the plane $z = 2 - x$ (see the accompanying figure).

- Find $\bar{x}$ and $\bar{y}$.

- Evaluate the integral

$$M_{xy} = \int_{-2}^2 \int_{-(1/2)\sqrt{4-x^2}}^{(1/2)\sqrt{4-x^2}} \int_0^{2-x} z \, dz \, dy \, dx$$

using integral tables to carry out the final integration with respect to x . Then divide M_{xy} by M to verify that $\bar{z} = 5/4$.



- a. Center of mass** Find the center of mass of a solid of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$.
- Find the plane $z = c$ that divides the solid into two parts of equal volume. This plane does not pass through the center of mass.
- Moments and radii of gyration** A solid cube, 2 units on a side, is bounded by the planes $x = \pm 1$, $z = \pm 1$, $y = 3$, and $y = 5$. Find the center of mass and the moments of inertia and radii of gyration about the coordinate axes.
- Moment of inertia and radius of gyration about a line** A wedge like the one in Exercise 2 has $a = 4$, $b = 6$, and $c = 3$. Make a quick sketch to check for yourself that the square of the distance from a typical point (x, y, z) of the wedge to the line $L: z = 0, y = 6$ is $r^2 = (y - 6)^2 + z^2$. Then calculate the moment of inertia and radius of gyration of the wedge about L .
- Moment of inertia and radius of gyration about a line** A wedge like the one in Exercise 2 has $a = 4$, $b = 6$, and $c = 3$. Make a quick sketch to check for yourself that the square of the distance from a typical point (x, y, z) of the wedge to the line $L: x = 4, y = 0$ is $r^2 = (x - 4)^2 + y^2$. Then calculate the moment of inertia and radius of gyration of the wedge about L .
- Moment of inertia and radius of gyration about a line** A solid like the one in Exercise 3 has $a = 4$, $b = 2$, and $c = 1$. Make a quick sketch to check for yourself that the square of the distance between a typical point (x, y, z) of the solid and the line $L: y = 2, z = 0$ is $r^2 = (y - 2)^2 + z^2$. Then find the moment of inertia and radius of gyration of the solid about L .



- 12. Moment of inertia and radius of gyration about a line** A solid like the one in Exercise 3 has $a = 4$, $b = 2$, and $c = 1$. Make a quick sketch to check for yourself that the square of the distance between a typical point (x, y, z) of the solid and the line $L: x = 4, y = 0$ is $r^2 = (x - 4)^2 + y^2$. Then find the moment of inertia and radius of gyration of the solid about L .

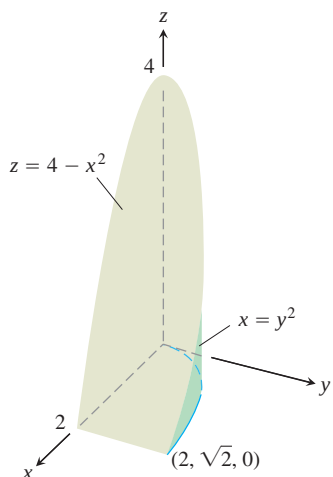
Variable Density

In Exercises 13 and 14, find

- the mass of the solid.
- the center of mass.



- 13.** A solid region in the first octant is bounded by the coordinate planes and the plane $x + y + z = 2$. The density of the solid is $\delta(x, y, z) = 2x$.
- 14.** A solid in the first octant is bounded by the planes $y = 0$ and $z = 0$ and by the surfaces $z = 4 - x^2$ and $x = y^2$ (see the accompanying figure). Its density function is $\delta(x, y, z) = kxy$, k a constant.



In Exercises 15 and 16, find

- the mass of the solid.
 - the center of mass.
 - the moments of inertia about the coordinate axes.
 - the radii of gyration about the coordinate axes.
- 15.** A solid cube in the first octant is bounded by the coordinate planes and by the planes $x = 1$, $y = 1$, and $z = 1$. The density of the cube is $\delta(x, y, z) = x + y + z + 1$.
- 16.** A wedge like the one in Exercise 2 has dimensions $a = 2$, $b = 6$, and $c = 3$. The density is $\delta(x, y, z) = x + 1$. Notice that if the density is constant, the center of mass will be $(0, 0, 0)$.
- 17. Mass** Find the mass of the solid bounded by the planes $x + z = 1$, $x - z = -1$, $y = 0$ and the surface $y = \sqrt{z}$. The density of the solid is $\delta(x, y, z) = 2y + 5$.



- 18. Mass** Find the mass of the solid region bounded by the parabolic surfaces $z = 16 - 2x^2 - 2y^2$ and $z = 2x^2 + 2y^2$ if the density of the solid is $\delta(x, y, z) = \sqrt{x^2 + y^2}$.

Work

In Exercises 19 and 20, calculate the following.

- The amount of work done by (constant) gravity g in moving the liquid filling in the container to the xy -plane. (*Hint:* Partition the liquid into small volume elements ΔV_i and find the work done (approximately) by gravity on each element. Summation and passage to the limit gives a triple integral to evaluate.)
 - The work done by gravity in moving the center of mass down to the xy -plane.
- 19.** The container is a cubical box in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$. The density of the liquid filling the box is $\delta(x, y, z) = x + y + z + 1$ (see Exercise 15).
- 20.** The container is in the shape of the region bounded by $y = 0$, $z = 0$, $z = 4 - x^2$, and $x = y^2$. The density of the liquid filling the region is $\delta(x, y, z) = kxy$, k a constant (see Exercise 14).



The Parallel Axis Theorem

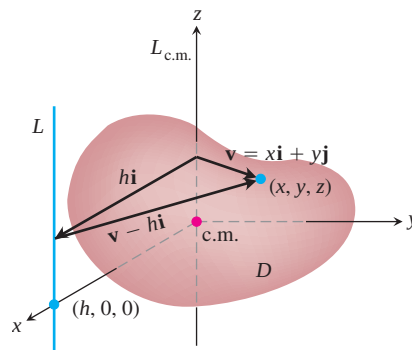
The Parallel Axis Theorem (Exercises 15.2) holds in three dimensions as well as in two. Let $L_{c.m.}$ be a line through the center of mass of a body of mass m and let L be a parallel line h units away from $L_{c.m.}$. The **Parallel Axis Theorem** says that the moments of inertia $I_{c.m.}$ and I_L of the body about $L_{c.m.}$ and L satisfy the equation

$$I_L = I_{c.m.} + mh^2. \quad (1)$$

As in the two-dimensional case, the theorem gives a quick way to calculate one moment when the other moment and the mass are known.

21. Proof of the Parallel Axis Theorem

- Show that the first moment of a body in space about any plane through the body's center of mass is zero. (*Hint:* Place the body's center of mass at the origin and let the plane be the yz -plane. What does the formula $\bar{x} = M_{yz}/M$ then tell you?)



- b. To prove the Parallel Axis Theorem, place the body with its center of mass at the origin, with the line $L_{c.m.}$ along the z -axis and the line L perpendicular to the xy -plane at the point $(h, 0, 0)$. Let D be the region of space occupied by the body. Then, in the notation of the figure,

$$I_L = \iiint_D |\mathbf{r} - h\mathbf{i}|^2 dm.$$

Expand the integrand in this integral and complete the proof.

22. The moment of inertia about a diameter of a solid sphere of constant density and radius a is $(2/5)ma^2$, where m is the mass of the sphere. Find the moment of inertia about a line tangent to the sphere.
23. The moment of inertia of the solid in Exercise 3 about the z -axis is $I_z = abc(a^2 + b^2)/3$.
- Use Equation (1) to find the moment of inertia and radius of gyration of the solid about the line parallel to the z -axis through the solid's center of mass.
 - Use Equation (1) and the result in part (a) to find the moment of inertia and radius of gyration of the solid about the line $x = 0, y = 2b$.
24. If $a = b = 6$ and $c = 4$, the moment of inertia of the solid wedge in Exercise 2 about the x -axis is $I_x = 208$. Find the moment of inertia of the wedge about the line $y = 4, z = -4/3$ (the edge of the wedge's narrow end).

Pappus's Formula

Pappus's formula (Exercises 15.2) holds in three dimensions as well as in two. Suppose that bodies B_1 and B_2 of mass m_1 and m_2 , respectively, occupy nonoverlapping regions in space and that $\mathbf{c}_1$ and $\mathbf{c}_2$ are the vectors from the origin to the bodies' respective centers of mass. Then the center of mass of the union $B_1 \cup B_2$ of the two bodies is determined by the vector

$$\mathbf{c} = \frac{m_1 \mathbf{c}_1 + m_2 \mathbf{c}_2}{m_1 + m_2}.$$

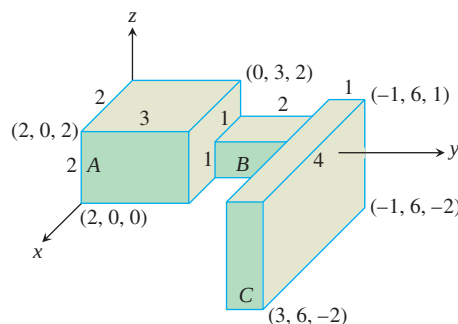
As before, this formula is called **Pappus's formula**. As in the two-dimensional case, the formula generalizes to

$$\mathbf{c} = \frac{m_1 \mathbf{c}_1 + m_2 \mathbf{c}_2 + \cdots + m_n \mathbf{c}_n}{m_1 + m_2 + \cdots + m_n}$$

for n bodies.

25. Derive Pappus's formula. (*Hint:* Sketch B_1 and B_2 as nonoverlapping regions in the first octant and label their centers of mass $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ and $(\bar{x}_2, \bar{y}_2, \bar{z}_2)$. Express the moments of $B_1 \cup B_2$ about the coordinate planes in terms of the masses m_1 and m_2 and the coordinates of these centers.)
26. The accompanying figure shows a solid made from three rectangular solids of constant density $\delta = 1$. Use Pappus's formula to find the center of mass of

- $A \cup B$
- $A \cup C$
- $B \cup C$
- $A \cup B \cup C$.



27. a. Suppose that a solid right circular cone C of base radius a and altitude h is constructed on the circular base of a solid hemisphere S of radius a so that the union of the two solids resembles an ice cream cone. The centroid of a solid cone lies one-fourth of the way from the base toward the vertex. The centroid of a solid hemisphere lies three-eighths of the way from the base to the top. What relation must hold between h and a to place the centroid of $C \cup S$ in the common base of the two solids?
- b. If you have not already done so, answer the analogous question about a triangle and a semicircle (Section 15.2, Exercise 55). The answers are not the same.
28. A solid pyramid P with height h and four congruent sides is built with its base as one face of a solid cube C whose edges have length s . The centroid of a solid pyramid lies one-fourth of the way from the base toward the vertex. What relation must hold between h and s to place the centroid of $P \cup C$ in the base of the pyramid? Compare your answer with the answer to Exercise 27. Also compare it with the answer to Exercise 56 in Section 15.2.

15.6

Triple Integrals in Cylindrical and Spherical Coordinates



When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this section. The procedure for transforming to these coordinates and evaluating the resulting triple integrals is similar to the transformation to polar coordinates in the plane studied in Section 15.3.

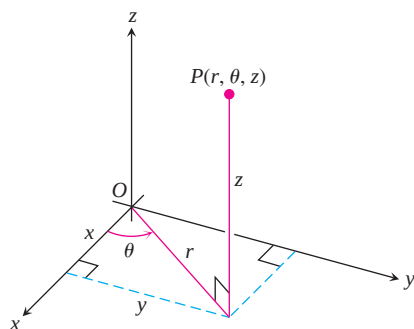


FIGURE 15.36 The cylindrical coordinates of a point in space are r , θ , and z .

Integration in Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy -plane with the usual z -axis. This assigns to every point in space one or more coordinate triples of the form (r, θ, z) , as shown in Figure 15.36.

DEFINITION Cylindrical Coordinates

Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy -plane
2. z is the rectangular vertical coordinate.

The values of x , y , r , and θ in rectangular and cylindrical coordinates are related by the usual equations.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$\begin{aligned} x &= r \cos \theta, & y &= r \sin \theta, & z &= z, \\ r^2 &= x^2 + y^2, & \tan \theta &= y/x \end{aligned}$$

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy -plane but an entire cylinder about the z -axis (Figure 15.37). The z -axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z -axis and makes an angle θ_0 with the positive x -axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z -axis.

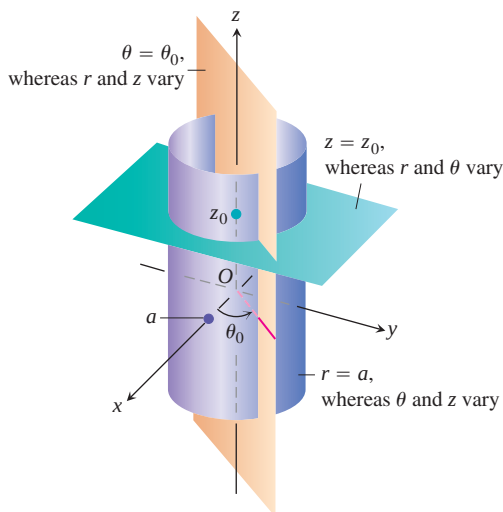


FIGURE 15.37 Constant-coordinate equations in cylindrical coordinates yield cylinders and planes.

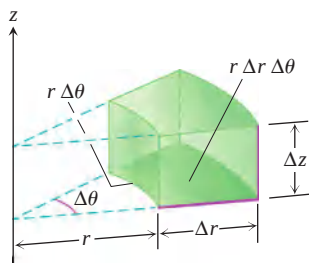


FIGURE 15.38 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z r \Delta r \Delta \theta$.

Cylindrical coordinates are good for describing cylinders whose axes run along the z -axis and planes that either contain the z -axis or lie perpendicular to the z -axis. Surfaces like these have equations of constant coordinate value:

$$r = 4. \quad \text{Cylinder, radius 4, axis the } z\text{-axis}$$

$$\theta = \frac{\pi}{3}. \quad \text{Plane containing the } z\text{-axis}$$

$$z = 2. \quad \text{Plane perpendicular to the } z\text{-axis}$$

When computing triple integrals over a region D in cylindrical coordinates, we partition the region into n small cylindrical wedges, rather than into rectangular boxes. In the k th cylindrical wedge, r , θ and z change by Δr_k , $\Delta \theta_k$, and Δz_k , and the largest of these numbers among all the cylindrical wedges is called the **norm** of the partition. We define the triple integral as a limit of Riemann sums using these wedges. The volume of such a cylindrical wedge ΔV_k is obtained by taking the area ΔA_k of its base in the $r\theta$ -plane and multiplying by the height Δz (Figure 15.38).

For a point (r_k, θ_k, z_k) in the center of the k th wedge, we calculated in polar coordinates that $\Delta A_k = r_k \Delta r_k \Delta \theta_k$. So $\Delta V_k = \Delta z_k r_k \Delta r_k \Delta \theta_k$ and a Riemann sum for f over D has the form

$$S_n = \sum_{k=1}^n f(r_k, \theta_k, z_k) \Delta z_k r_k \Delta r_k \Delta \theta_k.$$

The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero

$$\lim_{n \rightarrow \infty} S_n = \iiint_D f \, dV = \iiint_D f \, dz \, r \, dr \, d\theta.$$

Triple integrals in cylindrical coordinates are then evaluated as iterated integrals, as in the following example.

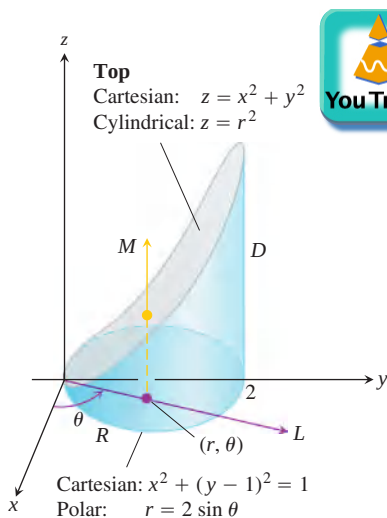


FIGURE 15.39 Finding the limits of integration for evaluating an integral in cylindrical coordinates (Example 1).



EXAMPLE 1 Finding Limits of Integration in Cylindrical Coordinates

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution The base of D is also the region's projection R on the xy -plane. The boundary of R is the circle $x^2 + (y - 1)^2 = 1$. Its polar coordinate equation is

$$x^2 + (y - 1)^2 = 1$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$r^2 - 2r \sin \theta = 0$$

$$r = 2 \sin \theta.$$

The region is sketched in Figure 15.39.

We find the limits of integration, starting with the z -limits. A line M through a typical point (r, θ) in R parallel to the z -axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next we find the r -limits of integration. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2 \sin \theta$.

Finally we find the θ -limits of integration. As L sweeps across R , the angle θ it makes with the positive x -axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\iiint_D f(r, \theta, z) \, dV = \int_0^\pi \int_0^{2 \sin \theta} \int_0^{r^2} f(r, \theta, z) \, dz \, r \, dr \, d\theta. \quad \blacksquare$$

Example 1 illustrates a good procedure for finding limits of integration in cylindrical coordinates. The procedure is summarized as follows.

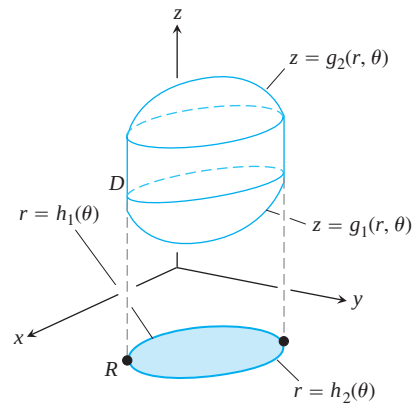
How to Integrate in Cylindrical Coordinates

To evaluate

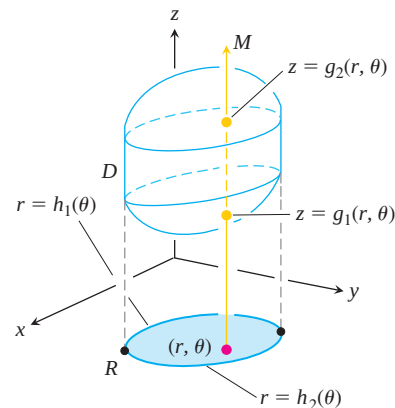
$$\iiint_D f(r, \theta, z) \, dV$$

over a region D in space in cylindrical coordinates, integrating first with respect to z , then with respect to r , and finally with respect to θ , take the following steps.

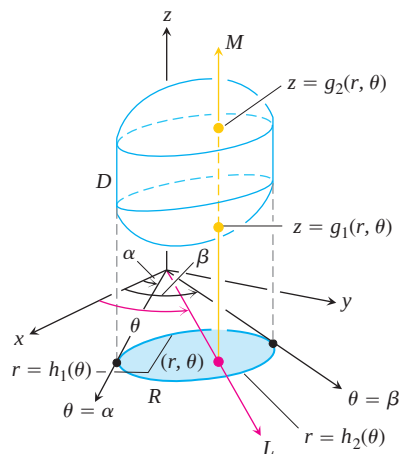
1. *Sketch.* Sketch the region D along with its projection R on the xy -plane. Label the surfaces and curves that bound D and R .



2. *Find the z -limits of integration.* Draw a line M through a typical point (r, θ) of R parallel to the z -axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z -limits of integration.



3. Find the r -limits of integration. Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r -limits of integration.



4. Find the θ -limits of integration. As L sweeps across R , the angle θ it makes with the positive x -axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ -limits of integration. The integral is

$$\iiint_D f(r, \theta, z) dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r, \theta)}^{z=g_2(r, \theta)} f(r, \theta, z) dz r dr d\theta.$$

EXAMPLE 2 Finding a Centroid

Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy -plane.



Solution We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$ (Figure 15.40). Its base R is the disk $0 \leq r \leq 2$ in the xy -plane.

The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z -axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M .

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

The z -limits. A line M through a typical point (r, θ) in the base parallel to the z -axis enters the solid at $z = 0$ and leaves at $z = r^2$.

The r -limits. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.

The θ -limits. As L sweeps over the base like a clock hand, the angle θ it makes with the positive x -axis runs from $\theta = 0$ to $\theta = 2\pi$. The value of M_{xy} is

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} z dz r dr d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2} \right]_0^{r^2} r dr d\theta \\ &= \int_0^{2\pi} \int_0^2 \frac{r^5}{2} dr d\theta = \int_0^{2\pi} \left[\frac{r^6}{12} \right]_0^2 d\theta = \int_0^{2\pi} \frac{16}{3} d\theta = \frac{32\pi}{3}. \end{aligned}$$

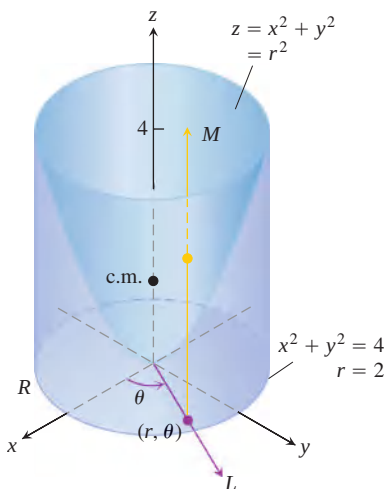


FIGURE 15.40 Example 2 shows how to find the centroid of this solid.

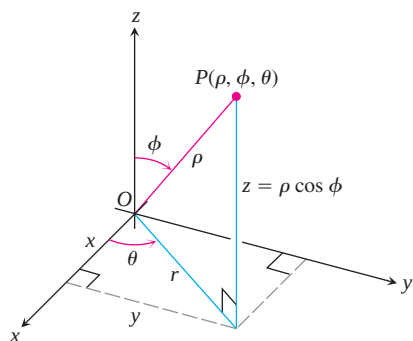


FIGURE 15.41 The spherical coordinates ρ , ϕ , and θ and their relation to x , y , z , and r .

The value of M is

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \left[z \right]_0^{r^2} r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 r^3 \, dr \, d\theta = \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\theta = \int_0^{2\pi} 4 \, d\theta = 8\pi. \end{aligned}$$

Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{8\pi} \frac{1}{3} = \frac{4}{3},$$

and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies outside the solid. ■

Spherical Coordinates and Integration

Spherical coordinates locate points in space with two angles and one distance, as shown in Figure 15.41. The first coordinate, $\rho = |\overrightarrow{OP}|$, is the point's distance from the origin. Unlike r , the variable ρ is never negative. The second coordinate, ϕ , is the angle $\overrightarrow{OP}$ makes with the positive z -axis. It is required to lie in the interval $[0, \pi]$. The third coordinate is the angle θ as measured in cylindrical coordinates.

DEFINITION Spherical Coordinates

Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z -axis ($0 \leq \phi \leq \pi$).
3. θ is the angle from cylindrical coordinates.

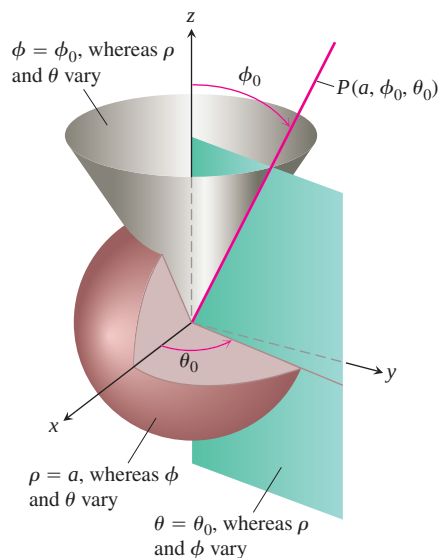


FIGURE 15.42 Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

On maps of the Earth, θ is related to the meridian of a point on the Earth and ϕ to its latitude, while ρ is related to elevation above the Earth's surface.

The equation $\rho = a$ describes the sphere of radius a centered at the origin (Figure 15.42). The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z -axis. (We broaden our interpretation to include the xy -plane as the cone $\phi = \pi/2$.) If ϕ_0 is greater than $\pi/2$, the cone $\phi = \phi_0$ opens downward. The equation $\theta = \theta_0$ describes the half-plane that contains the z -axis and makes an angle θ_0 with the positive x -axis.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$\begin{aligned} r &= \rho \sin \phi, & x &= r \cos \theta = \rho \sin \phi \cos \theta, \\ z &= \rho \cos \phi, & y &= r \sin \theta = \rho \sin \phi \sin \theta, \\ \rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}. \end{aligned} \quad (1)$$

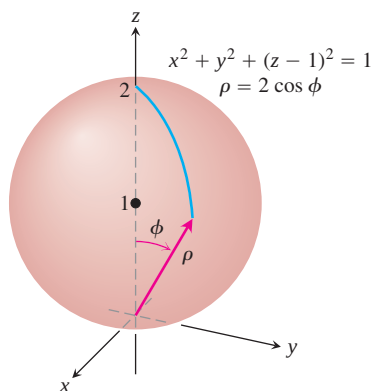


FIGURE 15.43 The sphere in Example 3.

EXAMPLE 3 Converting Cartesian to SphericalFind a spherical coordinate equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$.**Solution** We use Equations (1) to substitute for x , y , and z :

$$\begin{aligned}
 x^2 + y^2 + (z - 1)^2 &= 1 \\
 \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + (\rho \cos \phi - 1)^2 &= 1 && \text{Equations (1)} \\
 \rho^2 \sin^2 \phi (\underbrace{\cos^2 \theta + \sin^2 \theta}_1) + \rho^2 \cos^2 \phi - 2\rho \cos \phi + 1 &= 1 \\
 \rho^2 (\underbrace{\sin^2 \phi + \cos^2 \phi}_1) - 2\rho \cos \phi + 1 &= 1 \\
 \rho^2 - 2\rho \cos \phi + 1 &= 1 \\
 \rho^2 &= 2\rho \cos \phi \\
 \rho &= 2 \cos \phi.
 \end{aligned}$$

See Figure 15.43. ■

EXAMPLE 4 Converting Cartesian to SphericalFind a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$ (Figure 15.44).**Solution 1** *Use geometry.* The cone is symmetric with respect to the z -axis and cuts the first quadrant of the yz -plane along the line $z = y$. The angle between the cone and the positive z -axis is therefore $\pi/4$ radians. The cone consists of the points whose spherical coordinates have ϕ equal to $\pi/4$, so its equation is $\phi = \pi/4$.**Solution 2** *Use algebra.* If we use Equations (1) to substitute for x , y , and z we obtain the same result:

$$\begin{aligned}
 z &= \sqrt{x^2 + y^2} \\
 \rho \cos \phi &= \sqrt{\rho^2 \sin^2 \phi} && \text{Example 3} \\
 \rho \cos \phi &= \rho \sin \phi && \rho \geq 0, \sin \phi \geq 0 \\
 \cos \phi &= \sin \phi \\
 \phi &= \frac{\pi}{4}. && 0 \leq \phi \leq \pi
 \end{aligned}$$

Spherical coordinates are good for describing spheres centered at the origin, half-planes hinged along the z -axis, and cones whose vertices lie at the origin and whose axes lie along the z -axis. Surfaces like these have equations of constant coordinate value:

$$\begin{aligned}
 \rho &= 4 && \text{Sphere, radius 4, center at origin} \\
 \phi &= \frac{\pi}{3} && \text{Cone opening up from the origin, making an angle of } \pi/3 \text{ radians with the positive } z\text{-axis} \\
 \theta &= \frac{\pi}{3} && \text{Half-plane, hinged along the } z\text{-axis, making an angle of } \pi/3 \text{ radians with the positive } x\text{-axis}
 \end{aligned}$$

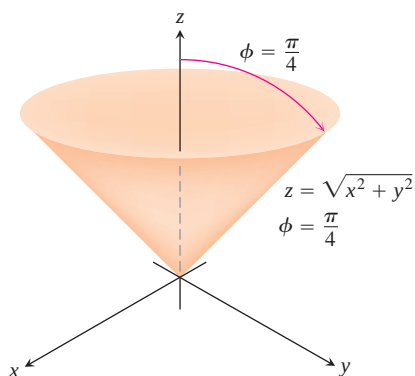
When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges. The size of the k th spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given by changes by $\Delta \rho_k$, $\Delta \theta_k$, and $\Delta \phi_k$ in ρ , θ , and ϕ . Such a spherical wedge has one edge a circular arc of length $\rho_k \Delta \phi_k$, another edge a circular arc of

FIGURE 15.44 The cone in Example 4.

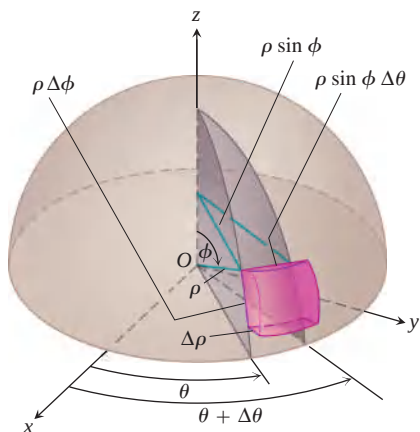


FIGURE 15.45 In spherical coordinates

$$\begin{aligned} dV &= d\rho \cdot \rho \, d\phi \cdot \rho \sin \phi \, d\theta \\ &= \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta. \end{aligned}$$

length $\rho_k \sin \phi_k \Delta\theta_k$, and thickness $\Delta\rho_k$. The spherical wedge closely approximates a cube of these dimensions when $\Delta\rho_k$, $\Delta\theta_k$, and $\Delta\phi_k$ are all small (Figure 15.45). It can be shown that the volume of this spherical wedge ΔV_k is $\Delta V_k = \rho_k^2 \sin \phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$ for $(\rho_k, \phi_k, \theta_k)$ a point chosen inside the wedge.

The corresponding Riemann sum for a function $F(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^n F(\rho_k, \phi_k, \theta_k) \rho_k^2 \sin \phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k.$$

As the norm of a partition approaches zero, and the spherical wedges get smaller, the Riemann sums have a limit when F is continuous:

$$\lim_{n \rightarrow \infty} S_n = \iiint_D F(\rho, \phi, \theta) \, dV = \iiint_D F(\rho, \phi, \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

In spherical coordinates, we have

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

To evaluate integrals in spherical coordinates, we usually integrate first with respect to ρ . The procedure for finding the limits of integration is shown below. We restrict our attention to integrating over domains that are solids of revolution about the z -axis (or portions thereof) and for which the limits for θ and ϕ are constant.

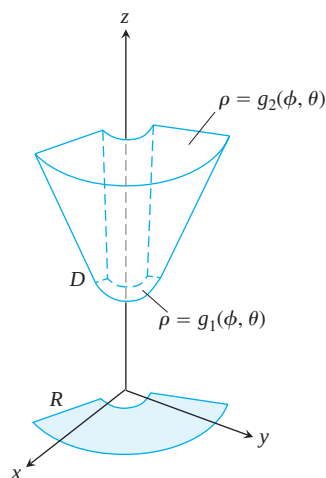
How to Integrate in Spherical Coordinates

To evaluate

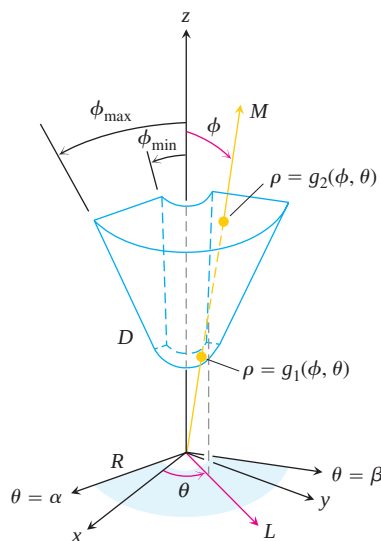
$$\iiint_D f(\rho, \phi, \theta) \, dV$$

over a region D in space in spherical coordinates, integrating first with respect to ρ , then with respect to ϕ , and finally with respect to θ , take the following steps.

1. *Sketch.* Sketch the region D along with its projection R on the xy -plane. Label the surfaces that bound D .



2. Find the ρ -limits of integration. Draw a ray M from the origin through D making an angle ϕ with the positive z -axis. Also draw the projection of M on the xy -plane (call the projection L). The ray L makes an angle θ with the positive x -axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ -limits of integration.



3. Find the ϕ -limits of integration. For any given θ , the angle ϕ that M makes with the z -axis runs from $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$. These are the ϕ -limits of integration.
4. Find the θ -limits of integration. The ray L sweeps over R as θ runs from α to β . These are the θ -limits of integration. The integral is



$$\iiint_D f(\rho, \phi, \theta) dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi, \theta)}^{\rho=g_2(\phi, \theta)} f(\rho, \phi, \theta) \rho^2 \sin \phi d\rho d\phi d\theta.$$

EXAMPLE 5 Finding a Volume in Spherical Coordinates

Find the volume of the “ice cream cone” D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.



Solution The volume is $V = \iiint_D \rho^2 \sin \phi d\rho d\phi d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over D .

To find the limits of integration for evaluating the integral, we begin by sketching D and its projection R on the xy -plane (Figure 15.46).

The ρ -limits of integration. We draw a ray M from the origin through D making an angle ϕ with the positive z -axis. We also draw L , the projection of M on the xy -plane, along with the angle θ that L makes with the positive x -axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

The ϕ -limits of integration. The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive z -axis. For any given θ , the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$.

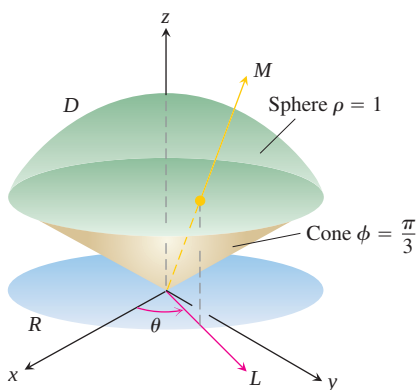


FIGURE 15.46 The ice cream cone in Example 5.

The θ -limits of integration. The ray L sweeps over R as θ runs from 0 to 2π . The volume is

$$\begin{aligned} V &= \iiint_D \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3} \right]_0^1 \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin \phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3} \cos \phi \right]_0^{\pi/3} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3} \right) d\theta = \frac{1}{6} (2\pi) = \frac{\pi}{3}. \end{aligned}$$

EXAMPLE 6 Finding a Moment of Inertia

A solid of constant density $\delta = 1$ occupies the region D in Example 5. Find the solid's moment of inertia about the z -axis.

Solution In rectangular coordinates, the moment is

$$I_z = \iiint (x^2 + y^2) \, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = \rho^2 \sin^2 \phi$. Hence,

$$I_z = \iiint (\rho^2 \sin^2 \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \iiint \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta.$$

For the region in Example 5, this becomes

$$\begin{aligned} I_z &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^5}{5} \right]_0^1 \sin^3 \phi \, d\phi \, d\theta \\ &= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi/3} (1 - \cos^2 \phi) \sin \phi \, d\phi \, d\theta = \frac{1}{5} \int_0^{2\pi} \left[-\cos \phi + \frac{\cos^3 \phi}{3} \right]_0^{\pi/3} d\theta \\ &= \frac{1}{5} \int_0^{2\pi} \left(-\frac{1}{2} + 1 + \frac{1}{24} - \frac{1}{3} \right) d\theta = \frac{1}{5} \int_0^{2\pi} \frac{5}{24} d\theta = \frac{1}{24} (2\pi) = \frac{\pi}{12}. \end{aligned}$$

Coordinate Conversion Formulas

CYLINDRICAL TO
RECTANGULAR

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

SPHERICAL TO
RECTANGULAR

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

SPHERICAL TO
CYLINDRICAL

$$r = \rho \sin \phi$$

$$z = \rho \cos \phi$$

$$\theta = \theta$$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx \, dy \, dz \\ &= dz \, r \, dr \, d\theta \\ &= \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \end{aligned}$$

In the next section we offer a more general procedure for determining dV in cylindrical and spherical coordinates. The results, of course, will be the same.

EXERCISES 15.6

Evaluating Integrals in Cylindrical Coordinates

Evaluate the cylindrical coordinate integrals in Exercises 1–6.

1. $\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz \, r \, dr \, d\theta$
2. $\int_0^{2\pi} \int_0^3 \int_{r^2/3}^{\sqrt{18-r^2}} dz \, r \, dr \, d\theta$
3. $\int_0^{2\pi} \int_0^{\theta/2\pi} \int_0^{3+24r^2} dz \, r \, dr \, d\theta$
4. $\int_0^\pi \int_0^{\theta/\pi} \int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z \, dz \, r \, dr \, d\theta$
5. $\int_0^{2\pi} \int_0^1 \int_r^{1/\sqrt{2-r^2}} 3 \, dz \, r \, dr \, d\theta$
6. $\int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} (r^2 \sin^2 \theta + z^2) \, dz \, r \, dr \, d\theta$

Changing Order of Integration in Cylindrical Coordinates

The integrals we have seen so far suggest that there are preferred orders of integration for cylindrical coordinates, but other orders usually work well and are occasionally easier to evaluate. Evaluate the integrals in Exercises 7–10.

7. $\int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 \, dr \, dz \, d\theta$
8. $\int_{-1}^1 \int_0^{2\pi} \int_0^{1+\cos \theta} 4r \, dr \, d\theta \, dz$
9. $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r \, d\theta \, dr \, dz$
10. $\int_0^2 \int_{r-2}^{\sqrt{4-r^2}} \int_0^{2\pi} (r \sin \theta + 1) r \, d\theta \, dz \, dr$
11. Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.
 - a. $dz \, dr \, d\theta$
 - b. $dr \, dz \, d\theta$
 - c. $d\theta \, dz \, dr$

12. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

- a. $dz \, dr \, d\theta$
- b. $dr \, dz \, d\theta$
- c. $d\theta \, dz \, dr$

13. Give the limits of integration for evaluating the integral

$$\iiint_D f(r, \theta, z) \, dz \, r \, dr \, d\theta$$

as an iterated integral over the region that is bounded below by the plane $z = 0$, on the side by the cylinder $r = \cos \theta$, and on top by the paraboloid $z = 3r^2$.

14. Convert the integral

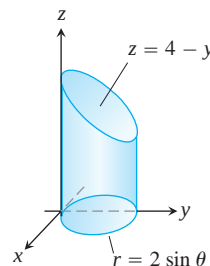
$$\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x (x^2 + y^2) \, dz \, dx \, dy$$

to an equivalent integral in cylindrical coordinates and evaluate the result.

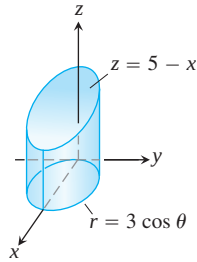
Finding Iterated Integrals in Cylindrical Coordinates

In Exercises 15–20, set up the iterated integral for evaluating $\iiint_D f(r, \theta, z) \, dz \, r \, dr \, d\theta$ over the given region D .

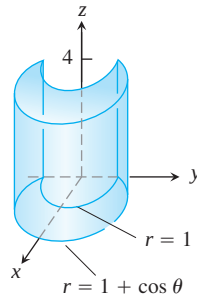
15. D is the right circular cylinder whose base is the circle $r = 2 \sin \theta$ in the xy -plane and whose top lies in the plane $z = 4 - y$.



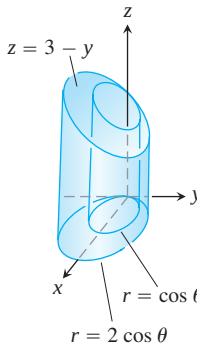
16. D is the right circular cylinder whose base is the circle $r = 3 \cos \theta$ and whose top lies in the plane $z = 5 - x$.



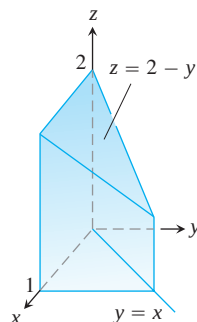
17. D is the solid right cylinder whose base is the region in the xy -plane that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



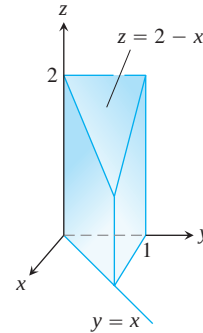
18. D is the solid right cylinder whose base is the region between the circles $r = \cos \theta$ and $r = 2 \cos \theta$ and whose top lies in the plane $z = 3 - y$.



19. D is the prism whose base is the triangle in the xy -plane bounded by the x -axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = 2 - y$.



20. D is the prism whose base is the triangle in the xy -plane bounded by the y -axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$.



Evaluating Integrals in Spherical Coordinates

Evaluate the spherical coordinate integrals in Exercises 21–26.

21. $\int_0^\pi \int_0^\pi \int_0^{2 \sin \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$
22. $\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$
23. $\int_0^{2\pi} \int_0^\pi \int_0^{(1-\cos \phi)/2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$
24. $\int_0^{3\pi/2} \int_0^{\pi/4} \int_0^1 5\rho^3 \sin^3 \phi \, d\rho \, d\phi \, d\theta$
25. $\int_0^{2\pi} \int_0^{\pi/3} \int_{\sec \phi}^2 3\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$
26. $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

Changing Order of Integration in Spherical Coordinates

The previous integrals suggest there are preferred orders of integration for spherical coordinates, but other orders are possible and occasionally easier to evaluate. Evaluate the integrals in Exercises 27–30.

27. $\int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$
28. $\int_{\pi/6}^{\pi/3} \int_{\csc \phi}^{2 \csc \phi} \int_0^{2\pi} \rho^2 \sin \phi \, d\theta \, d\rho \, d\phi$
29. $\int_0^1 \int_0^\pi \int_0^{\pi/4} 12\rho \sin^3 \phi \, d\phi \, d\theta \, d\rho$
30. $\int_{\pi/6}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{\csc \phi}^2 5\rho^4 \sin^3 \phi \, d\rho \, d\theta \, d\phi$

31. Let D be the region in Exercise 11. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

- a. $d\rho \, d\phi \, d\theta$
- b. $d\phi \, d\rho \, d\theta$



32. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

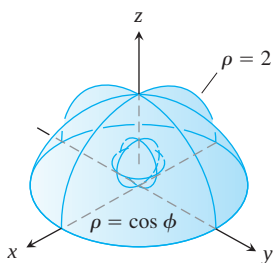
a. $dp \, d\phi \, d\theta$

b. $d\phi \, dp \, d\theta$

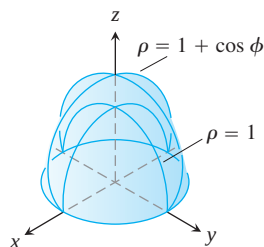
Finding Iterated Integrals in Spherical Coordinates

In Exercises 33–38, (a) find the spherical coordinate limits for the integral that calculates the volume of the given solid and (b) then evaluate the integral.

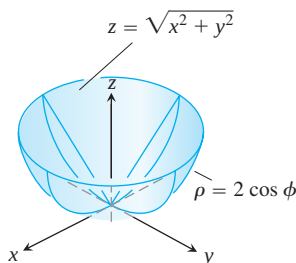
33. The solid between the sphere $\rho = \cos \phi$ and the hemisphere $\rho = 2, z \geq 0$



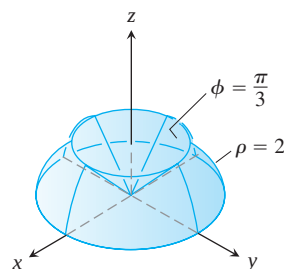
34. The solid bounded below by the hemisphere $\rho = 1, z \geq 0$, and above by the cardioid of revolution $\rho = 1 + \cos \phi$



35. The solid enclosed by the cardioid of revolution $\rho = 1 - \cos \phi$
 36. The upper portion cut from the solid in Exercise 35 by the xy -plane
 37. The solid bounded below by the sphere $\rho = 2 \cos \phi$ and above by the cone $z = \sqrt{x^2 + y^2}$



38. The solid bounded below by the xy -plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$



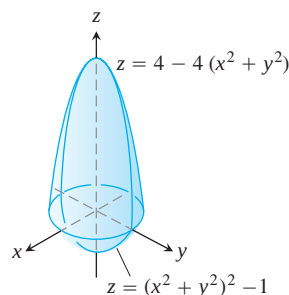
Rectangular, Cylindrical, and Spherical Coordinates

39. Set up triple integrals for the volume of the sphere $\rho = 2$ in (a) spherical, (b) cylindrical, and (c) rectangular coordinates.
 40. Let D be the region in the first octant that is bounded below by the cone $\phi = \pi/4$ and above by the sphere $\rho = 3$. Express the volume of D as an iterated triple integral in (a) cylindrical and (b) spherical coordinates. Then (c) find V .
 41. Let D be the smaller cap cut from a solid ball of radius 2 units by a plane 1 unit from the center of the sphere. Express the volume of D as an iterated triple integral in (a) spherical, (b) cylindrical, and (c) rectangular coordinates. Then (d) find the volume by evaluating one of the three triple integrals.
 42. Express the moment of inertia I_z of the solid hemisphere $x^2 + y^2 + z^2 \leq 1, z \geq 0$, as an iterated integral in (a) cylindrical and (b) spherical coordinates. Then (c) find I_z .

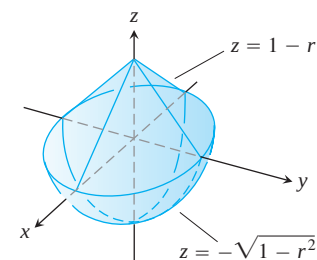
Volumes

Find the volumes of the solids in Exercises 43–48.

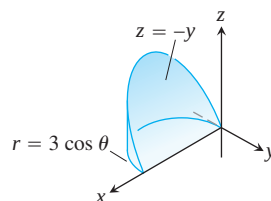
43.



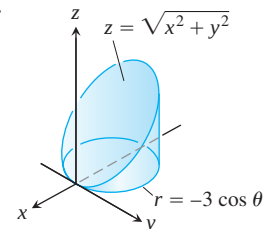
44.



45.

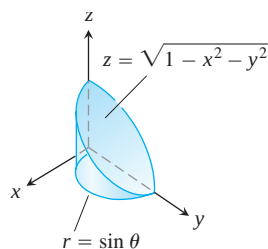


46.

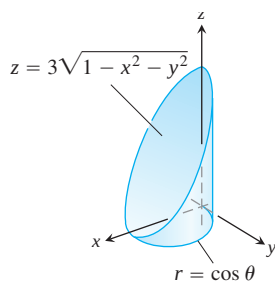




47.



48.



- 49. Sphere and cones** Find the volume of the portion of the solid sphere $\rho \leq a$ that lies between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$.
- 50. Sphere and half-planes** Find the volume of the region cut from the solid sphere $\rho \leq a$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.
- 51. Sphere and plane** Find the volume of the smaller region cut from the solid sphere $\rho \leq 2$ by the plane $z = 1$.
- 52. Cone and planes** Find the volume of the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.
- 53. Cylinder and paraboloid** Find the volume of the region bounded below by the plane $z = 0$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.
- 54. Cylinder and paraboloids** Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$.
- 55. Cylinder and cones** Find the volume of the solid cut from the thick-walled cylinder $1 \leq x^2 + y^2 \leq 2$ by the cones $z = \pm \sqrt{x^2 + y^2}$.
- 56. Sphere and cylinder** Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.
- 57. Cylinder and planes** Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.
- 58. Cylinder and planes** Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $x + y + z = 4$.
- 59. Region trapped by paraboloids** Find the volume of the region bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.
- 60. Paraboloid and cylinder** Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy -plane, and lying outside the cylinder $x^2 + y^2 = 1$.
- 61. Cylinder and sphere** Find the volume of the region cut from the solid cylinder $x^2 + y^2 \leq 1$ by the sphere $x^2 + y^2 + z^2 = 4$.
- 62. Sphere and paraboloid** Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

Average Values

- 63.** Find the average value of the function $f(r, \theta, z) = r$ over the region bounded by the cylinder $r = 1$ between the planes $z = -1$ and $z = 1$.
- 64.** Find the average value of the function $f(r, \theta, z) = r$ over the solid ball bounded by the sphere $r^2 + z^2 = 1$. (This is the sphere $x^2 + y^2 + z^2 = 1$.)
- 65.** Find the average value of the function $f(\rho, \phi, \theta) = \rho$ over the solid ball $\rho \leq 1$.
- 66.** Find the average value of the function $f(\rho, \phi, \theta) = \rho \cos \phi$ over the solid upper ball $\rho \leq 1, 0 \leq \phi \leq \pi/2$.

Masses, Moments, and Centroids

- 67. Center of mass** A solid of constant density is bounded below by the plane $z = 0$, above by the cone $z = r, r \geq 0$, and on the sides by the cylinder $r = 1$. Find the center of mass.
- 68. Centroid** Find the centroid of the region in the first octant that is bounded above by the cone $z = \sqrt{x^2 + y^2}$, below by the plane $z = 0$, and on the sides by the cylinder $x^2 + y^2 = 4$ and the planes $x = 0$ and $y = 0$.
- 69. Centroid** Find the centroid of the solid in Exercise 38.
- 70. Centroid** Find the centroid of the solid bounded above by the sphere $\rho = a$ and below by the cone $\phi = \pi/4$.
- 71. Centroid** Find the centroid of the region that is bounded above by the surface $z = \sqrt{r}$, on the sides by the cylinder $r = 4$, and below by the xy -plane.
- 72. Centroid** Find the centroid of the region cut from the solid ball $r^2 + z^2 \leq 1$ by the half-planes $\theta = -\pi/3, r \geq 0$, and $\theta = \pi/3, r \geq 0$.
- 73. Inertia and radius of gyration** Find the moment of inertia and radius of gyration about the z -axis of a thick-walled right circular cylinder bounded on the inside by the cylinder $r = 1$, on the outside by the cylinder $r = 2$, and on the top and bottom by the planes $z = 4$ and $z = 0$. (Take $\delta = 1$.)
- 74. Moments of inertia of solid circular cylinder** Find the moment of inertia of a solid circular cylinder of radius 1 and height 2 (a) about the axis of the cylinder and (b) about a line through the centroid perpendicular to the axis of the cylinder. (Take $\delta = 1$.)
- 75. Moment of inertia of solid cone** Find the moment of inertia of a right circular cone of base radius 1 and height 1 about an axis through the vertex parallel to the base. (Take $\delta = 1$.)
- 76. Moment of inertia of solid sphere** Find the moment of inertia of a solid sphere of radius a about a diameter. (Take $\delta = 1$.)
- 77. Moment of inertia of solid cone** Find the moment of inertia of a right circular cone of base radius a and height h about its axis. (Hint: Place the cone with its vertex at the origin and its axis along the z -axis.)
- 78. Variable density** A solid is bounded on the top by the paraboloid $z = r^2$, on the bottom by the plane $z = 0$, and on the sides by





the cylinder $r = 1$. Find the center of mass and the moment of inertia and radius of gyration about the z -axis if the density is

- a. $\delta(r, \theta, z) = z$
- b. $\delta(r, \theta, z) = r$.

79. Variable density A solid is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Find the center of mass and the moment of inertia and radius of gyration about the z -axis if the density is

- a. $\delta(r, \theta, z) = z$
- b. $\delta(r, \theta, z) = z^2$.

80. Variable density A solid ball is bounded by the sphere $\rho = a$. Find the moment of inertia and radius of gyration about the z -axis if the density is

- a. $\delta(\rho, \phi, \theta) = \rho^2$
- b. $\delta(\rho, \phi, \theta) = r = \rho \sin \phi$.

81. Centroid of solid semiellipsoid Show that the centroid of the solid semiellipsoid of revolution $(r^2/a^2) + (z^2/h^2) \leq 1, z \geq 0$, lies on the z -axis three-eighths of the way from the base to the top. The special case $h = a$ gives a solid hemisphere. Thus, the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base to the top.

82. Centroid of solid cone Show that the centroid of a solid right circular cone is one-fourth of the way from the base to the vertex. (In general, the centroid of a solid cone or pyramid is one-fourth of the way from the centroid of the base to the vertex.)

83. Variable density A solid right circular cylinder is bounded by the cylinder $r = a$ and the planes $z = 0$ and $z = h, h > 0$. Find the center of mass and the moment of inertia and radius of gyration about the z -axis if the density is $\delta(r, \theta, z) = z + 1$.

84. Mass of planet's atmosphere A spherical planet of radius R has an atmosphere whose density is $\mu = \mu_0 e^{-ch}$, where h is the altitude above the surface of the planet, μ_0 is the density at sea level, and c is a positive constant. Find the mass of the planet's atmosphere.

85. Density of center of a planet A planet is in the shape of a sphere of radius R and total mass M with spherically symmetric density distribution that increases linearly as one approaches its center. What is the density at the center of this planet if the density at its edge (surface) is taken to be zero?

Theory and Examples

86. Vertical circular cylinders in spherical coordinates Find an equation of the form $\rho = f(\phi)$ for the cylinder $x^2 + y^2 = a^2$.

87. Vertical planes in cylindrical coordinates

- a. Show that planes perpendicular to the x -axis have equations of the form $r = a \sec \theta$ in cylindrical coordinates.
- b. Show that planes perpendicular to the y -axis have equations of the form $r = b \csc \theta$.

88. (Continuation of Exercise 87.) Find an equation of the form $r = f(\theta)$ in cylindrical coordinates for the plane $ax + by = c, c \neq 0$.

89. Symmetry What symmetry will you find in a surface that has an equation of the form $r = f(z)$ in cylindrical coordinates? Give reasons for your answer.

90. Symmetry What symmetry will you find in a surface that has an equation of the form $\rho = f(\phi)$ in spherical coordinates? Give reasons for your answer.

15.7

Substitutions in Multiple Integrals

This section shows how to evaluate multiple integrals by substitution. As in single integration, the goal of substitution is to replace complicated integrals by ones that are easier to evaluate. Substitutions accomplish this by simplifying the integrand, the limits of integration, or both.

Substitutions in Double Integrals

The polar coordinate substitution of Section 15.3 is a special case of a more general substitution method for double integrals, a method that pictures changes in variables as transformations of regions.

Suppose that a region G in the uv -plane is transformed one-to-one into the region R in the xy -plane by equations of the form

$$x = g(u, v), \quad y = h(u, v),$$

as suggested in Figure 15.47. We call R the **image** of G under the transformation, and G the **preimage** of R . Any function $f(x, y)$ defined on R can be thought of as a function

$f(g(u, v), h(u, v))$ defined on G as well. How is the integral of $f(x, y)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G ?

The answer is: If g , h , and f have continuous partial derivatives and $J(u, v)$ (to be discussed in a moment) is zero only at isolated points, if at all, then

$$\iint_R f(x, y) \, dx \, dy = \iint_G f(g(u, v), h(u, v)) |J(u, v)| \, du \, dv. \quad (1)$$

The factor $J(u, v)$, whose absolute value appears in Equation (1), is the *Jacobian* of the coordinate transformation, named after German mathematician Carl Jacobi. It measures how much the transformation is expanding or contracting the area around a point in G as G is transformed into R .

HISTORICAL BIOGRAPHY

Carl Gustav Jacob Jacobi
(1804–1851)



Definition Jacobian

The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}. \quad (2)$$

The Jacobian is also denoted by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

to help remember how the determinant in Equation (2) is constructed from the partial derivatives of x and y . The derivation of Equation (1) is intricate and properly belongs to a course in advanced calculus. We do not give the derivation here.

For polar coordinates, we have r and θ in place of u and v . With $x = r \cos \theta$ and $y = r \sin \theta$, the Jacobian is

$$J(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r(\cos^2 \theta + \sin^2 \theta) = r.$$

Hence, Equation (1) becomes

$$\begin{aligned} \iint_R f(x, y) \, dx \, dy &= \iint_G f(r \cos \theta, r \sin \theta) |r| \, dr \, d\theta \\ &= \iint_G f(r \cos \theta, r \sin \theta) r \, dr \, d\theta, \quad \text{If } r \geq 0 \end{aligned} \quad (3)$$

which is the equation found in Section 15.3.

Figure 15.48 shows how the equations $x = r \cos \theta$, $y = r \sin \theta$ transform the rectangle G : $0 \leq r \leq 1$, $0 \leq \theta \leq \pi/2$ into the quarter circle R bounded by $x^2 + y^2 = 1$ in the first quadrant of the xy -plane.

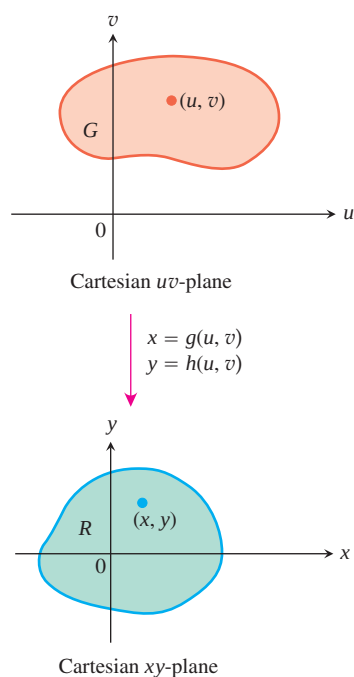


FIGURE 15.47 The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to change an integral over a region R in the xy -plane into an integral over a region G in the uv -plane.

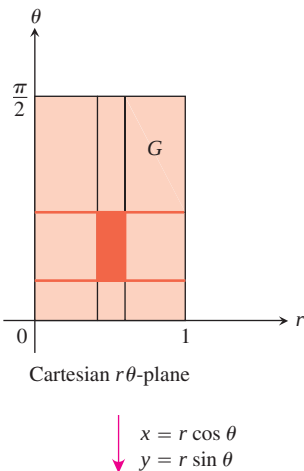


FIGURE 15.48 The equations $x = r \cos \theta$, $y = r \sin \theta$ transform G into R .

Notice that the integral on the right-hand side of Equation (3) is not the integral of $f(r \cos \theta, r \sin \theta)$ over a region in the polar coordinate plane. It is the integral of the product of $f(r \cos \theta, r \sin \theta)$ and r over a region G in the Cartesian $r\theta$ -plane.

Here is an example of another substitution.

EXAMPLE 1 Applying a Transformation to Integrate

Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} dx dy$$

by applying the transformation

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2} \tag{4}$$

and integrating over an appropriate region in the uv -plane.

Solution We sketch the region R of integration in the xy -plane and identify its boundaries (Figure 15.49).

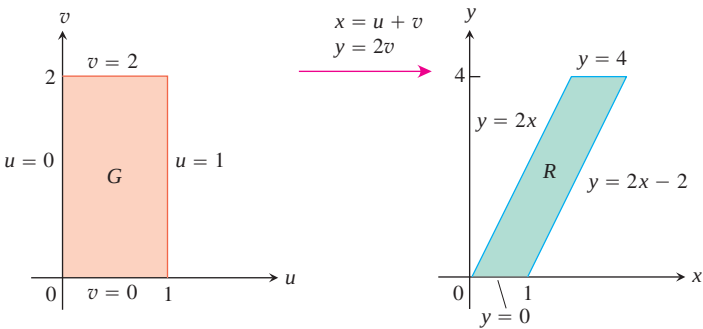


FIGURE 15.49 The equations $x = u + v$ and $y = 2v$ transform G into R . Reversing the transformation by the equations $u = (2x - y)/2$ and $v = y/2$ transforms R into G (Example 1).

To apply Equation (1), we need to find the corresponding uv -region G and the Jacobian of the transformation. To find them, we first solve Equations (4) for x and y in terms of u and v . Routine algebra gives

$$x = u + v \quad y = 2v. \tag{5}$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of R (Figure 15.49).

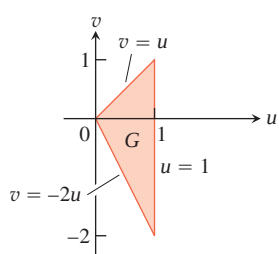
xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

The Jacobian of the transformation (again from Equations (5)) is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial u}(u + v) & \frac{\partial}{\partial v}(u + v) \\ \frac{\partial}{\partial u}(2v) & \frac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

We now have everything we need to apply Equation (1):

$$\begin{aligned} \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} dx dy &= \int_{v=0}^{v=2} \int_{u=0}^{u=1} u |J(u, v)| du dv \\ &= \int_0^2 \int_0^1 (u)(2) du dv = \int_0^2 \left[u^2 \right]_0^1 dv = \int_0^2 dv = 2. \end{aligned}$$



EXAMPLE 2 Applying a Transformation to Integrate

Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dy dx.$$

Solution We sketch the region R of integration in the xy -plane and identify its boundaries (Figure 15.50). The integrand suggests the transformation $u = x + y$ and $v = y - 2x$. Routine algebra produces x and y as functions of u and v :

$$x = \frac{u}{3} - \frac{v}{3}, \quad y = \frac{2u}{3} + \frac{v}{3}. \quad (6)$$

From Equations (6), we can find the boundaries of the uv -region G (Figure 15.50).

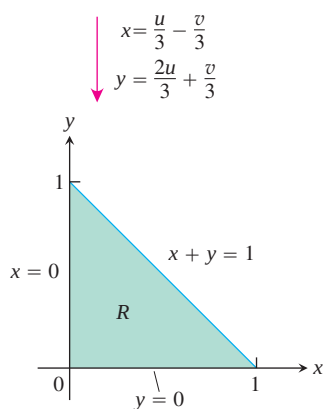


FIGURE 15.50 The equations $x = (u/3) - (v/3)$ and $y = (2u/3) + (v/3)$ transform G into R . Reversing the transformation by the equations $u = x + y$ and $v = y - 2x$ transforms R into G (Example 2).

xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

The Jacobian of the transformation in Equations (6) is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

Applying Equation (1), we evaluate the integral:

$$\begin{aligned}
 \int_0^1 \int_0^{1-x} \sqrt{x+y} (y-2x)^2 dy dx &= \int_{u=0}^1 \int_{v=-2u}^{v=u} u^{1/2} v^2 |J(u, v)| dv du \\
 &= \int_0^1 \int_{-2u}^u u^{1/2} v^2 \left(\frac{1}{3}\right) dv du = \frac{1}{3} \int_0^1 u^{1/2} \left[\frac{1}{3} v^3\right]_{v=-2u}^{v=u} du \\
 &= \frac{1}{9} \int_0^1 u^{1/2} (u^3 + 8u^3) du = \int_0^1 u^{7/2} du = \frac{2}{9} u^{9/2} \Big|_0^1 = \frac{2}{9}.
 \end{aligned}$$

Substitutions in Triple Integrals

The cylindrical and spherical coordinate substitutions in Section 15.6 are special cases of a substitution method that pictures changes of variables in triple integrals as transformations of three-dimensional regions. The method is like the method for double integrals except that now we work in three dimensions instead of two.

Suppose that a region G in uvw -space is transformed one-to-one into the region D in xyz -space by differentiable equations of the form

$$x = g(u, v, w), \quad y = h(u, v, w), \quad z = k(u, v, w),$$

as suggested in Figure 15.51. Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

defined on G . If g , h , and k have continuous first partial derivatives, then the integral of $F(x, y, z)$ over D is related to the integral of $H(u, v, w)$ over G by the equation

$$\iiint_D F(x, y, z) dx dy dz = \iiint_G H(u, v, w) |J(u, v, w)| du dv dw. \quad (7)$$

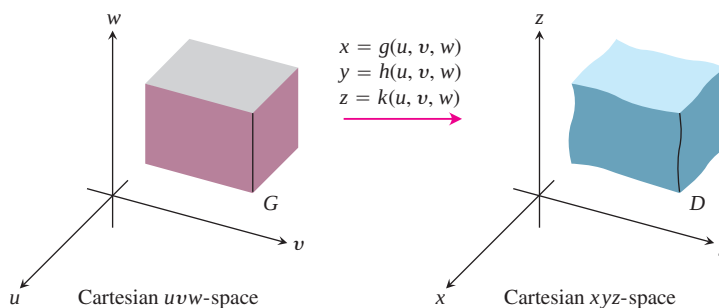


FIGURE 15.51 The equations $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$ allow us to change an integral over a region D in Cartesian xyz -space into an integral over a region G in Cartesian uvw -space.

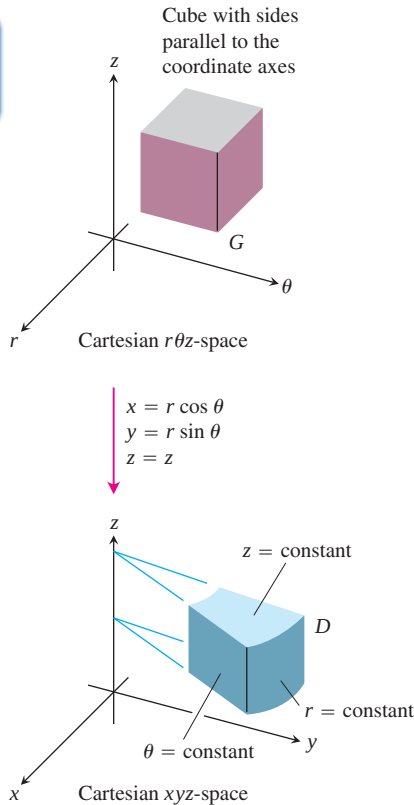


FIGURE 15.52 The equations $x = r \cos \theta$, $y = r \sin \theta$, and $z = z$ transform the cube G into a cylindrical wedge D .

The factor $J(u, v, w)$, whose absolute value appears in this equation, is the **Jacobian determinant**

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

This determinant measures how much the volume near a point in G is being expanded or contracted by the transformation from (u, v, w) to (x, y, z) coordinates. As in the two-dimensional case, the derivation of the change-of-variable formula in Equation (7) is complicated and we do not go into it here.

For cylindrical coordinates, r , θ , and z take the place of u , v , and w . The transformation from *Cartesian $r\theta z$ -space* to *Cartesian xyz -space* is given by the equations

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

(Figure 15.52). The Jacobian of the transformation is

$$J(r, \theta, z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r \cos^2 \theta + r \sin^2 \theta = r.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z) \, dx \, dy \, dz = \iiint_G H(r, \theta, z) |r| \, dr \, d\theta \, dz.$$

We can drop the absolute value signs whenever $r \geq 0$.

For spherical coordinates, ρ , ϕ , and θ take the place of u , v , and w . The transformation from *Cartesian $\rho\phi\theta$ -space* to *Cartesian xyz -space* is given by

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

(Figure 15.53). The Jacobian of the transformation is

$$J(\rho, \phi, \theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin \phi$$

(Exercise 17). The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z) \, dx \, dy \, dz = \iiint_G H(\rho, \phi, \theta) |\rho^2 \sin \phi| \, d\rho \, d\phi \, d\theta.$$

We can drop the absolute value signs because $\sin \phi$ is never negative for $0 \leq \phi \leq \pi$. Note that this is the same result we obtained in Section 15.6.

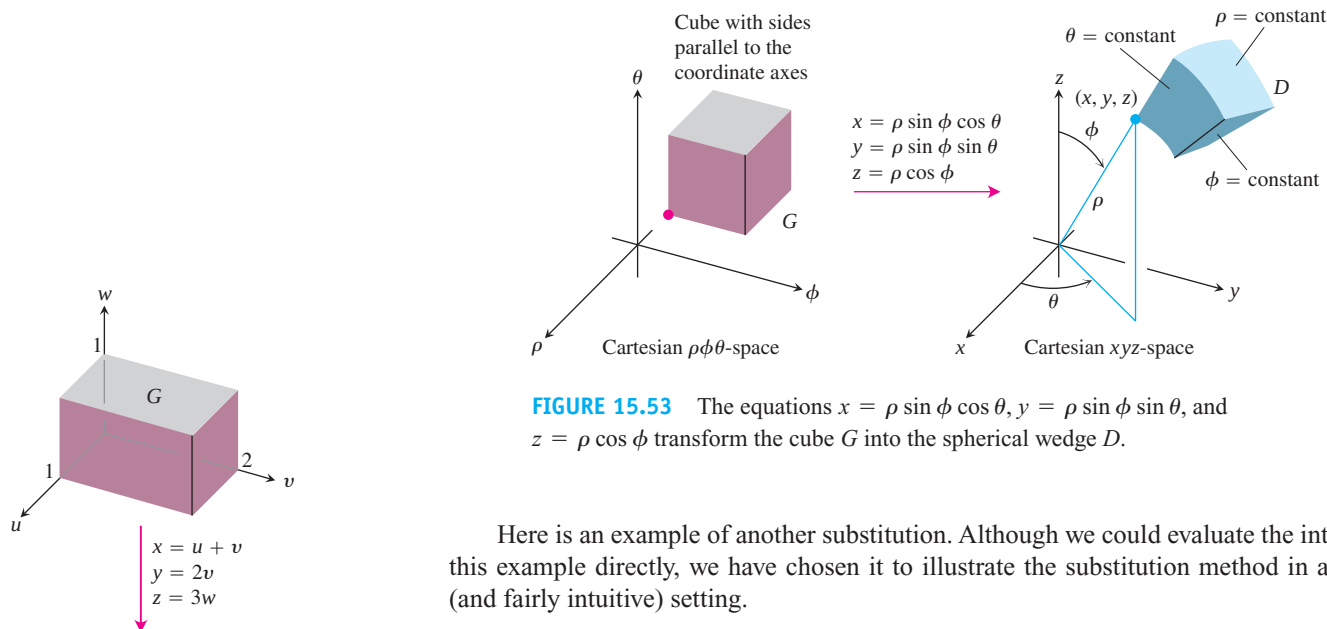


FIGURE 15.53 The equations $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$ transform the cube G into the spherical wedge D .

Here is an example of another substitution. Although we could evaluate the integral in this example directly, we have chosen it to illustrate the substitution method in a simple (and fairly intuitive) setting.

EXAMPLE 3 Applying a Transformation to Integrate

Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx \, dy \, dz$$

by applying the transformation

$$u = (2x - y)/2, \quad v = y/2, \quad w = z/3 \quad (8)$$

and integrating over an appropriate region in uvw -space.

Solution We sketch the region D of integration in xyz -space and identify its boundaries (Figure 15.54). In this case, the bounding surfaces are planes.

To apply Equation (7), we need to find the corresponding uvw -region G and the Jacobian of the transformation. To find them, we first solve Equations (8) for x , y , and z in terms of u , v , and w . Routine algebra gives

$$x = u + v, \quad y = 2v, \quad z = 3w. \quad (9)$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of D :

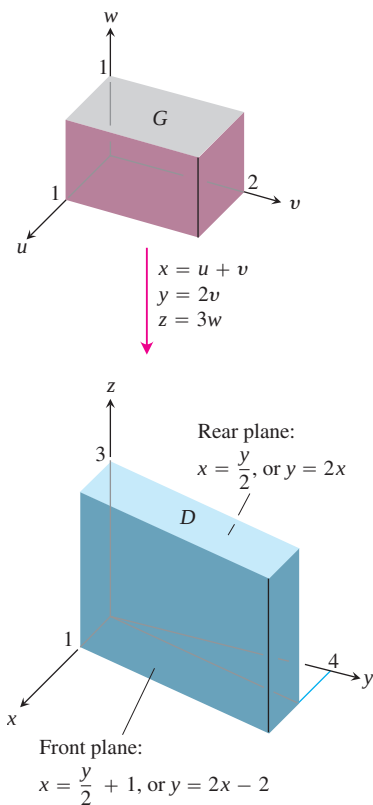


FIGURE 15.54 The equations $x = u + v$, $y = 2v$, and $z = 3w$ transform G into D . Reversing the transformation by the equations $u = (2x - y)/2$, $v = y/2$, and $w = z/3$ transforms D into G (Example 3).



<i>xyz</i>-equations for the boundary of <i>D</i>	Corresponding <i>uvw</i>-equations for the boundary of <i>G</i>	Simplified <i>uvw</i>-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Equation (7):

$$\begin{aligned} & \int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx dy dz \\ &= \int_0^1 \int_0^2 \int_0^1 (u+w) |J(u, v, w)| du dv dw \\ &= \int_0^1 \int_0^2 \int_0^1 (u+w)(6) du dv dw = 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv dw \\ &= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv dw = 6 \int_0^1 \left[\frac{v}{2} + vw \right]_0^2 dw = 6 \int_0^1 (1 + 2w) dw \\ &= 6[w + w^2]_0^1 = 6(2) = 12. \end{aligned}$$

The goal of this section was to introduce you to the ideas involved in coordinate transformations. A thorough discussion of transformations, the Jacobian, and multivariable substitution is best given in an advanced calculus course after a study of linear algebra.

EXERCISES 15.7

Finding Jacobians and Transformed Regions for Two Variables



1. a. Solve the system

$$u = x - y, \quad v = 2x + y$$

for x and y in terms of u and v . Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

- b. Find the image under the transformation $u = x - y$,

$v = 2x + y$ of the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(1, -2)$ in the xy -plane. Sketch the transformed region in the uv -plane.

2. a. Solve the system

$$u = x + 2y, \quad v = x - y$$

for x and y in terms of u and v . Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.



- b. Find the image under the transformation $u = x + 2y$, $v = x - y$ of the triangular region in the xy -plane bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$. Sketch the transformed region in the uv -plane.

3. a. Solve the system

$$u = 3x + 2y, \quad v = x + 4y$$

for x and y in terms of u and v . Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

- b. Find the image under the transformation $u = 3x + 2y$, $v = x + 4y$ of the triangular region in the xy -plane bounded by the x -axis, the y -axis, and the line $x + y = 1$. Sketch the transformed region in the uv -plane.
4. a. Solve the system

$$u = 2x - 3y, \quad v = -x + y$$

for x and y in terms of u and v . Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

- b. Find the image under the transformation $u = 2x - 3y$, $v = -x + y$ of the parallelogram R in the xy -plane with boundaries $x = -3$, $x = 0$, $y = x$, and $y = x + 1$. Sketch the transformed region in the uv -plane.

Applying Transformations to Evaluate Double Integrals



5. Evaluate the integral

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} dx dy$$

from Example 1 directly by integration with respect to x and y to confirm that its value is 2.

6. Use the transformation in Exercise 1 to evaluate the integral

$$\iint_R (2x^2 - xy - y^2) dx dy$$

for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

7. Use the transformation in Exercise 3 to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2) dx dy$$

for the region R in the first quadrant bounded by the lines $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, and $y = -(1/4)x + 1$.

8. Use the transformation and parallelogram R in Exercise 4 to evaluate the integral

$$\iint_R 2(x - y) dx dy.$$

9. Let R be the region in the first quadrant of the xy -plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy} \right) dx dy$$

as an integral over an appropriate region G in the uv -plane. Then evaluate the uv -integral over G .

10. a. Find the Jacobian of the transformation $x = u$, $y = uv$, and sketch the region G : $1 \leq u \leq 2$, $1 \leq uv \leq 2$ in the uv -plane.
- b. Then use Equation (1) to transform the integral

$$\int_1^2 \int_1^2 \frac{y}{x} dy dx$$

into an integral over G , and evaluate both integrals.

11. **Polar moment of inertia of an elliptical plate** A thin plate of constant density covers the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, $a > 0$, $b > 0$, in the xy -plane. Find the first moment of the plate about the origin. (Hint: Use the transformation $x = ar \cos \theta$, $y = br \sin \theta$.)

12. **The area of an ellipse** The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy -plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk G : $u^2 + v^2 \leq 1$ in the uv -plane. Find the area this way.

13. Use the transformation in Exercise 2 to evaluate the integral

$$\int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{(y-x)} dx dy$$

by first writing it as an integral over a region G in the uv -plane.

14. Use the transformation $x = u + (1/2)v$, $y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x - y)e^{(2x-y)^2} dx dy$$

by first writing it as an integral over a region G in the uv -plane.

Finding Jacobian Determinants

15. Find the Jacobian $\partial(x, y)/\partial(u, v)$ for the transformation

a. $x = u \cos v, \quad y = u \sin v$

b. $x = u \sin v, \quad y = u \cos v$

16. Find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ of the transformation

a. $x = u \cos v, \quad y = u \sin v, \quad z = w$

b. $x = 2u - 1, \quad y = 3v - 4, \quad z = (1/2)(w - 4)$

17. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$ -space to Cartesian xyz -space is $\rho^2 \sin \phi$.





- 18. Substitutions in single integrals** How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

Applying Transformations to Evaluate Triple Integrals



- 19.** Evaluate the integral in Example 3 by integrating with respect to x , y , and z .

- 20. Volume of an ellipsoid** Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(Hint: Let $x = au$, $y = bv$, and $z = cw$. Then find the volume of an appropriate region in uvw -space.)

- 21.** Evaluate

$$\iiint |xyz| \, dx \, dy \, dz$$

over the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

(Hint: Let $x = au$, $y = bv$, and $z = cw$. Then integrate over an appropriate region in uvw -space.)

- 22.** Let D be the region in xyz -space defined by the inequalities

$$1 \leq x \leq 2, \quad 0 \leq xy \leq 2, \quad 0 \leq z \leq 1.$$

Evaluate

$$\iiint_D (x^2y + 3xyz) \, dx \, dy \, dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw -space.

- 23. Centroid of a solid semiellipsoid** Assuming the result that the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base toward the top, show, by transforming the appropriate integrals, that the center of mass of a solid semiellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1$, $z \geq 0$, lies on the z -axis three-eighths of the way from the base toward the top. (You can do this without evaluating any of the integrals.)
- 24. Cylindrical shells** In Section 6.2, we learned how to find the volume of a solid of revolution using the shell method; namely, if the region between the curve $y = f(x)$ and the x -axis from a to b ($0 < a < b$) is revolved about the y -axis, the volume of the resulting solid is $\int_a^b 2\pi x f(x) \, dx$. Prove that finding volumes by using triple integrals gives the same result. (Hint: Use cylindrical coordinates with the roles of y and z changed.)

Chapter 15 Questions to Guide Your Review

1. Define the double integral of a function of two variables over a bounded region in the coordinate plane.
2. How are double integrals evaluated as iterated integrals? Does the order of integration matter? How are the limits of integration determined? Give examples.
3. How are double integrals used to calculate areas, average values, masses, moments, centers of mass, and radii of gyration? Give examples.
4. How can you change a double integral in rectangular coordinates into a double integral in polar coordinates? Why might it be worthwhile to do so? Give an example.
5. Define the triple integral of a function $f(x, y, z)$ over a bounded region in space.
6. How are triple integrals in rectangular coordinates evaluated? How are the limits of integration determined? Give an example.
7. How are triple integrals in rectangular coordinates used to calculate volumes, average values, masses, moments, centers of mass, and radii of gyration? Give examples.
8. How are triple integrals defined in cylindrical and spherical coordinates? Why might one prefer working in one of these coordinate systems to working in rectangular coordinates?
9. How are triple integrals in cylindrical and spherical coordinates evaluated? How are the limits of integration found? Give examples.
10. How are substitutions in double integrals pictured as transformations of two-dimensional regions? Give a sample calculation.
11. How are substitutions in triple integrals pictured as transformations of three-dimensional regions? Give a sample calculation.

Chapter 15 Practice Exercises

Planar Regions of Integration

In Exercises 1–4, sketch the region of integration and evaluate the double integral.

1. $\int_1^{10} \int_0^{1/y} ye^{xy} dx dy$
2. $\int_0^1 \int_0^{x^3} e^{y/x} dy dx$
3. $\int_0^{3/2} \int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t ds dt$
4. $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy dx dy$

Reversing the Order of Integration

In Exercises 5–8, sketch the region of integration and write an equivalent integral with the order of integration reversed. Then evaluate both integrals.

5. $\int_0^4 \int_{-\sqrt{4-y}}^{(y-4)/2} dx dy$
6. $\int_0^1 \int_{x^2}^x \sqrt{x} dy dx$
7. $\int_0^{3/2} \int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y dx dy$
8. $\int_0^2 \int_0^{4-x^2} 2x dy dx$

Evaluating Double Integrals

Evaluate the integrals in Exercises 9–12.

9. $\int_0^1 \int_{2y}^2 4 \cos(x^2) dx dy$
10. $\int_0^2 \int_{y/2}^1 e^{x^2} dx dy$
11. $\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy dx}{y^4 + 1}$
12. $\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{2\pi \sin \pi x^2}{x^2} dx dy$

Areas and Volumes

13. **Area between line and parabola** Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the xy -plane.
14. **Area bounded by lines and parabola** Find the area of the “triangular” region in the xy -plane that is bounded on the right by the parabola $y = x^2$, on the left by the line $x + y = 2$, and above by the line $y = 4$.
15. **Volume of the region under a paraboloid** Find the volume under the paraboloid $z = x^2 + y^2$ above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy -plane.
16. **Volume of the region under parabolic cylinder** Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the xy -plane.

Average Values

Find the average value of $f(x, y) = xy$ over the regions in Exercises 17 and 18.

17. The square bounded by the lines $x = 1$, $y = 1$ in the first quadrant
18. The quarter circle $x^2 + y^2 \leq 1$ in the first quadrant

Masses and Moments

19. **Centroid** Find the centroid of the “triangular” region bounded by the lines $x = 2$, $y = 2$ and the hyperbola $xy = 2$ in the xy -plane.
20. **Centroid** Find the centroid of the region between the parabola $x + y^2 - 2y = 0$ and the line $x + 2y = 0$ in the xy -plane.
21. **Polar moment** Find the polar moment of inertia about the origin of a thin triangular plate of constant density $\delta = 3$ bounded by the y -axis and the lines $y = 2x$ and $y = 4$ in the xy -plane.
22. **Polar moment** Find the polar moment of inertia about the center of a thin rectangular sheet of constant density $\delta = 1$ bounded by the lines
 - a. $x = \pm 2$, $y = \pm 1$ in the xy -plane
 - b. $x = \pm a$, $y = \pm b$ in the xy -plane.
 (Hint: Find I_x . Then use the formula for I_x to find I_y and add the two to find I_0 .)
23. **Inertial moment and radius of gyration** Find the moment of inertia and radius of gyration about the x -axis of a thin plate of constant density δ covering the triangle with vertices $(0, 0)$, $(3, 0)$, and $(3, 2)$ in the xy -plane.
24. **Plate with variable density** Find the center of mass and the moments of inertia and radii of gyration about the coordinate axes of a thin plate bounded by the line $y = x$ and the parabola $y = x^2$ in the xy -plane if the density is $\delta(x, y) = x + 1$.
25. **Plate with variable density** Find the mass and first moments about the coordinate axes of a thin square plate bounded by the lines $x = \pm 1$, $y = \pm 1$ in the xy -plane if the density is $\delta(x, y) = x^2 + y^2 + 1/3$.
26. **Triangles with same inertial moment and radius of gyration** Find the moment of inertia and radius of gyration about the x -axis of a thin triangular plate of constant density δ whose base lies along the interval $[0, b]$ on the x -axis and whose vertex lies on the line $y = h$ above the x -axis. As you will see, it does not matter where on the line this vertex lies. All such triangles have the same moment of inertia and radius of gyration about the x -axis.

Polar Coordinates

Evaluate the integrals in Exercises 27 and 28 by changing to polar coordinates.

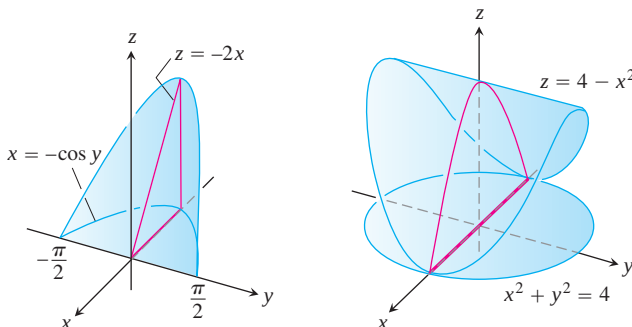
27. $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2 dy dx}{(1 + x^2 + y^2)^2}$
28. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1) dx dy$
29. **Centroid** Find the centroid of the region in the polar coordinate plane defined by the inequalities $0 \leq r \leq 3$, $-\pi/3 \leq \theta \leq \pi/3$.

30. **Centroid** Find the centroid of the region in the first quadrant bounded by the rays $\theta = 0$ and $\theta = \pi/2$ and the circles $r = 1$ and $r = 3$.
31. **a. Centroid** Find the centroid of the region in the polar coordinate plane that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.
- b.** Sketch the region and show the centroid in your sketch.
32. **a. Centroid** Find the centroid of the plane region defined by the polar coordinate inequalities $0 \leq r \leq a$, $-\alpha \leq \theta \leq \alpha$ ($0 < \alpha \leq \pi$). How does the centroid move as $\alpha \rightarrow \pi^-$?
- b.** Sketch the region for $\alpha = 5\pi/6$ and show the centroid in your sketch.
33. **Integrating over lemniscate** Integrate the function $f(x, y) = 1/(1 + x^2 + y^2)^2$ over the region enclosed by one loop of the lemniscate $(x^2 + y^2)^2 - (x^2 - y^2) = 0$.
34. Integrate $f(x, y) = 1/(1 + x^2 + y^2)^2$ over
- a. Triangular region** The triangle with vertices $(0, 0)$, $(1, 0)$, $(1, \sqrt{3})$.
- b. First quadrant** The first quadrant of the xy -plane.

Triple Integrals in Cartesian Coordinates

Evaluate the integrals in Exercises 35–38.

35. $\int_0^\pi \int_0^\pi \int_0^\pi \cos(x + y + z) \, dx \, dy \, dz$
36. $\int_{\ln 6}^{\ln 7} \int_0^{\ln 2} \int_{\ln 4}^{\ln 5} e^{(x+y+z)} \, dz \, dy \, dx$
37. $\int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z) \, dz \, dy \, dx$
38. $\int_1^e \int_1^x \int_0^z \frac{2y}{z^3} \, dy \, dz \, dx$
39. **Volume** Find the volume of the wedge-shaped region enclosed on the side by the cylinder $x = -\cos y$, $-\pi/2 \leq y \leq \pi/2$, on the top by the plane $z = -2x$, and below by the xy -plane.



40. **Volume** Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the xy -plane.

41. **Average value** Find the average value of $f(x, y, z) = 30xz \sqrt{x^2 + y}$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 3$, $z = 1$.
42. **Average value** Find the average value of ρ over the solid sphere $\rho \leq a$ (spherical coordinates).

Cylindrical and Spherical Coordinates

43. **Cylindrical to rectangular coordinates** Convert

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3 \, dz \, r \, dr \, d\theta, \quad r \geq 0$$

to **(a)** rectangular coordinates with the order of integration $dz \, dx \, dy$ and **(b)** spherical coordinates. Then **(c)** evaluate one of the integrals.

44. **Rectangular to cylindrical coordinates** **(a)** Convert to cylindrical coordinates. Then **(b)** evaluate the new integral.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-(x^2+y^2)}^{(x^2+y^2)} 21xy^2 \, dz \, dy \, dx$$

45. **Rectangular to spherical coordinates** **(a)** Convert to spherical coordinates. Then **(b)** evaluate the new integral.

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 dz \, dy \, dx$$

46. **Rectangular, cylindrical, and spherical coordinates** Write an iterated triple integral for the integral of $f(x, y, z) = 6 + 4y$ over the region in the first octant bounded by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 1$, and the coordinate planes in **(a)** rectangular coordinates, **(b)** cylindrical coordinates, and **(c)** spherical coordinates. Then **(d)** find the integral of f by evaluating one of the triple integrals.
47. **Cylindrical to rectangular coordinates** Set up an integral in rectangular coordinates equivalent to the integral

$$\int_0^{\pi/2} \int_1^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r^3 (\sin \theta \cos \theta) z^2 \, dz \, dr \, d\theta.$$

Arrange the order of integration to be z first, then y , then x .

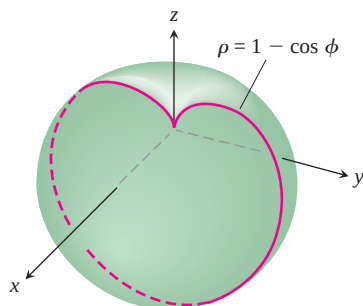
48. **Rectangular to cylindrical coordinates** The volume of a solid is

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz \, dy \, dx.$$

- a.** Describe the solid by giving equations for the surfaces that form its boundary.
- b.** Convert the integral to cylindrical coordinates but do not evaluate the integral.
49. **Spherical versus cylindrical coordinates** Triple integrals involving spherical shapes do not always require spherical coordinates for convenient evaluation. Some calculations may be accomplished more easily with cylindrical coordinates. As a case in point, find the volume of the region bounded above by the

sphere $x^2 + y^2 + z^2 = 8$ and below by the plane $z = 2$ by using (a) cylindrical coordinates and (b) spherical coordinates.

- 50. Finding I_z in spherical coordinates** Find the moment of inertia about the z -axis of a solid of constant density $\delta = 1$ that is bounded above by the sphere $\rho = 2$ and below by the cone $\phi = \pi/3$ (spherical coordinates).
- 51. Moment of inertia of a “thick” sphere** Find the moment of inertia of a solid of constant density δ bounded by two concentric spheres of radii a and b ($a < b$) about a diameter.
- 52. Moment of inertia of an apple** Find the moment of inertia about the z -axis of a solid of density $\delta = 1$ enclosed by the spherical coordinate surface $\rho = 1 - \cos \phi$. The solid is the red curve rotated about the z -axis in the accompanying figure.



Substitutions

- 53.** Show that if $u = x - y$ and $v = y$, then

$$\int_0^\infty \int_0^x e^{-sx} f(x - y, y) dy dx = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u, v) du dv.$$

- 54.** What relationship must hold between the constants a , b , and c to make

$$\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(ax^2 + 2bxy + cy^2)} dx dy = 1?$$

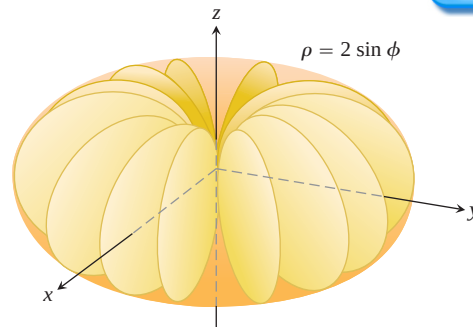
(Hint: Let $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$, where $(\alpha\delta - \beta\gamma)^2 = ac - b^2$. Then $ax^2 + 2bxy + cy^2 = s^2 + t^2$.)

Chapter 15 Additional and Advanced Exercises



Volumes

- Sand pile: double and triple integrals** The base of a sand pile covers the region in the xy -plane that is bounded by the parabola $x^2 + y = 6$ and the line $y = x$. The height of the sand above the point (x, y) is x^2 . Express the volume of sand as (a) a double integral, (b) a triple integral. Then (c) find the volume.
- Water in a hemispherical bowl** A hemispherical bowl of radius 5 cm is filled with water to within 3 cm of the top. Find the volume of water in the bowl.
- Solid cylindrical region between two planes** Find the volume of the portion of the solid cylinder $x^2 + y^2 \leq 1$ that lies between the planes $z = 0$ and $x + y + z = 2$.
- Sphere and paraboloid** Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.
- Two paraboloids** Find the volume of the region bounded above by the paraboloid $z = 3 - x^2 - y^2$ and below by the paraboloid $z = 2x^2 + 2y^2$.
- Spherical coordinates** Find the volume of the region enclosed by the spherical coordinate surface $\rho = 2 \sin \phi$ (see accompanying figure).



- Hole in sphere** A circular cylindrical hole is bored through a solid sphere, the axis of the hole being a diameter of the sphere. The volume of the remaining solid is

$$V = 2 \int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-z^2}} r \, dr \, dz \, d\theta.$$

- Find the radius of the hole and the radius of the sphere.
 - Evaluate the integral.
- Sphere and cylinder** Find the volume of material cut from the solid sphere $r^2 + z^2 \leq 9$ by the cylinder $r = 3 \sin \theta$.

- 9. Two paraboloids** Find the volume of the region enclosed by the surfaces $z = x^2 + y^2$ and $z = (x^2 + y^2 + 1)/2$.
- 10. Cylinder and surface $z = xy$** Find the volume of the region in the first octant that lies between the cylinders $r = 1$ and $r = 2$ and that is bounded below by the xy -plane and above by the surface $z = xy$.

Changing the Order of Integration

- 11.** Evaluate the integral

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx.$$

(Hint: Use the relation

$$\frac{e^{-ax} - e^{-bx}}{x} = \int_a^b e^{-xy} dy$$

to form a double integral and evaluate the integral by changing the order of integration.)

- 12. a. Polar coordinates** Show, by changing to polar coordinates, that

$$\int_0^{a \sin \beta} \int_{y \cot \beta}^{\sqrt{a^2 - y^2}} \ln(x^2 + y^2) dx dy = a^2 \beta \left(\ln a - \frac{1}{2} \right),$$

where $a > 0$ and $0 < \beta < \pi/2$.

- b.** Rewrite the Cartesian integral with the order of integration reversed.

- 13. Reducing a double to a single integral** By changing the order of integration, show that the following double integral can be reduced to a single integral:

$$\int_0^x \int_0^u e^{m(x-t)} f(t) dt du = \int_0^x (x-t) e^{m(x-t)} f(t) dt.$$

Similarly, it can be shown that

$$\int_0^x \int_0^v \int_0^u e^{m(x-t)} f(t) dt du dv = \int_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t) dt.$$

- 14. Transforming a double integral to obtain constant limits** Sometimes a multiple integral with variable limits can be changed into one with constant limits. By changing the order of integration, show that

$$\begin{aligned} \int_0^1 f(x) \left(\int_0^x g(x-y) f(y) dy \right) dx \\ &= \int_0^1 f(y) \left(\int_y^1 g(x-y) f(x) dx \right) dy \\ &= \frac{1}{2} \int_0^1 \int_0^1 g(|x-y|) f(x) f(y) dx dy. \end{aligned}$$

Masses and Moments

- 15. Minimizing polar inertia** A thin plate of constant density is to occupy the triangular region in the first quadrant of the xy -plane

having vertices $(0, 0)$, $(a, 0)$, and $(a, 1/a)$. What value of a will minimize the plate's polar moment of inertia about the origin?

- 16. Polar inertia of triangular plate** Find the polar moment of inertia about the origin of a thin triangular plate of constant density $\delta = 3$ bounded by the y -axis and the lines $y = 2x$ and $y = 4$ in the xy -plane.

- 17. Mass and polar inertia of a counterweight** The counterweight of a flywheel of constant density 1 has the form of the smaller segment cut from a circle of radius a by a chord at a distance b from the center ($b < a$). Find the mass of the counterweight and its polar moment of inertia about the center of the wheel.

- 18. Centroid of boomerang** Find the centroid of the boomerang-shaped region between the parabolas $y^2 = -4(x-1)$ and $y^2 = -2(x-2)$ in the xy -plane.

Theory and Applications

- 19.** Evaluate

$$\int_0^a \int_0^b e^{\max(b^2x^2, a^2y^2)} dy dx,$$

where a and b are positive numbers and

$$\max(b^2x^2, a^2y^2) = \begin{cases} b^2x^2 & \text{if } b^2x^2 \geq a^2y^2 \\ a^2y^2 & \text{if } b^2x^2 < a^2y^2. \end{cases}$$

- 20.** Show that

$$\iint \frac{\partial^2 F(x, y)}{\partial x \partial y} dx dy$$

over the rectangle $x_0 \leq x \leq x_1, y_0 \leq y \leq y_1$, is

$$F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0).$$

- 21.** Suppose that $f(x, y)$ can be written as a product $f(x, y) = F(x)G(y)$ of a function of x and a function of y . Then the integral of f over the rectangle $R: a \leq x \leq b, c \leq y \leq d$ can be evaluated as a product as well, by the formula

$$\iint_R f(x, y) dA = \left(\int_a^b F(x) dx \right) \left(\int_c^d G(y) dy \right). \quad (1)$$

The argument is that

$$\iint_R f(x, y) dA = \int_c^d \left(\int_a^b F(x)G(y) dx \right) dy \quad (i)$$

$$= \int_c^d \left(G(y) \int_a^b F(x) dx \right) dy \quad (ii)$$

$$= \int_c^d \left(\int_a^b F(x) dx \right) G(y) dy \quad (iii)$$

$$= \left(\int_a^b F(x) dx \right) \int_c^d G(y) dy. \quad (iv)$$

- a. Give reasons for steps (i) through (v).

When it applies, Equation (1) can be a time saver. Use it to evaluate the following integrals.

b. $\int_0^{\ln 2} \int_0^{\pi/2} e^x \cos y \, dy \, dx$ c. $\int_1^2 \int_{-1}^1 \frac{x}{y^2} \, dx \, dy$

22. Let $D_{\mathbf{u}}f$ denote the derivative of $f(x, y) = (x^2 + y^2)/2$ in the direction of the unit vector $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j}$.

a. **Finding average value** Find the average value of $D_{\mathbf{u}}f$ over the triangular region cut from the first quadrant by the line $x + y = 1$.

b. **Average value and centroid** Show in general that the average value of $D_{\mathbf{u}}f$ over a region in the xy -plane is the value of $D_{\mathbf{u}}f$ at the centroid of the region.

23. **The value of $\Gamma(1/2)$** The gamma function,

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, dt,$$

extends the factorial function from the nonnegative integers to other real values. Of particular interest in the theory of differential equations is the number

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{(1/2)-1} e^{-t} \, dt = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} \, dt. \quad (2)$$

- a. If you have not yet done Exercise 37 in Section 15.3, do it now to show that

$$I = \int_0^{\infty} e^{-y^2} \, dy = \frac{\sqrt{\pi}}{2}.$$

- b. Substitute $y = \sqrt{t}$ in Equation (2) to show that $\Gamma(1/2) = 2I = \sqrt{\pi}$.

24. **Total electrical charge over circular plate** The electrical charge distribution on a circular plate of radius R meters is $\sigma(r, \theta) = kr(1 - \sin \theta)$ coulomb/m² (k a constant). Integrate σ over the plate to find the total charge Q .

25. **A parabolic rain gauge** A bowl is in the shape of the graph of $z = x^2 + y^2$ from $z = 0$ to $z = 10$ in. You plan to calibrate the bowl to make it into a rain gauge. What height in the bowl would correspond to 1 in. of rain? 3 in. of rain?

26. **Water in a satellite dish** A parabolic satellite dish is 2 m wide and 1/2 m deep. Its axis of symmetry is tilted 30 degrees from the vertical.

a. Set up, but do not evaluate, a triple integral in rectangular coordinates that gives the amount of water the satellite dish will hold. (*Hint:* Put your coordinate system so that the satellite dish is in “standard position” and the plane of the water level is slanted.) (*Caution:* The limits of integration are not “nice.”)

b. What would be the smallest tilt of the satellite dish so that it holds no water?

27. **An infinite half-cylinder** Let D be the interior of the infinite right circular half-cylinder of radius 1 with its single-end face suspended 1 unit above the origin and its axis the ray from $(0, 0, 1)$ to ∞ . Use cylindrical coordinates to evaluate

$$\iiint_D z(r^2 + z^2)^{-5/2} \, dV.$$

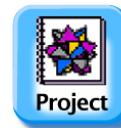
28. **Hypervolume** We have learned that $\int_a^b 1 \, dx$ is the length of the interval $[a, b]$ on the number line (one-dimensional space), $\iint_R 1 \, dA$ is the area of region R in the xy -plane (two-dimensional space), and $\iiint_D 1 \, dV$ is the volume of the region D in three-dimensional space (xyz -space). We could continue: If Q is a region in 4-space ($xyzw$ -space), then $\iiint\int_Q 1 \, dV$ is the “hypervolume” of Q . Use your generalizing abilities and a Cartesian coordinate system of 4-space to find the hypervolume inside the unit 4-sphere $x^2 + y^2 + z^2 + w^2 = 1$.

Chapter 15 Technology Application Projects

Mathematica/Maple Module

Take Your Chances: Try the Monte Carlo Technique for Numerical Integration in Three Dimensions

Use the Monte Carlo technique to integrate numerically in three dimensions.

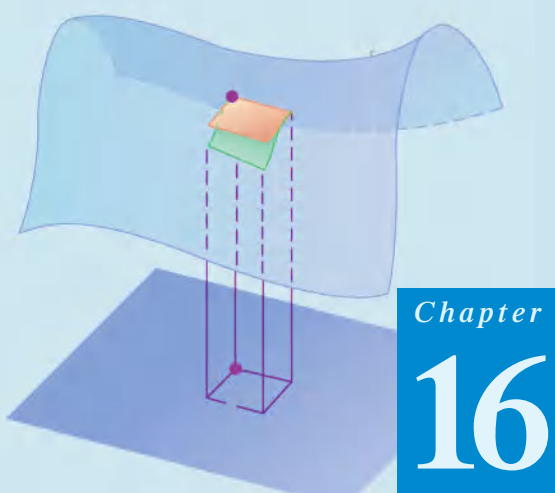


Mathematica/Maple Module

Means and Moments and Exploring New Plotting Techniques, Part II.

Use the method of moments in a form that makes use of geometric symmetry as well as multiple integration.





Chapter 16

INTEGRATION IN VECTOR FIELDS

OVERVIEW This chapter treats integration in vector fields. It is the mathematics that engineers and physicists use to describe fluid flow, design underwater transmission cables, explain the flow of heat in stars, and put satellites in orbit. In particular, we define line integrals, which are used to find the work done by a force field in moving an object along a path through the field. We also define surface integrals so we can find the rate that a fluid flows across a surface. Along the way we develop key concepts and results, such as *conservative* force fields and Green's Theorem, to simplify our calculations of these new integrals by connecting them to the single, double, and triple integrals we have already studied.

16.1

Line Integrals

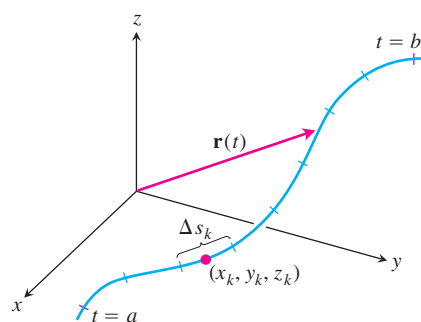


FIGURE 16.1 The curve $\mathbf{r}(t)$ partitioned into small arcs from $t = a$ to $t = b$. The length of a typical subarc is Δs_k .

In Chapter 5 we defined the definite integral of a function over a finite closed interval $[a, b]$ on the x -axis. We used definite integrals to find the mass of a thin straight rod, or the work done by a variable force directed along the x -axis. Now we would like to calculate the masses of thin rods or wires lying along a *curve* in the plane or in space. For these calculations we need a more general notion of a “line” integral than integrating over a line segment on the x -axis. Instead we need to integrate over a curve C in the plane or in space. These more general integrals are called *line integrals*, although “curve” integrals might be more descriptive. We make our definitions for space curves, remembering that curves in the xy -plane are just a special case with z -coordinate identically zero.

Suppose that $f(x, y, z)$ is a real-valued function we wish to integrate over the curve $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \leq t \leq b$, lying within the domain of f . The values of f along the curve are given by the composite function $f(g(t), h(t), k(t))$. We are going to integrate this composite with respect to arc length from $t = a$ to $t = b$. To begin, we first partition the curve into a finite number n of subarcs (Figure 16.1). The typical subarc has length Δs_k . In each subarc we choose a point (x_k, y_k, z_k) and form the sum

$$S_n = \sum_{k=1}^n f(x_k, y_k, z_k) \Delta s_k.$$

If f is continuous and the functions g , h , and k have continuous first derivatives, then these sums approach a limit as n increases and the lengths Δs_k approach zero. We call this limit the **line integral of f over the curve from a to b** . If the curve is denoted by a single letter, C for example, the notation for the integral is

$$\int_C f(x, y, z) \, ds \quad \text{“The integral of } f \text{ over } C\text{”} \quad (1)$$

If $\mathbf{r}(t)$ is smooth for $a \leq t \leq b$ ($\mathbf{v} = d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$), we can use the equation

$$s(t) = \int_a^t |\mathbf{v}(\tau)| d\tau \quad \begin{array}{l} \text{Equation (3) of Section 13.3} \\ \text{with } t_0 = a \end{array}$$

to express ds in Equation (1) as $ds = |\mathbf{v}(t)| dt$. A theorem from advanced calculus says that we can then evaluate the integral of f over C as

$$\int_C f(x, y, z) ds = \int_a^b f(g(t), h(t), k(t)) |\mathbf{v}(t)| dt.$$

Notice that the integral on the right side of this last equation is just an ordinary (single) definite integral, as defined in Chapter 5, where we are integrating with respect to the parameter t . The formula evaluates the line integral on the left side correctly no matter what parametrization is used, as long as the parametrization is smooth.

How to Evaluate a Line Integral

To integrate a continuous function $f(x, y, z)$ over a curve C :

1. Find a smooth parametrization of C ,

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \quad a \leq t \leq b$$

2. Evaluate the integral as

$$\int_C f(x, y, z) ds = \int_a^b f(g(t), h(t), k(t)) |\mathbf{v}(t)| dt. \quad (2)$$

If f has the constant value 1, then the integral of f over C gives the length of C .

EXAMPLE 1 Evaluating a Line Integral

Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1, 1, 1)$ (Figure 16.2).

Solution We choose the simplest parametrization we can think of:

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 1.$$

The components have continuous first derivatives and $|\mathbf{v}(t)| = |\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ is never 0, so the parametrization is smooth. The integral of f over C is

$$\begin{aligned} \int_C f(x, y, z) ds &= \int_0^1 f(t, t, t)(\sqrt{3}) dt && \text{Equation (2)} \\ &= \int_0^1 (t - 3t^2 + t)\sqrt{3} dt \\ &= \sqrt{3} \int_0^1 (2t - 3t^2) dt = \sqrt{3} [t^2 - t^3]_0^1 = 0. \end{aligned}$$

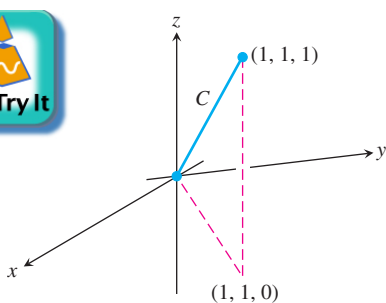


FIGURE 16.2 The integration path in Example 1.



Additivity

Line integrals have the useful property that if a curve C is made by joining a finite number of curves $C_1, C_2, \dots, C_n$ end to end, then the integral of a function over C is the sum of the integrals over the curves that make it up:

$$\int_C f \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds + \cdots + \int_{C_n} f \, ds. \quad (3)$$

EXAMPLE 2 Line Integral for Two Joined Paths

Figure 16.3 shows another path from the origin to $(1, 1, 1)$, the union of line segments C_1 and C_2 . Integrate $f(x, y, z) = x - 3y^2 + z$ over $C_1 \cup C_2$.

Solution We choose the simplest parametrizations for C_1 and C_2 we can think of, checking the lengths of the velocity vectors as we go along:

$$C_1: \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \leq t \leq 1; \quad |\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$C_2: \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 1; \quad |\mathbf{v}| = \sqrt{0^2 + 0^2 + 1^2} = 1.$$

With these parametrizations we find that

$$\int_{C_1 \cup C_2} f(x, y, z) \, ds = \int_{C_1} f(x, y, z) \, ds + \int_{C_2} f(x, y, z) \, ds \quad \text{Equation (3)}$$

$$= \int_0^1 f(t, t, 0) \sqrt{2} \, dt + \int_0^1 f(1, 1, t) (1) \, dt \quad \text{Equation (2)}$$

$$= \int_0^1 (t - 3t^2 + 0) \sqrt{2} \, dt + \int_0^1 (1 - 3 + t)(1) \, dt$$

$$= \sqrt{2} \left[\frac{t^2}{2} - t^3 \right]_0^1 + \left[\frac{t^2}{2} - 2t \right]_0^1 = -\frac{\sqrt{2}}{2} - \frac{3}{2}. \quad \blacksquare$$

Notice three things about the integrations in Examples 1 and 2. First, as soon as the components of the appropriate curve were substituted into the formula for f , the integration became a standard integration with respect to t . Second, the integral of f over $C_1 \cup C_2$ was obtained by integrating f over each section of the path and adding the results. Third, the integrals of f over C and $C_1 \cup C_2$ had different values. For most functions, the value of the integral along a path joining two points changes if you change the path between them. For some functions, however, the value remains the same, as we will see in Section 16.3.

Mass and Moment Calculations

We treat coil springs and wires like masses distributed along smooth curves in space. The distribution is described by a continuous density function $\delta(x, y, z)$ (mass per unit length). The spring's or wire's mass, center of mass, and moments are then calculated with the formulas in Table 16.1. The formulas also apply to thin rods.

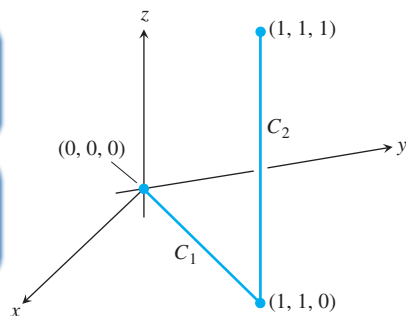


FIGURE 16.3 The path of integration in Example 2.

TABLE 16.1 Mass and moment formulas for coil springs, thin rods, and wires lying along a smooth curve C in space

Mass: $M = \int_C \delta(x, y, z) \, ds$ ($\delta = \delta(x, y, z) = \text{density}$)

First moments about the coordinate planes:

$$M_{yz} = \int_C x \delta \, ds, \quad M_{xz} = \int_C y \delta \, ds, \quad M_{xy} = \int_C z \delta \, ds$$

Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \quad \bar{y} = M_{xz}/M, \quad \bar{z} = M_{xy}/M$$

Moments of inertia about axes and other lines:

$$I_x = \int_C (y^2 + z^2) \delta \, ds, \quad I_y = \int_C (x^2 + z^2) \delta \, ds$$

$$I_z = \int_C (x^2 + y^2) \delta \, ds, \quad I_L = \int_C r^2 \delta \, ds$$

$$r(x, y, z) = \text{distance from the point } (x, y, z) \text{ to line } L$$

Radius of gyration about a line L : $R_L = \sqrt{I_L/M}$



EXAMPLE 3 Finding Mass, Center of Mass, Moment of Inertia, Radius of Gyration

A coil spring lies along the helix

$$\mathbf{r}(t) = (\cos 4t)\mathbf{i} + (\sin 4t)\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 2\pi.$$

The spring's density is a constant, $\delta = 1$. Find the spring's mass and center of mass, and its moment of inertia and radius of gyration about the z -axis.

Solution We sketch the spring (Figure 16.4). Because of the symmetries involved, the center of mass lies at the point $(0, 0, \pi)$ on the z -axis.

For the remaining calculations, we first find $|\mathbf{v}(t)|$:

$$\begin{aligned} |\mathbf{v}(t)| &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \\ &= \sqrt{(-4 \sin 4t)^2 + (4 \cos 4t)^2 + 1} = \sqrt{17}. \end{aligned}$$

We then evaluate the formulas from Table 16.1 using Equation (2):

$$M = \int_{\text{Helix}} \delta \, ds = \int_0^{2\pi} (1) \sqrt{17} \, dt = 2\pi \sqrt{17}$$

$$\begin{aligned} I_z &= \int_{\text{Helix}} (x^2 + y^2) \delta \, ds = \int_0^{2\pi} (\cos^2 4t + \sin^2 4t)(1) \sqrt{17} \, dt \\ &= \int_0^{2\pi} \sqrt{17} \, dt = 2\pi \sqrt{17} \end{aligned}$$

$$R_z = \sqrt{I_z/M} = \sqrt{2\pi \sqrt{17}/(2\pi \sqrt{17})} = 1.$$

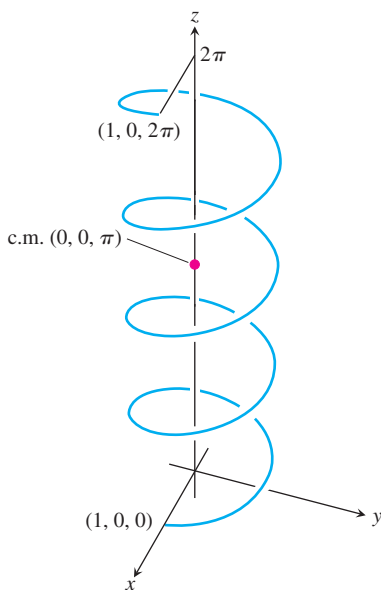


FIGURE 16.4 The helical spring in Example 3.

Notice that the radius of gyration about the z -axis is the radius of the cylinder around which the helix winds. ■

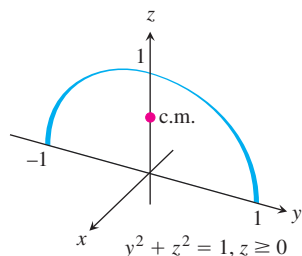


FIGURE 16.5 Example 4 shows how to find the center of mass of a circular arch of variable density.

EXAMPLE 4 Finding an Arch's Center of Mass

A slender metal arch, denser at the bottom than top, lies along the semicircle $y^2 + z^2 = 1$, $z \geq 0$, in the yz -plane (Figure 16.5). Find the center of the arch's mass if the density at the point (x, y, z) on the arch is $\delta(x, y, z) = 2 - z$.

Solution We know that $\bar{x} = 0$ and $\bar{y} = 0$ because the arch lies in the yz -plane with its mass distributed symmetrically about the z -axis. To find $\bar{z}$, we parametrize the circle as

$$\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}, \quad 0 \leq t \leq \pi.$$

For this parametrization,

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(0)^2 + (-\sin t)^2 + (\cos t)^2} = 1.$$

The formulas in Table 16.1 then give

$$M = \int_C \delta \, ds = \int_C (2 - z) \, ds = \int_0^\pi (2 - \sin t)(1) \, dt = 2\pi - 2$$

$$\begin{aligned} M_{xy} &= \int_C z\delta \, ds = \int_C z(2 - z) \, ds = \int_0^\pi (\sin t)(2 - \sin t) \, dt \\ &= \int_0^\pi (2 \sin t - \sin^2 t) \, dt = \frac{8 - \pi}{2} \end{aligned}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{8 - \pi}{2} \cdot \frac{1}{2\pi - 2} = \frac{8 - \pi}{4\pi - 4} \approx 0.57.$$

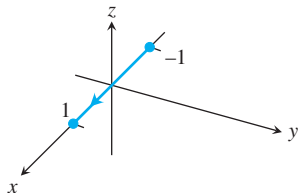
With $\bar{z}$ to the nearest hundredth, the center of mass is $(0, 0, 0.57)$. ■

EXERCISES 16.1

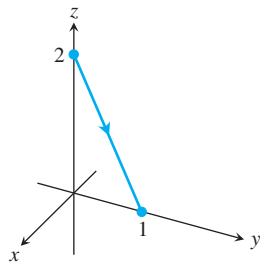
Graphs of Vector Equations

Match the vector equations in Exercises 1–8 with the graphs (a)–(h) given here.

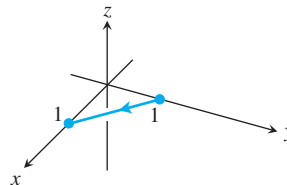
a.



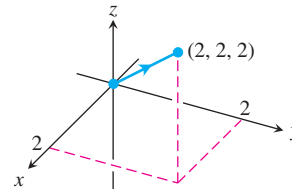
b.



c.



d.



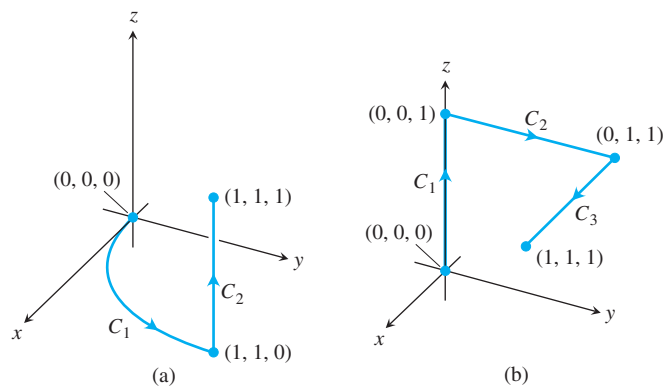
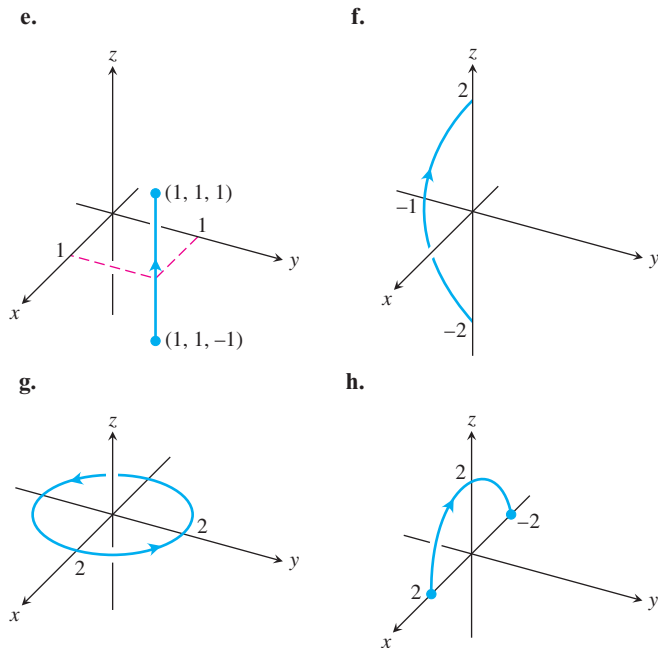


FIGURE 16.6 The paths of integration for Exercises 15 and 16.

16. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ (Figure 16.6b) given by

$$C_1: \mathbf{r}(t) = t\mathbf{k}, \quad 0 \leq t \leq 1$$

$$C_2: \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, \quad 0 \leq t \leq 1$$

$$C_3: \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad 0 \leq t \leq 1$$

17. Integrate $f(x, y, z) = (x + y + z)/(x^2 + y^2 + z^2)$ over the path $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 0 \leq t \leq b$.

18. Integrate $f(x, y, z) = -\sqrt{x^2 + z^2}$ over the circle

$$\mathbf{r}(t) = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k}, \quad 0 \leq t \leq 2\pi.$$

1. $\mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j}, \quad 0 \leq t \leq 1$
 2. $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad -1 \leq t \leq 1$
 3. $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi$
 4. $\mathbf{r}(t) = t\mathbf{i}, \quad -1 \leq t \leq 1$
 5. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 2$
 6. $\mathbf{r}(t) = t\mathbf{j} + (2 - 2t)\mathbf{k}, \quad 0 \leq t \leq 1$
 7. $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, \quad -1 \leq t \leq 1$
 8. $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{k}, \quad 0 \leq t \leq \pi$

Evaluating Line Integrals over Space Curves

9. Evaluate $\int_C (x + y) ds$ where C is the straight-line segment $x = t, y = (1 - t), z = 0$, from $(0, 1, 0)$ to $(1, 0, 0)$.
 10. Evaluate $\int_C (x - y + z - 2) ds$ where C is the straight-line segment $x = t, y = (1 - t), z = 1$, from $(0, 1, 1)$ to $(1, 0, 1)$.
 11. Evaluate $\int_C (xy + y + z) ds$ along the curve $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}, 0 \leq t \leq 1$.
 12. Evaluate $\int_C \sqrt{x^2 + y^2} ds$ along the curve $\mathbf{r}(t) = (4 \cos t)\mathbf{i} + (4 \sin t)\mathbf{j} + 3t\mathbf{k}, -2\pi \leq t \leq 2\pi$.
 13. Find the line integral of $f(x, y, z) = x + y + z$ over the straight-line segment from $(1, 2, 3)$ to $(0, -1, 1)$.
 14. Find the line integral of $f(x, y, z) = \sqrt{3}/(x^2 + y^2 + z^2)$ over the curve $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 1 \leq t \leq \infty$.
 15. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ (Figure 16.6a) given by

$$C_1: \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 1$$

$$C_2: \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 1$$

Line Integrals over Plane Curves

In Exercises 19–22, integrate f over the given curve.

19. $f(x, y) = x^3/y, \quad C: y = x^2/2, \quad 0 \leq x \leq 2$
 20. $f(x, y) = (x + y^2)/\sqrt{1 + x^2}, \quad C: y = x^2/2$ from $(1, 1/2)$ to $(0, 0)$
 21. $f(x, y) = x + y, \quad C: x^2 + y^2 = 4$ in the first quadrant from $(2, 0)$ to $(0, 2)$
 22. $f(x, y) = x^2 - y, \quad C: x^2 + y^2 = 4$ in the first quadrant from $(0, 2)$ to $(\sqrt{2}, \sqrt{2})$

Mass and Moments

23. **Mass of a wire** Find the mass of a wire that lies along the curve $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, 0 \leq t \leq 1$, if the density is $\delta = (3/2)t$.
 24. **Center of mass of a curved wire** A wire of density $\delta(x, y, z) = 15\sqrt{y} + 2$ lies along the curve $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, -1 \leq t \leq 1$. Find its center of mass. Then sketch the curve and center of mass together.
 25. **Mass of wire with variable density** Find the mass of a thin wire lying along the curve $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}, 0 \leq t \leq 1$, if the density is (a) $\delta = 3t$ and (b) $\delta = 1$.



- 26. Center of mass of wire with variable density** Find the center of mass of a thin wire lying along the curve $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + (2/3)t^{3/2}\mathbf{k}$, $0 \leq t \leq 2$, if the density is $\delta = 3\sqrt{5+t}$.
- 27. Moment of inertia and radius of gyration of wire hoop** A circular wire hoop of constant density δ lies along the circle $x^2 + y^2 = a^2$ in the xy -plane. Find the hoop's moment of inertia and radius of gyration about the z -axis.
- 28. Inertia and radii of gyration of slender rod** A slender rod of constant density lies along the line segment $\mathbf{r}(t) = t\mathbf{j} + (2 - 2t)\mathbf{k}$, $0 \leq t \leq 1$, in the yz -plane. Find the moments of inertia and radii of gyration of the rod about the three coordinate axes.
- 29. Two springs of constant density** A spring of constant density δ lies along the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 2\pi.$$

- Find I_z and R_z .
 - Suppose that you have another spring of constant density δ that is twice as long as the spring in part (a) and lies along the helix for $0 \leq t \leq 4\pi$. Do you expect I_z and R_z for the longer spring to be the same as those for the shorter one, or should they be different? Check your predictions by calculating I_z and R_z for the longer spring.
- 30. Wire of constant density** A wire of constant density $\delta = 1$ lies along the curve
- $$\mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + (2\sqrt{2}/3)t^{3/2}\mathbf{k}, \quad 0 \leq t \leq 1.$$
- Find $\bar{z}$, I_z , and R_z .
- 31. The arch in Example 4** Find I_x and R_x for the arch in Example 4.

- 32. Center of mass, moments of inertia, and radii of gyration for wire with variable density** Find the center of mass, and the moments of inertia and radii of gyration about the coordinate axes of a thin wire lying along the curve

$$\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}, \quad 0 \leq t \leq 2,$$

if the density is $\delta = 1/(t + 1)$

COMPUTER EXPLORATIONS

Evaluating Line Integrals Numerically

In Exercises 33–36, use a CAS to perform the following steps to evaluate the line integrals.

- Find $ds = |\mathbf{v}(t)| dt$ for the path $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$.
 - Express the integrand $f(g(t), h(t), k(t))|\mathbf{v}(t)|$ as a function of the parameter t .
 - Evaluate $\int_C f ds$ using Equation (2) in the text.
- 33.** $f(x, y, z) = \sqrt{1 + 30x^2 + 10y}$; $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3t^2\mathbf{k}$, $0 \leq t \leq 2$
- 34.** $f(x, y, z) = \sqrt{1 + x^3 + 5y^3}$; $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^2\mathbf{j} + \sqrt{t}\mathbf{k}$, $0 \leq t \leq 2$
- 35.** $f(x, y, z) = x\sqrt{y} - 3z^2$; $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j} + 5t\mathbf{k}$, $0 \leq t \leq 2\pi$
- 36.** $f(x, y, z) = \left(1 + \frac{9}{4}z^{1/3}\right)^{1/4}$; $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j} + t^{5/2}\mathbf{k}$, $0 \leq t \leq 2\pi$

16.2

Vector Fields, Work, Circulation, and Flux

When we study physical phenomena that are represented by vectors, we replace integrals over closed intervals by integrals over paths through vector fields. We use such integrals to find the work done in moving an object along a path against a variable force (such as a vehicle sent into space against Earth's gravitational field) or to find the work done by a vector field in moving an object along a path through the field (such as the work done by an accelerator in raising the energy of a particle). We also use line integrals to find the rates at which fluids flow along and across curves.

Vector Fields

Suppose a region in the plane or in space is occupied by a moving fluid such as air or water. Imagine that the fluid is made up of a very large number of particles, and that at any instant of time a particle has a velocity $\mathbf{v}$. If we take a picture of the velocities of some particles at

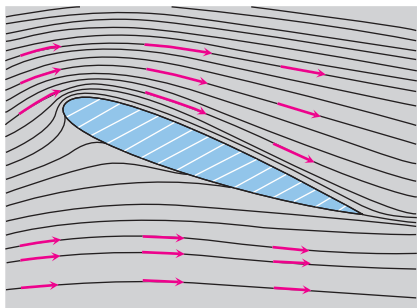


FIGURE 16.7 Velocity vectors of a flow around an airfoil in a wind tunnel. The streamlines were made visible by kerosene smoke.

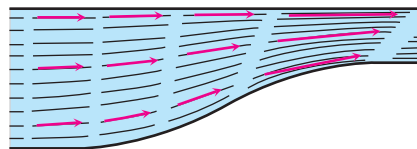


FIGURE 16.8 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

different position points at the same instant, we would expect to find that these velocities vary from position to position. We can think of a velocity vector as being attached to each point of the fluid. Such a fluid flow exemplifies a *vector field*. For example, Figure 16.7 shows a velocity vector field obtained by attaching a velocity vector to each point of air flowing around an airfoil in a wind tunnel. Figure 16.8 shows another vector field of velocity vectors along the streamlines of water moving through a contracting channel. In addition to vector fields associated with fluid flows, there are vector force fields that are associated with gravitational attraction (Figure 16.9), magnetic force fields, electric fields, and even purely mathematical fields.

Generally, a **vector field** on a domain in the plane or in space is a function that assigns a vector to each point in the domain. A field of three-dimensional vectors might have a formula like

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

The field is **continuous** if the **component functions** M , N , and P are continuous, **differentiable** if M , N , and P are differentiable, and so on. A field of two-dimensional vectors might have a formula like

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}.$$

If we attach a projectile's velocity vector to each point of the projectile's trajectory in the plane of motion, we have a two-dimensional field defined along the trajectory. If we attach the gradient vector of a scalar function to each point of a level surface of the function, we have a three-dimensional field on the surface. If we attach the velocity vector to each point of a flowing fluid, we have a three-dimensional field defined on a region in space. These and other fields are illustrated in Figures 16.10–16.15. Some of the illustrations give formulas for the fields as well.

To sketch the fields that had formulas, we picked a representative selection of domain points and sketched the vectors attached to them. The arrows representing the vectors are drawn with their tails, not their heads, at the points where the vector functions are

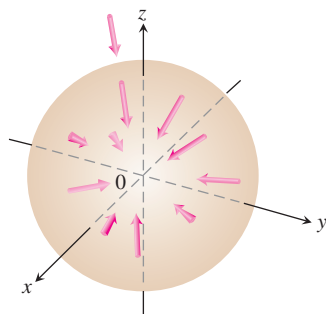


FIGURE 16.9 Vectors in a gravitational field point toward the center of mass that gives the source of the field.

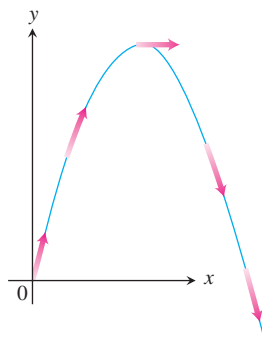


FIGURE 16.10 The velocity vectors $\mathbf{v}(t)$ of a projectile's motion make a vector field along the trajectory.

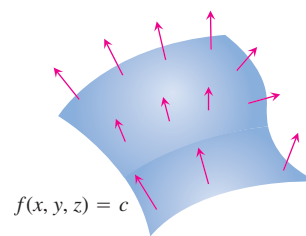


FIGURE 16.11 The field of gradient vectors ∇f on a surface $f(x, y, z) = c$.

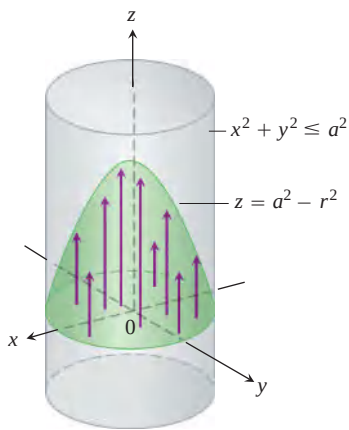


FIGURE 16.12 The flow of fluid in a long cylindrical pipe. The vectors $\mathbf{v} = (a^2 - r^2)\mathbf{k}$ inside the cylinder that have their bases in the xy -plane have their tips on the paraboloid $z = a^2 - r^2$.

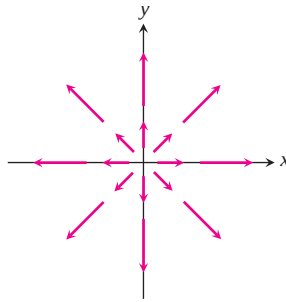


FIGURE 16.13 The radial field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ of position vectors of points in the plane. Notice the convention that an arrow is drawn with its tail, not its head, at the point where $\mathbf{F}$ is evaluated.

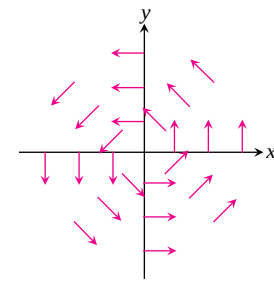


FIGURE 16.14 The circumferential or “spin” field of unit vectors $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$ in the plane. The field is not defined at the origin.

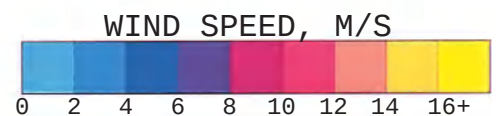
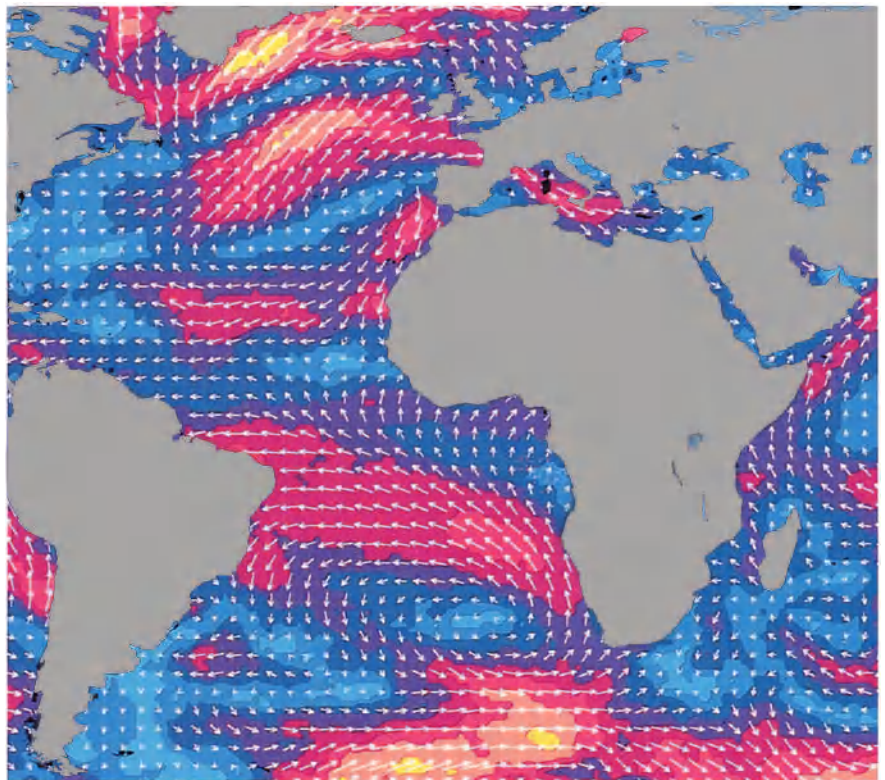


FIGURE 16.15 NASA’s *Seasat* used radar to take 350,000 wind measurements over the world’s oceans. The arrows show wind direction; their length and the color contouring indicate speed. Notice the heavy storm south of Greenland.

evaluated. This is different from the way we draw position vectors of planets and projectiles, with their tails at the origin and their heads at the planet's and projectile's locations.

Gradient Fields

DEFINITION Gradient Field

The **gradient field** of a differentiable function $f(x, y, z)$ is the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$

EXAMPLE 1 Finding a Gradient Field

Find the gradient field of $f(x, y, z) = xyz$.

Solution The gradient field of f is the field $\mathbf{F} = \nabla f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$. ■

As we will see in Section 16.3, gradient fields are of special importance in engineering, mathematics, and physics.



Work Done by a Force over a Curve in Space

Suppose that the vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ represents a force throughout a region in space (it might be the force of gravity or an electromagnetic force of some kind) and that

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \quad a \leq t \leq b,$$

is a smooth curve in the region. Then the integral of $\mathbf{F} \cdot \mathbf{T}$, the scalar component of $\mathbf{F}$ in the direction of the curve's unit tangent vector, over the curve is called the work done by $\mathbf{F}$ over the curve from a to b (Figure 16.16).

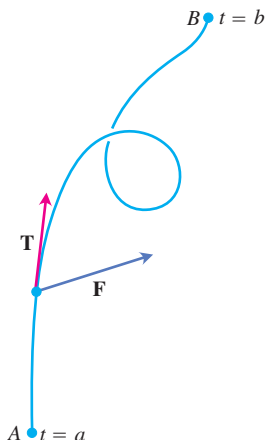


FIGURE 16.16 The work done by a force $\mathbf{F}$ is the line integral of the scalar component $\mathbf{F} \cdot \mathbf{T}$ over the smooth curve from A to B .

DEFINITION Work over a Smooth Curve

The **work** done by a force $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over a smooth curve $\mathbf{r}(t)$ from $t = a$ to $t = b$ is

$$W = \int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{T} \, ds. \quad (1)$$

We motivate Equation (1) with the same kind of reasoning we used in Chapter 6 to derive the formula $W = \int_a^b F(x) \, dx$ for the work done by a continuous force of magnitude $F(x)$ directed along an interval of the x -axis. We divide the curve into short segments, apply the (constant-force) $\times$ (distance) formula for work to approximate the work over each curved segment, add the results to approximate the work over the entire curve, and calculate

the work as the limit of the approximating sums as the segments become shorter and more numerous. To find exactly what the limiting integral should be, we partition the parameter interval $[a, b]$ in the usual way and choose a point c_k in each subinterval $[t_k, t_{k+1}]$. The partition of $[a, b]$ determines (“induces,” we say) a partition of the curve, with the point P_k being the tip of the position vector $\mathbf{r}(t_k)$ and Δs_k being the length of the curve segment $P_k P_{k+1}$ (Figure 16.17).

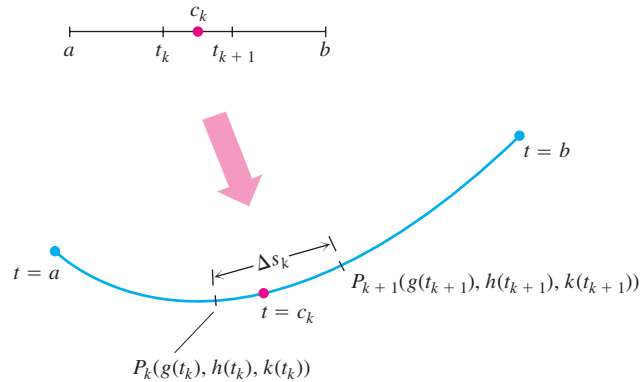


FIGURE 16.17 Each partition of $[a, b]$ induces a partition of the curve $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$.

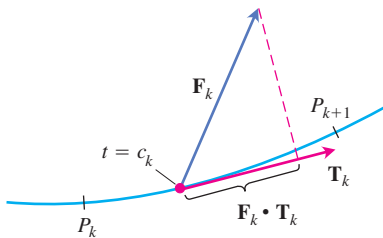


FIGURE 16.18 An enlarged view of the curve segment $P_k P_{k+1}$ in Figure 16.17, showing the force and unit tangent vectors at the point on the curve where $t = c_k$.

If $\mathbf{F}_k$ denotes the value of $\mathbf{F}$ at the point on the curve corresponding to $t = c_k$ and $\mathbf{T}_k$ denotes the curve’s unit tangent vector at this point, then $\mathbf{F}_k \cdot \mathbf{T}_k$ is the scalar component of $\mathbf{F}$ in the direction of $\mathbf{T}$ at $t = c_k$ (Figure 16.18). The work done by $\mathbf{F}$ along the curve segment $P_k P_{k+1}$ is approximately

$$\left(\begin{array}{c} \text{Force component in} \\ \text{direction of motion} \end{array} \right) \times \left(\begin{array}{c} \text{distance} \\ \text{applied} \end{array} \right) = \mathbf{F}_k \cdot \mathbf{T}_k \Delta s_k.$$

The work done by $\mathbf{F}$ along the curve from $t = a$ to $t = b$ is approximately

$$\sum_{k=1}^n \mathbf{F}_k \cdot \mathbf{T}_k \Delta s_k.$$

As the norm of the partition of $[a, b]$ approaches zero, the norm of the induced partition of the curve approaches zero and these sums approach the line integral

$$\int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{T} \, ds.$$

The sign of the number we calculate with this integral depends on the direction in which the curve is traversed as t increases. If we reverse the direction of motion, we reverse the direction of $\mathbf{T}$ and change the sign of $\mathbf{F} \cdot \mathbf{T}$ and its integral.

Table 16.2 shows six ways to write the work integral in Equation (1). Despite their variety, the formulas in Table 16.2 are all evaluated the same way. In the table, $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is a smooth curve, and

$$d\mathbf{r} = \frac{d\mathbf{r}}{dt} dt = dg\mathbf{i} + dh\mathbf{j} + dk\mathbf{k}$$

is its differential.

TABLE 16.2 Six different ways to write the work integral

$\mathbf{W} = \int_{t=a}^{t=b} \mathbf{F} \cdot \mathbf{T} \, ds$	The definition
$= \int_{t=a}^{t=b} \mathbf{F} \cdot d\mathbf{r}$	Compact differential form
$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$	Expanded to include dt ; emphasizes the parameter t and velocity vector $d\mathbf{r}/dt$
$= \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$	Emphasizes the component functions
$= \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt} \right) dt$	Abbreviates the components of $\mathbf{r}$
$= \int_a^b M \, dx + N \, dy + P \, dz$	dt 's canceled; the most common form

Evaluating a Work Integral

To evaluate the work integral along a smooth curve $\mathbf{r}(t)$, take these steps:

1. Evaluate $\mathbf{F}$ on the curve as a function of the parameter t .
2. Find $d\mathbf{r}/dt$
3. Integrate $\mathbf{F} \cdot d\mathbf{r}/dt$ from $t = a$ to $t = b$.

**EXAMPLE 2** Finding Work Done by a Variable Force over a Space Curve

Find the work done by $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ over the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \leq t \leq 1$, from $(0, 0, 0)$ to $(1, 1, 1)$ (Figure 16.19).

Solution First we evaluate $\mathbf{F}$ on the curve:

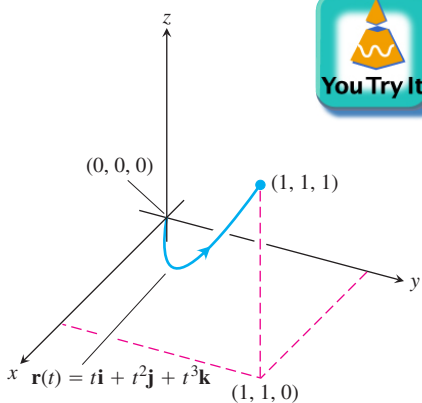
$$\begin{aligned} \mathbf{F} &= (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k} \\ &= \underbrace{(t^2 - t^2)}_0 \mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k} \end{aligned}$$

Then we find $d\mathbf{r}/dt$,

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$

Finally, we find $\mathbf{F} \cdot d\mathbf{r}/dt$ and integrate from $t = 0$ to $t = 1$:

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= [(t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}] \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) \\ &= (t^3 - t^4)(2t) + (t - t^6)(3t^2) = 2t^4 - 2t^5 + 3t^3 - 3t^8, \end{aligned}$$

**FIGURE 16.19** The curve in Example 2.

so

$$\begin{aligned}\text{Work} &= \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8) dt \\ &= \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9 \right]_0^1 = \frac{29}{60}.\end{aligned}$$

Flow Integrals and Circulation for Velocity Fields

Instead of being a force field, suppose that $\mathbf{F}$ represents the velocity field of a fluid flowing through a region in space (a tidal basin or the turbine chamber of a hydroelectric generator, for example). Under these circumstances, the integral of $\mathbf{F} \cdot \mathbf{T}$ along a curve in the region gives the fluid's flow along the curve.

DEFINITIONS Flow Integral, Circulation

If $\mathbf{r}(t)$ is a smooth curve in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $t = a$ to $t = b$ is

$$\text{Flow} = \int_a^b \mathbf{F} \cdot \mathbf{T} ds. \quad (2)$$

The integral in this case is called a **flow integral**. If the curve is a closed loop, the flow is called the **circulation** around the curve.

We evaluate flow integrals the same way we evaluate work integrals.



EXAMPLE 3 Finding Flow Along a Helix

A fluid's velocity field is $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$. Find the flow along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq \pi/2$.

Solution We evaluate $\mathbf{F}$ on the curve,

$$\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k} = (\cos t)\mathbf{i} + t\mathbf{j} + (\sin t)\mathbf{k}$$

and then find $d\mathbf{r}/dt$:

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}.$$

Then we integrate $\mathbf{F} \cdot (d\mathbf{r}/dt)$ from $t = 0$ to $t = \frac{\pi}{2}$:

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= (\cos t)(-\sin t) + (t)(\cos t) + (\sin t)(1) \\ &= -\sin t \cos t + t \cos t + \sin t\end{aligned}$$

so,

$$\begin{aligned}\text{Flow} &= \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{\pi/2} (-\sin t \cos t + t \cos t + \sin t) dt \\ &= \left[\frac{\cos^2 t}{2} + t \sin t \right]_0^{\pi/2} = \left(0 + \frac{\pi}{2} \right) - \left(\frac{1}{2} + 0 \right) = \frac{\pi}{2} - \frac{1}{2}. \quad \blacksquare\end{aligned}$$



EXAMPLE 4 Finding Circulation Around a Circle

Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$.

Solution On the circle, $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Then

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_1$$

gives

$$\begin{aligned}\text{Circulation} &= \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^{2\pi} (1 - \sin t \cos t) dt \\ &= \left[t - \frac{\sin^2 t}{2} \right]_0^{2\pi} = 2\pi. \quad \blacksquare\end{aligned}$$

Flux Across a Plane Curve

To find the rate at which a fluid is entering or leaving a region enclosed by a smooth curve C in the xy -plane, we calculate the line integral over C of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector. The value of this integral is the *flux* of $\mathbf{F}$ across C . *Flux* is Latin for *flow*, but many flux calculations involve no motion at all. If $\mathbf{F}$ were an electric field or a magnetic field, for instance, the integral of $\mathbf{F} \cdot \mathbf{n}$ would still be called the flux of the field across C .

DEFINITION Flux Across a Closed Curve in the Plane

If C is a smooth closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane and if $\mathbf{n}$ is the outward-pointing unit normal vector on C , the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n} \, ds. \quad (3)$$

Notice the difference between flux and circulation. The flux of $\mathbf{F}$ across C is the line integral with respect to arc length of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of $\mathbf{F}$ in the direction of the

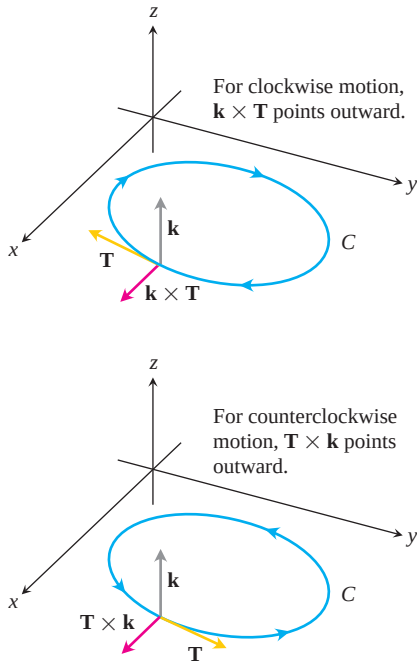


FIGURE 16.20 To find an outward unit normal vector for a smooth curve C in the xy -plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

outward normal. The circulation of $\mathbf{F}$ around C is the line integral with respect to arc length of $\mathbf{F} \cdot \mathbf{T}$, the scalar component of $\mathbf{F}$ in the direction of the unit tangent vector. Flux is the integral of the normal component of $\mathbf{F}$; circulation is the integral of the tangential component of $\mathbf{F}$.

To evaluate the integral in Equation (3), we begin with a smooth parametrization

$$x = g(t), \quad y = h(t), \quad a \leq t \leq b,$$

that traces the curve C exactly once as t increases from a to b . We can find the outward unit normal vector $\mathbf{n}$ by crossing the curve's unit tangent vector $\mathbf{T}$ with the vector $\mathbf{k}$. But which order do we choose, $\mathbf{T} \times \mathbf{k}$ or $\mathbf{k} \times \mathbf{T}$? Which one points outward? It depends on which way C is traversed as t increases. If the motion is clockwise, $\mathbf{k} \times \mathbf{T}$ points outward; if the motion is counterclockwise, $\mathbf{T} \times \mathbf{k}$ points outward (Figure 16.20). The usual choice is $\mathbf{n} = \mathbf{T} \times \mathbf{k}$, the choice that assumes counterclockwise motion. Thus, although the value of the arc length integral in the definition of flux in Equation (3) does not depend on which way C is traversed, the formulas we are about to derive for evaluating the integral in Equation (3) will assume counterclockwise motion.

In terms of components,

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \left(\frac{dx}{ds} \mathbf{i} + \frac{dy}{ds} \mathbf{j} \right) \times \mathbf{k} = \frac{dy}{ds} \mathbf{i} - \frac{dx}{ds} \mathbf{j}.$$

If $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, then

$$\mathbf{F} \cdot \mathbf{n} = M(x, y) \frac{dy}{ds} - N(x, y) \frac{dx}{ds}.$$

Hence,

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C \left(M \frac{dy}{ds} - N \frac{dx}{ds} \right) ds = \oint_C M \, dy - N \, dx.$$

We put a directed circle $\oint$ on the last integral as a reminder that the integration around the closed curve C is to be in the counterclockwise direction. To evaluate this integral, we express M , dy , N , and dx in terms of t and integrate from $t = a$ to $t = b$. We do not need to know either $\mathbf{n}$ or ds to find the flux.

Calculating Flux Across a Smooth Closed Plane Curve

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M \, dy - N \, dx \quad (4)$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \leq t \leq b$, that traces C counterclockwise exactly once.

EXAMPLE 5 Finding Flux Across a Circle

Find the flux of $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ across the circle $x^2 + y^2 = 1$ in the xy -plane.

Solution The parametrization $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$, traces the circle counterclockwise exactly once. We can therefore use this parametrization in Equation (4). With

$$\begin{aligned} M = x - y &= \cos t - \sin t, & dy &= d(\sin t) = \cos t \, dt \\ N = x &= \cos t, & dx &= d(\cos t) = -\sin t \, dt, \end{aligned}$$

We find

$$\begin{aligned} \text{Flux} &= \int_C M \, dy - N \, dx = \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \cos t \sin t) \, dt && \text{Equation (4)} \\ &= \int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt = \left[\frac{t}{2} + \frac{\sin 2t}{4} \right]_0^{2\pi} = \pi. \end{aligned}$$

The flux of $\mathbf{F}$ across the circle is π . Since the answer is positive, the net flow across the curve is outward. A net inward flow would have given a negative flux. ■

EXERCISES 16.2

Vector and Gradient Fields

Find the gradient fields of the functions in Exercises 1–4.

1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
2. $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$
3. $g(x, y, z) = e^z - \ln(x^2 + y^2)$
4. $g(x, y, z) = xy + yz + xz$
5. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the property that $\mathbf{F}$ points toward the origin with magnitude inversely proportional to the square of the distance from (x, y) to the origin. (The field is not defined at $(0, 0)$.)
6. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the properties that $\mathbf{F} = \mathbf{0}$ at $(0, 0)$ and that at any other point (a, b) , $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.

Work

In Exercises 7–12, find the work done by force $\mathbf{F}$ from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths (Figure 16.21):

- a. The straight-line path C_1 : $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 1$
 - b. The curved path C_2 : $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $0 \leq t \leq 1$
 - c. The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the segment from $(1, 1, 0)$ to $(1, 1, 1)$
7. $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$
 8. $\mathbf{F} = [1/(x^2 + 1)]\mathbf{j}$
 9. $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$
 10. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
 11. $\mathbf{F} = (3x^2 - 3x)\mathbf{i} + 3z\mathbf{j} + \mathbf{k}$
 12. $\mathbf{F} = (y + z)\mathbf{i} + (z + x)\mathbf{j} + (x + y)\mathbf{k}$

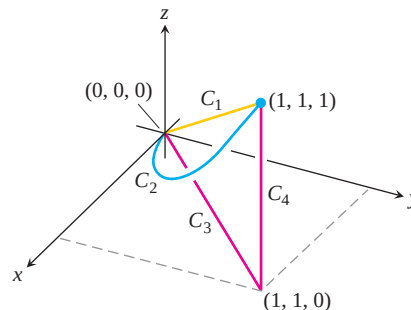


FIGURE 16.21 The paths from $(0, 0, 0)$ to $(1, 1, 1)$.

In Exercises 13–16, find the work done by $\mathbf{F}$ over the curve in the direction of increasing t .

13. $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$
 $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 1$
14. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$
 $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}$, $0 \leq t \leq 2\pi$
15. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
 $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq 2\pi$
16. $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k}$
 $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k}$, $0 \leq t \leq 2\pi$

Line Integrals and Vector Fields in the Plane

17. Evaluate $\int_C xy \, dx + (x + y) \, dy$ along the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$.
18. Evaluate $\int_C (x - y) \, dx + (x + y) \, dy$ counterclockwise around the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.



19. Evaluate $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$ for the vector field $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ along the curve $x = y^2$ from $(4, 2)$ to $(1, -1)$.
20. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ counterclockwise along the unit circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$.
21. **Work** Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$ over the straight line from $(1, 1)$ to $(2, 3)$.
22. **Work** Find the work done by the gradient of $f(x, y) = (x + y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from $(2, 0)$ to itself.
23. **Circulation and flux** Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

- a. The circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$
- b. The ellipse $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$
24. **Flux across a circle** Find the flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi.$$

Circulation and Flux

In Exercises 25–28, find the circulation and flux of the field $\mathbf{F}$ around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \leq t \leq \pi$, followed by the line segment $\mathbf{r}_2(t) = t\mathbf{i}$, $-a \leq t \leq a$.

25. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ 26. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$
27. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ 28. $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$
29. **Flow integrals** Find the flow of the velocity field $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$ along each of the following paths from $(1, 0)$ to $(-1, 0)$ in the xy -plane.
- a. The upper half of the circle $x^2 + y^2 = 1$
- b. The line segment from $(1, 0)$ to $(-1, 0)$
- c. The line segment from $(1, 0)$ to $(0, -1)$ followed by the line segment from $(0, -1)$ to $(-1, 0)$.
30. **Flux across a triangle** Find the flux of the field $\mathbf{F}$ in Exercise 29 outward across the triangle with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$.

Sketching and Finding Fields in the Plane

31. **Spin field** Draw the spin field

$$\mathbf{F} = -\frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$$

(see Figure 16.14) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 4$.

32. **Radial field** Draw the radial field

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$

(see Figure 16.13) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 1$.

33. **A field of tangent vectors**

- a. Find a field $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the xy -plane with the property that at any point $(a, b) \neq (0, 0)$, $\mathbf{G}$ is a vector of magnitude $\sqrt{a^2 + b^2}$ tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the counterclockwise direction. (The field is undefined at $(0, 0)$.)
- b. How is $\mathbf{G}$ related to the spin field $\mathbf{F}$ in Figure 16.14?

34. **A field of tangent vectors**

- a. Find a field $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the xy -plane with the property that at any point $(a, b) \neq (0, 0)$, $\mathbf{G}$ is a unit vector tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the clockwise direction.
- b. How is $\mathbf{G}$ related to the spin field $\mathbf{F}$ in Figure 16.14?

35. **Unit vectors pointing toward the origin** Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy -plane with the property that at each point $(x, y) \neq (0, 0)$, $\mathbf{F}$ is a unit vector pointing toward the origin. (The field is undefined at $(0, 0)$.)

36. **Two “central” fields** Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy -plane with the property that at each point $(x, y) \neq (0, 0)$, $\mathbf{F}$ points toward the origin and $|\mathbf{F}|$ is (a) the distance from (x, y) to the origin, (b) inversely proportional to the distance from (x, y) to the origin. (The field is undefined at $(0, 0)$.)

Flow Integrals in Space

In Exercises 37–40, $\mathbf{F}$ is the velocity field of a fluid flowing through a region in space. Find the flow along the given curve in the direction of increasing t .

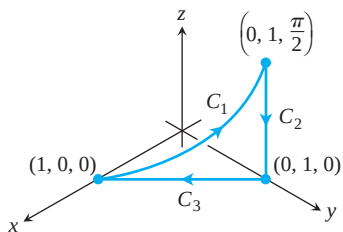
37. $\mathbf{F} = -4xy\mathbf{i} + 8y\mathbf{j} + 2\mathbf{k}$
 $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, \quad 0 \leq t \leq 2$
38. $\mathbf{F} = x^2\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k}$
 $\mathbf{r}(t) = 3t\mathbf{j} + 4t\mathbf{k}, \quad 0 \leq t \leq 1$
39. $\mathbf{F} = (x - z)\mathbf{i} + x\mathbf{k}$
 $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{k}, \quad 0 \leq t \leq \pi$
40. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 2\mathbf{k}$
 $\mathbf{r}(t) = (-2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + 2t\mathbf{k}, \quad 0 \leq t \leq 2\pi$
41. **Circulation** Find the circulation of $\mathbf{F} = 2x\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing t :

$$C_1: \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq \pi/2$$

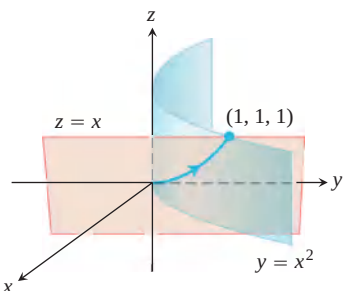
$$C_2: \mathbf{r}(t) = \mathbf{j} + (\pi/2)(1 - t)\mathbf{k}, \quad 0 \leq t \leq 1$$

$$C_3: \mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j}, \quad 0 \leq t \leq 1$$





42. **Zero circulation** Let C be the ellipse in which the plane $2x + 3y - z = 0$ meets the cylinder $x^2 + y^2 = 12$. Show, without evaluating either line integral directly, that the circulation of the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ around C in either direction is zero.
43. **Flow along a curve** The field $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ is the velocity field of a flow in space. Find the flow from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (Hint: Use $t = x$ as the parameter.)



44. **Flow of a gradient field** Find the flow of the field $\mathbf{F} = \nabla(xy^2z^3)$:
- Once around the curve C in Exercise 42, clockwise as viewed from above
 - Along the line segment from $(1, 1, 1)$ to $(2, 1, -1)$,

Theory and Examples

45. **Work and area** Suppose that $f(t)$ is differentiable and positive for $a \leq t \leq b$. Let C be the path $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}$, $a \leq t \leq b$, and $\mathbf{F} = y\mathbf{i}$. Is there any relation between the value of the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and the area of the region bounded by the t -axis, the graph of f , and the lines $t = a$ and $t = b$? Give reasons for your answer.

46. **Work done by a radial force with constant magnitude** A particle moves along the smooth curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$. The force moving the particle has constant magnitude k and always points away from the origin. Show that the work done by the force is

$$\int_C \mathbf{F} \cdot \mathbf{T} \, ds = k[(b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2}].$$

COMPUTER EXPLORATIONS

Finding Work Numerically

In Exercises 47–52, use a CAS to perform the following steps for finding the work done by force $\mathbf{F}$ over the given path:

- Find $d\mathbf{r}$ for the path $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$.
 - Evaluate the force $\mathbf{F}$ along the path.
 - Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.
47. $\mathbf{F} = xy^6\mathbf{i} + 3x(xy^5 + 2)\mathbf{j}$; $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$
48. $\mathbf{F} = \frac{3}{1+x^2}\mathbf{i} + \frac{2}{1+y^2}\mathbf{j}$; $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \leq t \leq \pi$
49. $\mathbf{F} = (y + yz \cos xyz)\mathbf{i} + (x^2 + xz \cos xyz)\mathbf{j} + (z + xy \cos xyz)\mathbf{k}$; $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 2\pi$
50. $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j} + ze^x\mathbf{k}$; $\mathbf{r}(t) = -t\mathbf{i} + \sqrt{t}\mathbf{j} + 3t\mathbf{k}$, $1 \leq t \leq 4$
51. $\mathbf{F} = (2y + \sin x)\mathbf{i} + (z^2 + (1/3)\cos y)\mathbf{j} + x^4\mathbf{k}$; $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin 2t)\mathbf{k}$, $-\pi/2 \leq t \leq \pi/2$
52. $\mathbf{F} = (x^2y)\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$; $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (2 \sin^2 t - 1)\mathbf{k}$, $0 \leq t \leq 2\pi$

16.3

Path Independence, Potential Functions, and Conservative Fields



In gravitational and electric fields, the amount of work it takes to move a mass or a charge from one point to another depends only on the object's initial and final positions and not on the path taken in between. This section discusses the notion of path independence of work integrals and describes the properties of fields in which work integrals are path independent. Work integrals are often easier to evaluate if they are path independent.

Path Independence

If A and B are two points in an open region D in space, the work $\int \mathbf{F} \cdot d\mathbf{r}$ done in moving a particle from A to B by a field $\mathbf{F}$ defined on D usually depends on the path taken. For some special fields, however, the integral's value is the same for all paths from A to B .

DEFINITIONS Path Independence, Conservative Field

Let $\mathbf{F}$ be a field defined on an open region D in space, and suppose that for any two points A and B in D the work $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ done in moving from A to B is the same over all paths from A to B . Then the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D** .

The word *conservative* comes from physics, where it refers to fields in which the principle of conservation of energy holds (it does, in conservative fields).

Under differentiability conditions normally met in practice, a field $\mathbf{F}$ is conservative if and only if it is the gradient field of a scalar function f ; that is, if and only if $\mathbf{F} = \nabla f$ for some f . The function f then has a special name.

DEFINITION Potential Function

If $\mathbf{F}$ is a field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D , then f is called a **potential function for $\mathbf{F}$** .

An electric potential is a scalar function whose gradient field is an electric field. A gravitational potential is a scalar function whose gradient field is a gravitational field, and so on. As we will see, once we have found a potential function f for a field $\mathbf{F}$, we can evaluate all the work integrals in the domain of $\mathbf{F}$ over any path between A and B by

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A). \quad (1)$$

If you think of ∇f for functions of several variables as being something like the derivative f' for functions of a single variable, then you see that Equation (1) is the vector calculus analogue of the Fundamental Theorem of Calculus formula

$$\int_a^b f'(x) dx = f(b) - f(a).$$

Conservative fields have other remarkable properties we will study as we go along. For example, saying that $\mathbf{F}$ is conservative on D is equivalent to saying that the integral of $\mathbf{F}$ around every closed path in D is zero. Naturally, certain conditions on the curves, fields, and domains must be satisfied for Equation (1) to be valid. We discuss these conditions below.

Assumptions in Effect from Now On: Connectivity and Simple Connectivity

We assume that all curves are **piecewise smooth**, that is, made up of finitely many smooth pieces connected end to end, as discussed in Section 13.1. We also assume that

the components of $\mathbf{F}$ have continuous first partial derivatives. When $\mathbf{F} = \nabla f$, this continuity requirement guarantees that the mixed second derivatives of the potential function f are equal, a result we will find revealing in studying conservative fields $\mathbf{F}$.

We assume D to be an *open* region in space. This means that every point in D is the center of an open ball that lies entirely in D . We assume D to be **connected**, which in an open region means that every point can be connected to every other point by a smooth curve that lies in the region. Finally, we assume D is **simply connected**, which means every loop in D can be contracted to a point in D without ever leaving D . (If D consisted of space with a line segment removed, for example, D would not be simply connected. There would be no way to contract a loop around the line segment to a point without leaving D .)

Connectivity and simple connectivity are not the same, and neither implies the other. Think of connected regions as being in “one piece” and simply connected regions as not having any “holes that catch loops.” All of space itself is both connected and simply connected. Some of the results in this chapter can fail to hold if applied to domains where these conditions do not hold. For example, the component test for conservative fields, given later in this section, is not valid on domains that are not simply connected.

Line Integrals in Conservative Fields

The following result provides a convenient way to evaluate a line integral in a conservative field. The result establishes that the value of the integral depends only on the endpoints and not on the specific path joining them.

THEOREM 1 The Fundamental Theorem of Line Integrals

1. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D .

2. If the integral is independent of the path from A to B , its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Proof that $\mathbf{F} = \nabla f$ Implies Path Independence of the Integral Suppose that A and B are two points in D and that $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \leq t \leq b$, is a smooth curve in D joining A and B . Along the curve, f is a differentiable function of t and

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \quad \text{Chain Rule with } x = g(t), \\ y = h(t), z = k(t)$$

$$= \nabla f \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \right) = \nabla f \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}. \quad \text{Because } \mathbf{F} = \nabla f$$

Therefore,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_a^b \frac{df}{dt} dt \\ &= f(g(t), h(t), k(t)) \Big|_a^b = f(B) - f(A).\end{aligned}$$

Thus, the value of the work integral depends only on the values of f at A and B and not on the path in between. This proves Part 2 as well as the forward implication in Part 1. We omit the more technical proof of the reverse implication. ■

EXAMPLE 1 Finding Work Done by a Conservative Field

Find the work done by the conservative field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla(xyz)$$

along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

Solution With $f(x, y, z) = xyz$, we have

$$\begin{aligned}\int_A^B \mathbf{F} \cdot d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} && \mathbf{F} = \nabla f \\ &= f(B) - f(A) && \text{Fundamental Theorem, Part 2} \\ &= xyz|_{(1,6,-4)} - xyz|_{(-1,3,9)} \\ &= (1)(6)(-4) - (-1)(3)(9) \\ &= -24 + 27 = 3.\end{aligned}$$

■

THEOREM 2 Closed-Loop Property of Conservative Fields

The following statements are equivalent.

1. $\int \mathbf{F} \cdot d\mathbf{r} = 0$ around every closed loop in D .
2. The field $\mathbf{F}$ is conservative on D .

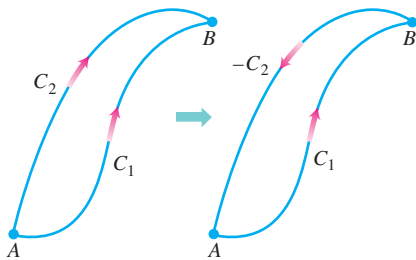


FIGURE 16.22 If we have two paths from A to B , one of them can be reversed to make a loop.

Proof that Part 1 $\Rightarrow$ Part 2 We want to show that for any two points A and B in D , the integral of $\mathbf{F} \cdot d\mathbf{r}$ has the same value over any two paths C_1 and C_2 from A to B . We reverse the direction on C_2 to make a path $-C_2$ from B to A (Figure 16.22). Together, C_1 and $-C_2$ make a closed loop C , and

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Thus, the integrals over C_1 and C_2 give the same value. Note that the definition of line integral shows that changing the direction along a curve reverses the sign of the line integral.

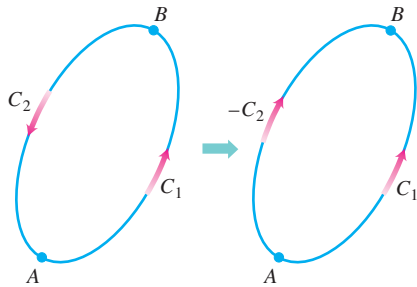


FIGURE 16.23 If A and B lie on a loop, we can reverse part of the loop to make two paths from A to B .

Proof that Part 2 $\Rightarrow$ Part 1 We want to show that the integral of $\mathbf{F} \cdot d\mathbf{r}$ is zero over any closed loop C . We pick two points A and B on C and use them to break C into two pieces: C_1 from A to B followed by C_2 from B back to A (Figure 16.23). Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r} - \int_A^B \mathbf{F} \cdot d\mathbf{r} = 0. \quad \blacksquare$$

The following diagram summarizes the results of Theorems 1 and 2.

Theorem 1	Theorem 2
$\mathbf{F} = \nabla f$ on D	$\mathbf{F}$ conservative on D
$\Leftrightarrow$	$\Leftrightarrow$
$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ over any closed path in D	

Now that we see how convenient it is to evaluate line integrals in conservative fields, two questions remain.

1. How do we know when a given field $\mathbf{F}$ is conservative?
2. If $\mathbf{F}$ is in fact conservative, how do we find a potential function f (so that $\mathbf{F} = \nabla f$)?

Finding Potentials for Conservative Fields

The test for being conservative is the following. Keep in mind our assumption that the domain of $\mathbf{F}$ is connected and simply connected.

Component Test for Conservative Fields

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \quad (2)$$

Proof that Equations (2) hold if $\mathbf{F}$ is conservative There is a potential function f such that

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Hence,

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial y \partial z} \\ &= \frac{\partial^2 f}{\partial z \partial y} \\ &= \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial N}{\partial z}. \end{aligned}$$

Continuity implies that the mixed partial derivatives are equal.

The others in Equations (2) are proved similarly. $\blacksquare$

The second half of the proof, that Equations (2) imply that $\mathbf{F}$ is conservative, is a consequence of Stokes' Theorem, taken up in Section 16.7, and requires our assumption that the domain of $\mathbf{F}$ be simply connected.

Once we know that $\mathbf{F}$ is conservative, we usually want to find a potential function for $\mathbf{F}$. This requires solving the equation $\nabla f = \mathbf{F}$ or

$$\frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

for f . We accomplish this by integrating the three equations

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = P,$$

as illustrated in the next example.



EXAMPLE 2 Finding a Potential Function

Show that $\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$ is conservative and find a potential function for it.

Solution We apply the test in Equations (2) to

$$M = e^x \cos y + yz, \quad N = xz - e^x \sin y, \quad P = xy + z$$

and calculate

$$\frac{\partial P}{\partial y} = x = \frac{\partial M}{\partial z}, \quad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y}.$$

Together, these equalities tell us that there is a function f with $\nabla f = \mathbf{F}$.

We find f by integrating the equations

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \quad \frac{\partial f}{\partial y} = xz - e^x \sin y, \quad \frac{\partial f}{\partial z} = xy + z. \quad (3)$$

We integrate the first equation with respect to x , holding y and z fixed, to get

$$f(x, y, z) = e^x \cos y + xyz + g(y, z).$$

We write the constant of integration as a function of y and z because its value may change if y and z change. We then calculate $\partial f / \partial y$ from this equation and match it with the expression for $\partial f / \partial y$ in Equations (3). This gives

$$-e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y,$$

so $\partial g / \partial y = 0$. Therefore, g is a function of z alone, and

$$f(x, y, z) = e^x \cos y + xyz + h(z).$$

We now calculate $\partial f / \partial z$ from this equation and match it to the formula for $\partial f / \partial z$ in Equations (3). This gives

$$xy + \frac{dh}{dz} = xy + z, \quad \text{or} \quad \frac{dh}{dz} = z,$$

so

$$h(z) = \frac{z^2}{2} + C.$$

Hence,

$$f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + C.$$

We have infinitely many potential functions of $\mathbf{F}$, one for each value of C . ■



EXAMPLE 3 Showing That a Field Is Not Conservative

Show that $\mathbf{F} = (2x - 3)\mathbf{i} - z\mathbf{j} + (\cos z)\mathbf{k}$ is not conservative.

Solution We apply the component test in Equations (2) and find immediately that

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(\cos z) = 0, \quad \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(-z) = -1.$$

The two are unequal, so $\mathbf{F}$ is not conservative. No further testing is required. ■

Exact Differential Forms

As we see in the next section and again later on, it is often convenient to express work and circulation integrals in the “differential” form

$$\int_A^B M dx + N dy + P dz$$

mentioned in Section 16.2. Such integrals are relatively easy to evaluate if $M dx + N dy + P dz$ is the total differential of a function f . For then

$$\begin{aligned} \int_A^B M dx + N dy + P dz &= \int_A^B \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \\ &= \int_A^B \nabla f \cdot d\mathbf{r} \\ &= f(B) - f(A). \end{aligned} \quad \text{Theorem 1}$$

Thus,

$$\int_A^B df = f(B) - f(A),$$

just as with differentiable functions of a single variable.

DEFINITIONS Exact Differential Form

Any expression $M(x, y, z) dx + N(x, y, z) dy + P(x, y, z) dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$M dx + N dy + P dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

for some scalar function f throughout D .

Notice that if $M dx + N dy + P dz = df$ on D , then $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the gradient field of f on D . Conversely, if $\mathbf{F} = \nabla f$, then the form $M dx + N dy + P dz$ is exact. The test for the form’s being exact is therefore the same as the test for $\mathbf{F}$ ’s being conservative.

Component Test for Exactness of $M dx + N dy + P dz$

The differential form $M dx + N dy + P dz$ is exact if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

**EXAMPLE 4** Showing That a Differential Form Is Exact

Show that $y dx + x dy + 4 dz$ is exact and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y dx + x dy + 4 dz$$

over the line segment from $(1, 1, 1)$ to $(2, 3, -1)$.

Solution We let $M = y$, $N = x$, $P = 4$ and apply the Test for Exactness:

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}.$$

These equalities tell us that $y dx + x dy + 4 dz$ is exact, so

$$y dx + x dy + 4 dz = df$$

for some function f , and the integral's value is $f(2, 3, -1) - f(1, 1, 1)$.

We find f up to a constant by integrating the equations

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 4. \quad (4)$$

From the first equation we get

$$f(x, y, z) = xy + g(y, z).$$

The second equation tells us that

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x, \quad \text{or} \quad \frac{\partial g}{\partial y} = 0.$$

Hence, g is a function of z alone, and

$$f(x, y, z) = xy + h(z).$$

The third of Equations (4) tells us that

$$\frac{\partial f}{\partial z} = 0 + \frac{dh}{dz} = 4, \quad \text{or} \quad h(z) = 4z + C.$$

Therefore,

$$f(x, y, z) = xy + 4z + C.$$

The value of the integral is

$$f(2, 3, -1) - f(1, 1, 1) = 2 + C - (5 + C) = -3. \quad \blacksquare$$

EXERCISES 16.3

Testing for Conservative Fields

Which fields in Exercises 1–6 are conservative, and which are not?

1. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
2. $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$
3. $\mathbf{F} = y\mathbf{i} + (x + z)\mathbf{j} - y\mathbf{k}$
4. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$
5. $\mathbf{F} = (z + y)\mathbf{i} + z\mathbf{j} + (y + x)\mathbf{k}$
6. $\mathbf{F} = (e^x \cos y)\mathbf{i} - (e^x \sin y)\mathbf{j} + z\mathbf{k}$

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$
8. $\mathbf{F} = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$
9. $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$
10. $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$
11. $\mathbf{F} = (\ln x + \sec^2(x + y))\mathbf{i} +$
 $\left(\sec^2(x + y) + \frac{y}{y^2 + z^2}\right)\mathbf{j} + \frac{z}{y^2 + z^2}\mathbf{k}$
12. $\mathbf{F} = \frac{y}{1 + x^2 y^2}\mathbf{i} + \left(\frac{x}{1 + x^2 y^2} + \frac{z}{\sqrt{1 - y^2 z^2}}\right)\mathbf{j} +$
 $\left(\frac{y}{\sqrt{1 - y^2 z^2}} + \frac{1}{z}\right)\mathbf{k}$

Evaluating Line Integrals

In Exercises 13–17, show that the differential forms in the integrals are exact. Then evaluate the integrals.

13. $\int_{(0,0,0)}^{(2,3,-6)} 2x \, dx + 2y \, dy + 2z \, dz$
14. $\int_{(1,1,2)}^{(3,5,0)} yz \, dx + xz \, dy + xy \, dz$
15. $\int_{(0,0,0)}^{(1,2,3)} 2xy \, dx + (x^2 - z^2) \, dy - 2yz \, dz$
16. $\int_{(0,0,0)}^{(3,3,1)} 2x \, dx - y^2 \, dy - \frac{4}{1 + z^2} \, dz$
17. $\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x \, dx + \cos y \sin x \, dy + dz$

Although they are not defined on all of space R^3 , the fields associated with Exercises 18–22 are simply connected and the Component Test can be used to show they are conservative. Find a potential function for each field and evaluate the integrals as in Example 4.

18. $\int_{(0,2,1)}^{(1,\pi/2,2)} 2 \cos y \, dx + \left(\frac{1}{y} - 2x \sin y\right) \, dy + \frac{1}{z} \, dz$

19. $\int_{(1,1,1)}^{(1,2,3)} 3x^2 \, dx + \frac{z^2}{y} \, dy + 2z \ln y \, dz$
20. $\int_{(1,2,1)}^{(2,1,1)} (2x \ln y - yz) \, dx + \left(\frac{x^2}{y} - xz\right) \, dy - xy \, dz$
21. $\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y} \, dx + \left(\frac{1}{z} - \frac{x}{y^2}\right) \, dy - \frac{y}{z^2} \, dz$
22. $\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x \, dx + 2y \, dy + 2z \, dz}{x^2 + y^2 + z^2}$
23. **Revisiting Example 4** Evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y \, dx + x \, dy + 4 \, dz$$

from Example 4 by finding parametric equations for the line segment from $(1, 1, 1)$ to $(2, 3, -1)$ and evaluating the line integral of $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + 4\mathbf{k}$ along the segment. Since $\mathbf{F}$ is conservative, the integral is independent of the path.

24. Evaluate

$$\int_C x^2 \, dx + yz \, dy + (y^2/2) \, dz$$

along the line segment C joining $(0, 0, 0)$ to $(0, 3, 4)$.

Theory, Applications, and Examples

Independence of path Show that the values of the integrals in Exercises 25 and 26 do not depend on the path taken from A to B .

25. $\int_A^B z^2 \, dx + 2y \, dy + 2xz \, dz$
26. $\int_A^B \frac{x \, dx + y \, dy + z \, dz}{\sqrt{x^2 + y^2 + z^2}}$

In Exercises 27 and 28, find a potential function for $\mathbf{F}$.

27. $\mathbf{F} = \frac{2x}{y}\mathbf{i} + \left(\frac{1 - x^2}{y^2}\right)\mathbf{j}$
28. $\mathbf{F} = (e^x \ln y)\mathbf{i} + \left(\frac{e^x}{y} + \sin z\right)\mathbf{j} + (y \cos z)\mathbf{k}$
29. **Work along different paths** Find the work done by $\mathbf{F} = (x^2 + y)\mathbf{i} + (y^2 + x)\mathbf{j} + ze^z\mathbf{k}$ over the following paths from $(1, 0, 0)$ to $(1, 0, 1)$.
 - a. The line segment $x = 1, y = 0, 0 \leq z \leq 1$
 - b. The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/2\pi)\mathbf{k}, 0 \leq t \leq 2\pi$
 - c. The x -axis from $(1, 0, 0)$ to $(0, 0, 0)$ followed by the parabola $z = x^2, y = 0$ from $(0, 0, 0)$ to $(1, 0, 1)$
30. **Work along different paths** Find the work done by $\mathbf{F} = e^{yz}\mathbf{i} + (xze^{yz} + z \cos y)\mathbf{j} + (xye^{yz} + \sin y)\mathbf{k}$ over the following paths from $(1, 0, 1)$ to $(1, \pi/2, 0)$.



- a. The line segment $x = 1, y = \pi t/2, z = 1 - t, 0 \leq t \leq 1$
- b. The line segment from $(1, 0, 1)$ to the origin followed by the line segment from the origin to $(1, \pi/2, 0)$
- c. The line segment from $(1, 0, 1)$ to $(1, 0, 0)$, followed by the x -axis from $(1, 0, 0)$ to the origin, followed by the parabola $y = \pi x^2/2, z = 0$ from there to $(1, \pi/2, 0)$
- 31. Evaluating a work integral two ways** Let $\mathbf{F} = \nabla(x^3y^2)$ and let C be the path in the xy -plane from $(-1, 1)$ to $(1, 1)$ that consists of the line segment from $(-1, 1)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(1, 1)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.
- a. Find parametrizations for the segments that make up C and evaluate the integral.
- b. Using $f(x, y) = x^3y^2$ as a potential function for $\mathbf{F}$.
- 32. Integral along different paths** Evaluate $\int_C 2x \cos y \, dx - x^2 \sin y \, dy$ along the following paths C in the xy -plane.
- a. The parabola $y = (x - 1)^2$ from $(1, 0)$ to $(0, 1)$
- b. The line segment from $(-1, \pi)$ to $(1, 0)$
- c. The x -axis from $(-1, 0)$ to $(1, 0)$
- d. The astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, 0 \leq t \leq 2\pi$, counterclockwise from $(1, 0)$ back to $(1, 0)$
- 33. a. Exact differential form** How are the constants a, b , and c related if the following differential form is exact?
- $$(ay^2 + 2czx) \, dx + y(bx + cz) \, dy + (ay^2 + cx^2) \, dz$$
- b. **Gradient field** For what values of b and c will
- $$\mathbf{F} = (y^2 + 2czx)\mathbf{i} + y(bx + cz)\mathbf{j} + (y^2 + cx^2)\mathbf{k}$$
- be a gradient field?

- 34. Gradient of a line integral** Suppose that $\mathbf{F} = \nabla f$ is a conservative vector field and

$$g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r}.$$

Show that $\nabla g = \mathbf{F}$.

- 35. Path of least work** You have been asked to find the path along which a force field $\mathbf{F}$ will perform the least work in moving a particle between two locations. A quick calculation on your part shows $\mathbf{F}$ to be conservative. How should you respond? Give reasons for your answer.
- 36. A revealing experiment** By experiment, you find that a force field $\mathbf{F}$ performs only half as much work in moving an object along path C_1 from A to B as it does in moving the object along path C_2 from A to B . What can you conclude about $\mathbf{F}$? Give reasons for your answer.
- 37. Work by a constant force** Show that the work done by a constant force field $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in moving a particle along any path from A to B is $W = \mathbf{F} \cdot \overrightarrow{AB}$.
- 38. Gravitational field**
- a. Find a potential function for the gravitational field
- $$\mathbf{F} = -GmM \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} \quad (G, m, \text{ and } M \text{ are constants}).$$
- b. Let P_1 and P_2 be points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is

$$GmM \left(\frac{1}{s_2} - \frac{1}{s_1} \right).$$

16.4

Green's Theorem in the Plane



From Table 16.2 in Section 16.2, we know that every line integral $\int_C M dx + N dy$ can be written as a flow integral $\int_a^b \mathbf{F} \cdot \mathbf{T} ds$. If the integral is independent of path, so the field $\mathbf{F}$ is conservative (over a domain satisfying the basic assumptions), we can evaluate the integral easily from a potential function for the field. In this section we consider how to evaluate the integral if it is *not* associated with a conservative vector field, but is a flow or flux integral across a closed curve in the xy -plane. The means for doing so is a result known as Green's Theorem, which converts the line integral into a double integral over the region enclosed by the path.

We frame our discussion in terms of velocity fields of fluid flows because they are easy to picture. However, Green's Theorem applies to any vector field satisfying certain mathematical conditions. It does not depend for its validity on the field's having a particular physical interpretation.

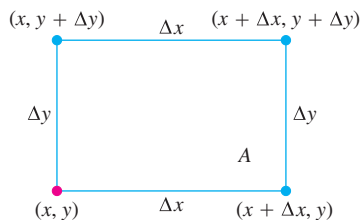


FIGURE 16.24 The rectangle for defining the divergence (flux density) of a vector field at a point (x, y) .

Divergence

We need two new ideas for Green's Theorem. The first is the idea of the *divergence* of a vector field at a point, sometimes called the *flux density* of the vector field by physicists and engineers. We obtain it in the following way.

Suppose that $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is the velocity field of a fluid flow in the plane and that the first partial derivatives of M and N are continuous at each point of a region R . Let (x, y) be a point in R and let A be a small rectangle with one corner at (x, y) that, along with its interior, lies entirely in R (Figure 16.24). The sides of the rectangle, parallel to the coordinate axes, have lengths of Δx and Δy . The rate at which fluid leaves the rectangle across the bottom edge is approximately

$$\mathbf{F}(x, y) \cdot (-\mathbf{j}) \Delta x = -N(x, y) \Delta x.$$

This is the scalar component of the velocity at (x, y) in the direction of the outward normal times the length of the segment. If the velocity is in meters per second, for example, the exit rate will be in meters per second times meters or square meters per second. The rates at which the fluid crosses the other three sides in the directions of their outward normals can be estimated in a similar way. All told, we have

Exit Rates:	Top:	$\mathbf{F}(x, y + \Delta y) \cdot \mathbf{j} \Delta x = N(x, y + \Delta y) \Delta x$
	Bottom:	$\mathbf{F}(x, y) \cdot (-\mathbf{j}) \Delta x = -N(x, y) \Delta x$
	Right:	$\mathbf{F}(x + \Delta x, y) \cdot \mathbf{i} \Delta y = M(x + \Delta x, y) \Delta y$
	Left:	$\mathbf{F}(x, y) \cdot (-\mathbf{i}) \Delta y = -M(x, y) \Delta y.$

Combining opposite pairs gives

$$\begin{aligned} \text{Top and bottom:} \quad & (N(x, y + \Delta y) - N(x, y)) \Delta x \approx \left(\frac{\partial N}{\partial y} \Delta y \right) \Delta x \\ \text{Right and left:} \quad & (M(x + \Delta x, y) - M(x, y)) \Delta y \approx \left(\frac{\partial M}{\partial x} \Delta x \right) \Delta y. \end{aligned}$$

Adding these last two equations gives

$$\text{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) \Delta x \Delta y.$$

We now divide by $\Delta x \Delta y$ to estimate the total flux per unit area or flux density for the rectangle:

$$\frac{\text{Flux across rectangle boundary}}{\text{rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right).$$

Finally, we let Δx and Δy approach zero to define what we call the *flux density* of $\mathbf{F}$ at the point (x, y) . In mathematics, we call the flux density the *divergence* of $\mathbf{F}$. The symbol for it is $\text{div } \mathbf{F}$, pronounced “divergence of $\mathbf{F}$ ” or “ $\text{div } \mathbf{F}$.”

DEFINITION Divergence (Flux Density)

The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \quad (1)$$

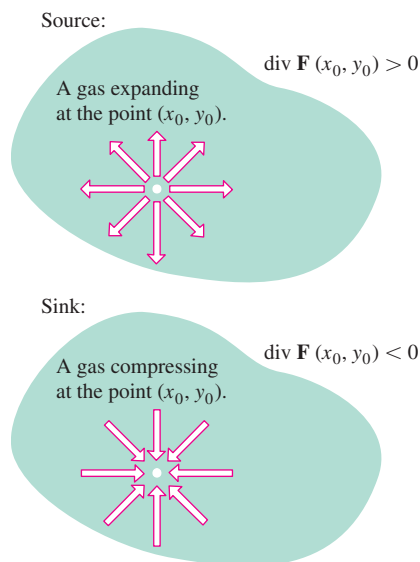


FIGURE 16.25 If a gas is expanding at a point (x_0, y_0) , the lines of flow have positive divergence; if the gas is compressing, the divergence is negative.

Intuitively, if a gas is expanding at the point (x_0, y_0) , the lines of flow would diverge there (hence the name) and, since the gas would be flowing out of a small rectangle about (x_0, y_0) the divergence of $\mathbf{F}$ at (x_0, y_0) would be positive. If the gas were compressing instead of expanding, the divergence would be negative (see Figure 16.25).

EXAMPLE 1 Finding Divergence

Find the divergence of $\mathbf{F}(x, y) = (x^2 - y)\mathbf{i} + (xy - y^2)\mathbf{j}$.

Solution We use the formula in Equation (1):

$$\begin{aligned}\operatorname{div} \mathbf{F} &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{\partial}{\partial x}(x^2 - y) + \frac{\partial}{\partial y}(xy - y^2) \\ &= 2x + x - 2y = 3x - 2y.\end{aligned}$$

Spin Around an Axis: The k-Component of Curl

The second idea we need for Green's Theorem has to do with measuring how a paddle wheel spins at a point in a fluid flowing in a plane region. This idea gives some sense of how the fluid is circulating around axes located at different points and perpendicular to the region. Physicists sometimes refer to this as the *circulation density* of a vector field $\mathbf{F}$ at a point. To obtain it, we return to the velocity field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

and the rectangle A . The rectangle is redrawn here as Figure 16.26.

The counterclockwise circulation of $\mathbf{F}$ around the boundary of A is the sum of flow rates along the sides. For the bottom edge, the flow rate is approximately

$$\mathbf{F}(x, y) \cdot \mathbf{i} \Delta x = M(x, y)\Delta x.$$

This is the scalar component of the velocity $\mathbf{F}(x, y)$ in the direction of the tangent vector $\mathbf{i}$ times the length of the segment. The rates of flow along the other sides in the counterclockwise direction are expressed in a similar way. In all, we have

$$\text{Top:} \quad \mathbf{F}(x, y + \Delta y) \cdot (-\mathbf{i}) \Delta x = -M(x, y + \Delta y)\Delta x$$

$$\text{Bottom:} \quad \mathbf{F}(x, y) \cdot \mathbf{i} \Delta x = M(x, y)\Delta x$$

$$\text{Right:} \quad \mathbf{F}(x + \Delta x, y) \cdot \mathbf{j} \Delta y = N(x + \Delta x, y)\Delta y$$

$$\text{Left:} \quad \mathbf{F}(x, y) \cdot (-\mathbf{j}) \Delta y = -N(x, y)\Delta y.$$

We add opposite pairs to get

Top and bottom:

$$-(M(x, y + \Delta y) - M(x, y))\Delta x \approx -\left(\frac{\partial M}{\partial y} \Delta y\right)\Delta x$$

Right and left:

$$(N(x + \Delta x, y) - N(x, y))\Delta y \approx \left(\frac{\partial N}{\partial x} \Delta x\right)\Delta y.$$

Adding these last two equations and dividing by $\Delta x \Delta y$ gives an estimate of the circulation density for the rectangle:

$$\frac{\text{Circulation around rectangle}}{\text{rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

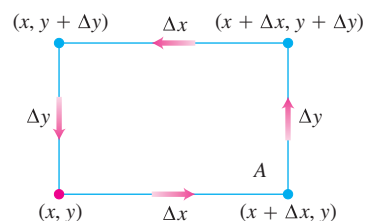


FIGURE 16.26 The rectangle for defining the curl (circulation density) of a vector field at a point (x, y) .

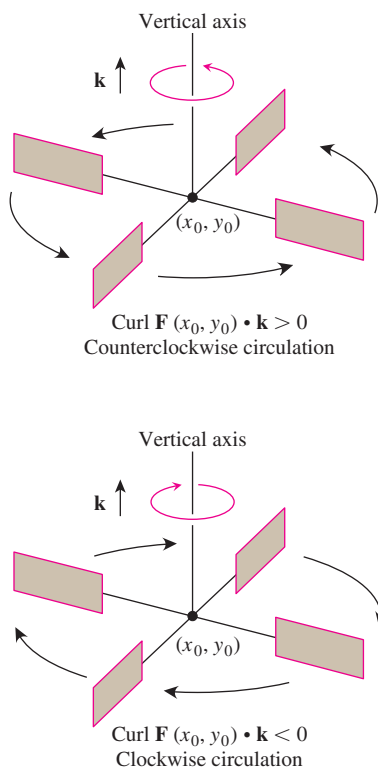


FIGURE 16.27 In the flow of an incompressible fluid over a plane region, the $\mathbf{k}$ -component of the curl measures the rate of the fluid's rotation at a point. The $\mathbf{k}$ -component of the curl is positive at points where the rotation is counterclockwise and negative where the rotation is clockwise.

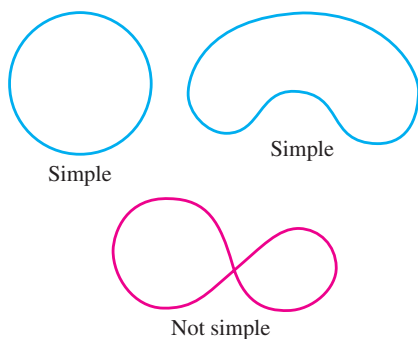


FIGURE 16.28 In proving Green's Theorem, we distinguish between two kinds of closed curves, simple and not simple. Simple curves do not cross themselves. A circle is simple but a figure 8 is not.

We let Δx and Δy approach zero to define what we call the *circulation density* of $\mathbf{F}$ at the point (x, y) .

The positive orientation of the circulation density for the plane is the *counterclockwise* rotation around the vertical axis, looking downward on the xy -plane from the tip of the (vertical) unit vector $\mathbf{k}$ (Figure 16.27). The circulation value is actually the $\mathbf{k}$ -component of a more general circulation vector we define in Section 16.7, called the *curl* of the vector field $\mathbf{F}$. For Green's Theorem, we need only this $\mathbf{k}$ -component.

DEFINITION $\mathbf{k}$ -Component of Curl (Circulation Density)

The $\mathbf{k}$ -component of the curl (circulation density) of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \quad (2)$$

If water is moving about a region in the xy -plane in a thin layer, then the $\mathbf{k}$ -component of the circulation, or curl, at a point (x_0, y_0) gives a way to measure how fast and in what direction a small paddle wheel will spin if it is put into the water at (x_0, y_0) with its axis perpendicular to the plane, parallel to $\mathbf{k}$ (Figure 16.27).

EXAMPLE 2 Finding the $\mathbf{k}$ -Component of the Curl

Find the $\mathbf{k}$ -component of the curl for the vector field

$$\mathbf{F}(x, y) = (x^2 - y)\mathbf{i} + (xy - y^2)\mathbf{j}.$$

Solution We use the formula in Equation (2):

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial}{\partial x}(xy - y^2) - \frac{\partial}{\partial y}(x^2 - y) = y + 1. \quad \blacksquare$$

Two Forms for Green's Theorem

In one form, Green's Theorem says that under suitable conditions the outward flux of a vector field across a simple closed curve in the plane (Figure 16.28) equals the double integral of the divergence of the field over the region enclosed by the curve. Recall the formulas for flux in Equations (3) and (4) in Section 16.2.

THEOREM 3 Green's Theorem (Flux-Divergence or Normal Form)

The outward flux of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ across a simple closed curve C equals the double integral of $\text{div } \mathbf{F}$ over the region R enclosed by C .

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C M \, dy - N \, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy \quad (3)$$

Outward flux Divergence integral

In another form, Green's Theorem says that the counterclockwise circulation of a vector field around a simple closed curve is the double integral of the $\mathbf{k}$ -component of the curl of the field over the region enclosed by the curve. Recall the defining Equation (2) for circulation in Section 16.2.



THEOREM 4 Green's Theorem (Circulation-Curl or Tangential Form)

The counterclockwise circulation of a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around a simple closed curve C in the plane equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over the region R enclosed by C .

$$\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \oint_C M \, dx + N \, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy \quad (4)$$

Counterclockwise circulation Curl integral

The two forms of Green's Theorem are equivalent. Applying Equation (3) to the field $\mathbf{G}_1 = N\mathbf{i} - M\mathbf{j}$ gives Equation (4), and applying Equation (4) to $\mathbf{G}_2 = -N\mathbf{i} + M\mathbf{j}$ gives Equation (3).

Mathematical Assumptions

We need two kinds of assumptions for Green's Theorem to hold. First, we need conditions on M and N to ensure the existence of the integrals. The usual assumptions are that M , N , and their first partial derivatives are continuous at every point of some open region containing C and R . Second, we need geometric conditions on the curve C . It must be simple, closed, and made up of pieces along which we can integrate M and N . The usual assumptions are that C is piecewise smooth. The proof we give for Green's Theorem, however, assumes things about the shape of R as well. You can find proofs that are less restrictive in more advanced texts. First let's look at examples.



EXAMPLE 3 Supporting Green's Theorem

Verify both forms of Green's Theorem for the field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region R bounded by the unit circle

$$C: \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi.$$

Solution We have

$$M = \cos t - \sin t, \quad dx = d(\cos t) = -\sin t \, dt,$$

$$N = \cos t, \quad dy = d(\sin t) = \cos t \, dt,$$

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0.$$

The two sides of Equation (3) are

$$\begin{aligned}\oint_C M dy - N dx &= \int_{t=0}^{t=2\pi} (\cos t - \sin t)(\cos t dt) - (\cos t)(-\sin t dt) \\ &= \int_0^{2\pi} \cos^2 t dt = \pi \\ \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy &= \iint_R (1 + 0) dx dy \\ &= \iint_R dx dy = \text{area inside the unit circle} = \pi.\end{aligned}$$

The two sides of Equation (4) are

$$\begin{aligned}\oint_C M dx + N dy &= \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t dt) + (\cos t)(\cos t dt) \\ &= \int_0^{2\pi} (-\sin t \cos t + 1) dt = 2\pi \\ \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \iint_R (1 - (-1)) dx dy = 2 \iint_R dx dy = 2\pi. \quad \blacksquare\end{aligned}$$

Using Green's Theorem to Evaluate Line Integrals

If we construct a closed curve C by piecing a number of different curves end to end, the process of evaluating a line integral over C can be lengthy because there are so many different integrals to evaluate. If C bounds a region R to which Green's Theorem applies, however, we can use Green's Theorem to change the line integral around C into one double integral over R .



EXAMPLE 4 Evaluating a Line Integral Using Green's Theorem

Evaluate the integral

$$\oint_C xy dy - y^2 dx,$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Solution We can use either form of Green's Theorem to change the line integral into a double integral over the square.

1. *With the Normal Form Equation (3):* Taking $M = xy$, $N = y^2$, and C and R as the square's boundary and interior gives

$$\begin{aligned}\oint_C xy dy - y^2 dx &= \iint_R (y + 2y) dx dy = \int_0^1 \int_0^1 3y dx dy \\ &= \int_0^1 \left[3xy \right]_{x=0}^{x=1} dy = \int_0^1 3y dy = \left[\frac{3}{2} y^2 \right]_0^1 = \frac{3}{2}.\end{aligned}$$

2. *With the Tangential Form Equation (4):* Taking $M = -y^2$ and $N = xy$ gives the same result:

$$\oint_C -y^2 dx + xy dy = \iint_R (y - (-2y)) dx dy = \frac{3}{2}. \quad \blacksquare$$



EXAMPLE 5 Finding Outward Flux

Calculate the outward flux of the field $\mathbf{F}(x, y) = x\mathbf{i} + y^2\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution Calculating the flux with a line integral would take four integrations, one for each side of the square. With Green's Theorem, we can change the line integral to one double integral. With $M = x$, $N = y^2$, C the square, and R the square's interior, we have

$$\begin{aligned} \text{Flux} &= \oint_C \mathbf{F} \cdot \mathbf{n} ds = \oint_C M dy - N dx \\ &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy && \text{Green's Theorem} \\ &= \int_{-1}^1 \int_{-1}^1 (1 + 2y) dx dy = \int_{-1}^1 \left[x + 2xy \right]_{x=-1}^{x=1} dy \\ &= \int_{-1}^1 (2 + 4y) dy = \left[2y + 2y^2 \right]_{-1}^1 = 4. \quad \blacksquare \end{aligned}$$

Proof of Green's Theorem for Special Regions

Let C be a smooth simple closed curve in the xy -plane with the property that lines parallel to the axes cut it in no more than two points. Let R be the region enclosed by C and suppose that M , N , and their first partial derivatives are continuous at every point of some open region containing C and R . We want to prove the circulation-curl form of Green's Theorem,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy. \quad (5)$$

Figure 16.29 shows C made up of two directed parts:

$$C_1: y = f_1(x), \quad a \leq x \leq b, \quad C_2: y = f_2(x), \quad b \geq x \geq a.$$

For any x between a and b , we can integrate $\partial M / \partial y$ with respect to y from $y = f_1(x)$ to $y = f_2(x)$ and obtain

$$\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy = M(x, y) \Big|_{y=f_1(x)}^{y=f_2(x)} = M(x, f_2(x)) - M(x, f_1(x)).$$

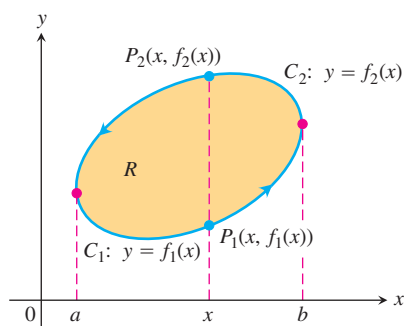


FIGURE 16.29 The boundary curve C is made up of C_1 , the graph of $y = f_1(x)$, and C_2 , the graph of $y = f_2(x)$.

We can then integrate this with respect to x from a to b :

$$\begin{aligned}
 \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy dx &= \int_a^b [M(x, f_2(x)) - M(x, f_1(x))] dx \\
 &= - \int_b^a M(x, f_2(x)) dx - \int_a^b M(x, f_1(x)) dx \\
 &= - \int_{C_2} M dx - \int_{C_1} M dx \\
 &= - \oint_C M dx.
 \end{aligned}$$

Therefore

$$\oint_C M dx = \iint_R \left(-\frac{\partial M}{\partial y} \right) dx dy. \quad (6)$$

Equation (6) is half the result we need for Equation (5). We derive the other half by integrating $\partial N/\partial x$ first with respect to x and then with respect to y , as suggested by Figure 16.30. This shows the curve C of Figure 16.29 decomposed into the two directed parts $C'_1: x = g_1(y)$, $d \geq y \geq c$ and $C'_2: x = g_2(y)$, $c \leq y \leq d$. The result of this double integration is

$$\oint_C N dy = \iint_R \frac{\partial N}{\partial x} dx dy. \quad (7)$$

Summing Equations (6) and (7) gives Equation (5). This concludes the proof. ■

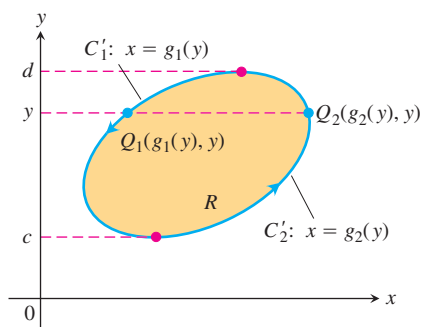


FIGURE 16.30 The boundary curve C is made up of C'_1 , the graph of $x = g_1(y)$, and C'_2 , the graph of $x = g_2(y)$.

Extending the Proof to Other Regions

The argument we just gave does not apply directly to the rectangular region in Figure 16.31 because the lines $x = a$, $x = b$, $y = c$, and $y = d$ meet the region's boundary in more than two points. If we divide the boundary C into four directed line segments, however,

$$\begin{aligned}
 C_1: y = c, \quad a \leq x \leq b, & \quad C_2: x = b, \quad c \leq y \leq d \\
 C_3: y = d, \quad b \geq x \geq a, & \quad C_4: x = a, \quad d \geq y \geq c,
 \end{aligned}$$

we can modify the argument in the following way.

Proceeding as in the proof of Equation (7), we have

$$\begin{aligned}
 \int_c^d \int_a^b \frac{\partial N}{\partial x} dx dy &= \int_c^d (N(b, y) - N(a, y)) dy \\
 &= \int_c^d N(b, y) dy + \int_d^c N(a, y) dy \\
 &= \int_{C_2} N dy + \int_{C_4} N dy.
 \end{aligned} \quad (8)$$

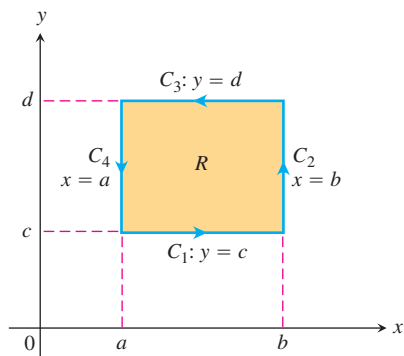


FIGURE 16.31 To prove Green's Theorem for a rectangle, we divide the boundary into four directed line segments.

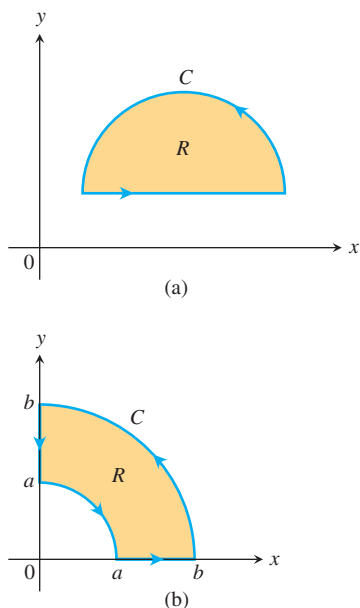


FIGURE 16.32 Other regions to which Green's Theorem applies.

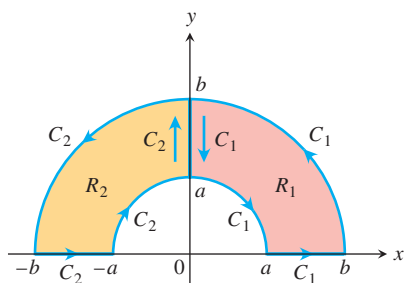


FIGURE 16.33 A region R that combines regions R_1 and R_2 .

Because y is constant along C_1 and C_3 , $\int_{C_1} N dy = \int_{C_3} N dy = 0$, so we can add $\int_{C_1} N dy = \int_{C_3} N dy$ to the right-hand side of Equation (8) without changing the equality. Doing so, we have

$$\int_c^d \int_a^b \frac{\partial N}{\partial x} dx dy = \oint_C N dy. \quad (9)$$

Similarly, we can show that

$$\int_a^b \int_c^d \frac{\partial M}{\partial y} dy dx = - \oint_C M dx. \quad (10)$$

Subtracting Equation (10) from Equation (9), we again arrive at

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

Regions like those in Figure 16.32 can be handled with no greater difficulty. Equation (5) still applies. It also applies to the horseshoe-shaped region R shown in Figure 16.33, as we see by putting together the regions R_1 and R_2 and their boundaries. Green's Theorem applies to C_1, R_1 and to C_2, R_2 , yielding

$$\int_{C_1} M dx + N dy = \iint_{R_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\int_{C_2} M dx + N dy = \iint_{R_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy.$$

When we add these two equations, the line integral along the y -axis from b to a for C_1 cancels the integral over the same segment but in the opposite direction for C_2 . Hence,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy,$$

where C consists of the two segments of the x -axis from $-b$ to $-a$ and from a to b and of the two semicircles, and where R is the region inside C .

The device of adding line integrals over separate boundaries to build up an integral over a single boundary can be extended to any finite number of subregions. In Figure 16.34a let C_1 be the boundary, oriented counterclockwise, of the region R_1 in the first quadrant. Similarly, for the other three quadrants, C_i is the boundary of the region R_i , $i = 2, 3, 4$. By Green's Theorem,

$$\oint_{C_i} M dx + N dy = \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy. \quad (11)$$

We sum Equation (11) over $i = 1, 2, 3, 4$, and get (Figure 16.34b):

$$\oint_{r=b} (M dx + N dy) + \oint_{r=a} (M dx + N dy) = \iint_{\bigcup R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy. \quad (12)$$

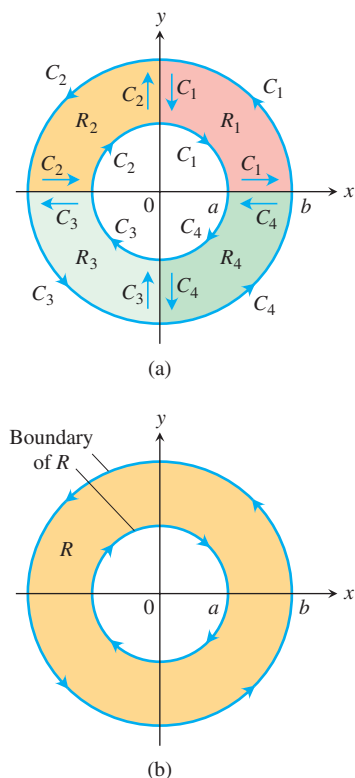


FIGURE 16.34 The annular region R combines four smaller regions. In polar coordinates, $r = a$ for the inner circle, $r = b$ for the outer circle, and $a \leq r \leq b$ for the region itself.

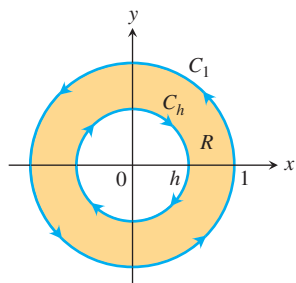


FIGURE 16.35 Green's Theorem may be applied to the annular region R by integrating along the boundaries as shown (Example 6).

Equation (12) says that the double integral of $(\partial N/\partial x) - (\partial M/\partial y)$ over the annular ring R equals the line integral of $M dx + N dy$ over the complete boundary of R in the direction that keeps R on our left as we progress (Figure 16.34b).

EXAMPLE 6 Verifying Green's Theorem for an Annular Ring

Verify the circulation form of Green's Theorem (Equation 4) on the annular ring $R: h^2 \leq x^2 + y^2 \leq 1, 0 < h < 1$ (Figure 16.35), if

$$M = \frac{-y}{x^2 + y^2}, \quad N = \frac{x}{x^2 + y^2}.$$

Solution The boundary of R consists of the circle

$$C_1: x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq 2\pi,$$

traversed counterclockwise as t increases, and the circle

$$C_h: x = h \cos \theta, \quad y = -h \sin \theta, \quad 0 \leq \theta \leq 2\pi,$$

traversed clockwise as θ increases. The functions M and N and their partial derivatives are continuous throughout R . Moreover,

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{(x^2 + y^2)(-1) + y(2y)}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x}, \end{aligned}$$

so

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R 0 dx dy = 0.$$

The integral of $M dx + N dy$ over the boundary of R is

$$\begin{aligned} \int_C M dx + N dy &= \oint_{C_1} \frac{x dy - y dx}{x^2 + y^2} + \oint_{C_h} \frac{x dy - y dx}{x^2 + y^2} \\ &= \int_0^{2\pi} (\cos^2 t + \sin^2 t) dt - \int_0^{2\pi} \frac{h^2(\cos^2 \theta + \sin^2 \theta)}{h^2} d\theta \\ &= 2\pi - 2\pi = 0. \end{aligned}$$

The functions M and N in Example 6 are discontinuous at $(0, 0)$, so we cannot apply Green's Theorem to the circle C_1 and the region inside it. We must exclude the origin. We do so by excluding the points interior to C_h .

We could replace the circle C_1 in Example 6 by an ellipse or any other simple closed curve K surrounding C_h (Figure 16.36). The result would still be

$$\oint_K (M dx + N dy) + \oint_{C_h} (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy dx = 0,$$

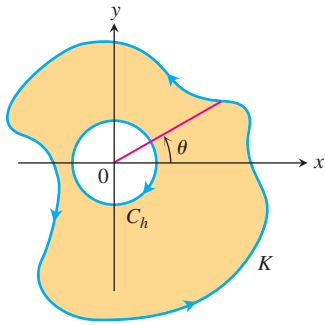


FIGURE 16.36 The region bounded by the circle C_h and the curve K .

which leads to the conclusion that

$$\oint_K (M dx + N dy) = 2\pi$$

for any such curve K . We can explain this result by changing to polar coordinates. With

$$\begin{aligned} x &= r \cos \theta, & y &= r \sin \theta, \\ dx &= -r \sin \theta d\theta + \cos \theta dr, & dy &= r \cos \theta d\theta + \sin \theta dr, \end{aligned}$$

we have

$$\frac{x dy - y dx}{x^2 + y^2} = \frac{r^2(\cos^2 \theta + \sin^2 \theta) d\theta}{r^2} = d\theta,$$

and θ increases by 2π as we traverse K once counterclockwise.

EXERCISES 16.4

Verifying Green's Theorem

In Exercises 1–4, verify the conclusion of Green's Theorem by evaluating both sides of Equations (3) and (4) for the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. Take the domains of integration in each case to be the disk $R: x^2 + y^2 \leq a^2$ and its bounding circle $C: \mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$.

1. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$
2. $\mathbf{F} = y\mathbf{i}$
3. $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j}$
4. $\mathbf{F} = -x^2y\mathbf{i} + xy^2\mathbf{j}$

Counterclockwise Circulation and Outward Flux

In Exercises 5–10, use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C .

5. $\mathbf{F} = (x - y)\mathbf{i} + (y - x)\mathbf{j}$
 C : The square bounded by $x = 0, x = 1, y = 0, y = 1$
6. $\mathbf{F} = (x^2 + 4y)\mathbf{i} + (x + y^2)\mathbf{j}$
 C : The square bounded by $x = 0, x = 1, y = 0, y = 1$
7. $\mathbf{F} = (y^2 - x^2)\mathbf{i} + (x^2 + y^2)\mathbf{j}$
 C : The triangle bounded by $y = 0, x = 3$, and $y = x$
8. $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$
 C : The triangle bounded by $y = 0, x = 1$, and $y = x$
9. $\mathbf{F} = (x + e^x \sin y)\mathbf{i} + (x + e^x \cos y)\mathbf{j}$
 C : The right-hand loop of the lemniscate $r^2 = \cos 2\theta$
10. $\mathbf{F} = \left(\tan^{-1} \frac{y}{x}\right)\mathbf{i} + \ln(x^2 + y^2)\mathbf{j}$
 C : The boundary of the region defined by the polar coordinate inequalities $1 \leq r \leq 2, 0 \leq \theta \leq \pi$
11. Find the counterclockwise circulation and outward flux of the field $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ around and over the boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.

12. Find the counterclockwise circulation and the outward flux of the field $\mathbf{F} = (-\sin y)\mathbf{i} + (x \cos y)\mathbf{j}$ around and over the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.

13. Find the outward flux of the field

$$\mathbf{F} = \left(3xy - \frac{x}{1 + y^2}\right)\mathbf{i} + (e^x + \tan^{-1} y)\mathbf{j}$$

across the cardioid $r = a(1 + \cos \theta)$, $a > 0$.

14. Find the counterclockwise circulation of $\mathbf{F} = (y + e^x \ln y)\mathbf{i} + (e^x/y)\mathbf{j}$ around the boundary of the region that is bounded above by the curve $y = 3 - x^2$ and below by the curve $y = x^4 + 1$.

Work

In Exercises 15 and 16, find the work done by $\mathbf{F}$ in moving a particle once counterclockwise around the given curve.

15. $\mathbf{F} = 2xy^3\mathbf{i} + 4x^2y^2\mathbf{j}$
 C : The boundary of the “triangular” region in the first quadrant enclosed by the x -axis, the line $x = 1$, and the curve $y = x^3$
16. $\mathbf{F} = (4x - 2y)\mathbf{i} + (2x - 4y)\mathbf{j}$
 C : The circle $(x - 2)^2 + (y - 2)^2 = 4$

Evaluating Line Integrals in the Plane

Apply Green's Theorem to evaluate the integrals in Exercises 17–20.

17. $\oint_C (y^2 dx + x^2 dy)$
 C : The triangle bounded by $x = 0, x + y = 1, y = 0$
18. $\oint_C (3y dx + 2x dy)$
 C : The boundary of $0 \leq x \leq \pi, 0 \leq y \leq \sin x$





19. $\oint_C (6y + x) dx + (y + 2x) dy$

C : The circle $(x - 2)^2 + (y - 3)^2 = 4$

20. $\oint_C (2x + y^2) dx + (2xy + 3y) dy$

C : Any simple closed curve in the plane for which Green's Theorem holds

Calculating Area with Green's Theorem

If a simple closed curve C in the plane and the region R it encloses satisfy the hypotheses of Green's Theorem, the area of R is given by

Green's Theorem Area Formula

$$\text{Area of } R = \frac{1}{2} \oint_C x dy - y dx \quad (13)$$

The reason is that by Equation (3), run backward,

$$\begin{aligned} \text{Area of } R &= \iint_R dy dx = \iint_R \left(\frac{1}{2} + \frac{1}{2} \right) dy dx \\ &= \oint_C \frac{1}{2} x dy - \frac{1}{2} y dx. \end{aligned}$$

Use the Green's Theorem area formula (Equation 13) to find the areas of the regions enclosed by the curves in Exercises 21–24.

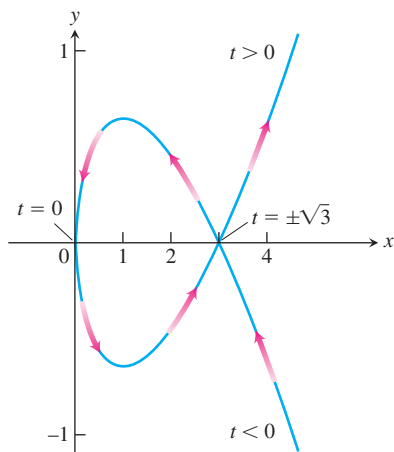


21. The circle $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$

22. The ellipse $\mathbf{r}(t) = (a \cos t)\mathbf{i} + (b \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$

23. The astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}$, $0 \leq t \leq 2\pi$

24. The curve $\mathbf{r}(t) = t^2\mathbf{i} + ((t^3/3) - t)\mathbf{j}$, $-\sqrt{3} \leq t \leq \sqrt{3}$ (see accompanying figure).



Theory and Examples

25. Let C be the boundary of a region on which Green's Theorem holds. Use Green's Theorem to calculate

a. $\oint_C f(x) dx + g(y) dy$

b. $\oint_C ky dx + hx dy$ (k and h constants).

26. **Integral dependent only on area** Show that the value of

$$\oint_C xy^2 dx + (x^2y + 2x) dy$$

around any square depends only on the area of the square and not on its location in the plane.

27. What is special about the integral

$$\oint_C 4x^3y dx + x^4 dy?$$

Give reasons for your answer.

28. What is special about the integral

$$\oint_C -y^3 dy + x^3 dx?$$

Give reasons for your answer.

29. **Area as a line integral** Show that if R is a region in the plane bounded by a piecewise-smooth simple closed curve C , then

$$\text{Area of } R = \oint_C x dy = - \oint_C y dx.$$

30. **Definite integral as a line integral** Suppose that a nonnegative function $y = f(x)$ has a continuous first derivative on $[a, b]$. Let C be the boundary of the region in the xy -plane that is bounded below by the x -axis, above by the graph of f , and on the sides by the lines $x = a$ and $x = b$. Show that

$$\int_a^b f(x) dx = - \oint_C y dx.$$

31. **Area and the centroid** Let A be the area and $\bar{x}$ the x -coordinate of the centroid of a region R that is bounded by a piecewise-smooth simple closed curve C in the xy -plane. Show that

$$\frac{1}{2} \oint_C x^2 dy = - \oint_C xy dx = \frac{1}{3} \oint_C x^2 dy - xy dx = A\bar{x}.$$

32. **Moment of inertia** Let I_y be the moment of inertia about the y -axis of the region in Exercise 31. Show that

$$\frac{1}{3} \oint_C x^3 dy = - \oint_C x^2y dx = \frac{1}{4} \oint_C x^3 dy - x^2y dx = I_y.$$

- 33. Green's Theorem and Laplace's equation** Assuming that all the necessary derivatives exist and are continuous, show that if $f(x, y)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

then

$$\oint_C \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy = 0$$

for all closed curves C to which Green's Theorem applies. (The converse is also true: If the line integral is always zero, then f satisfies the Laplace equation.)

- 34. Maximizing work** Among all smooth simple closed curves in the plane, oriented counterclockwise, find the one along which the work done by

$$\mathbf{F} = \left(\frac{1}{4}x^2y + \frac{1}{3}y^3 \right) \mathbf{i} + xy \mathbf{j}$$

is greatest. (*Hint:* Where is $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ positive?)

- 35. Regions with many holes** Green's Theorem holds for a region R with any finite number of holes as long as the bounding curves are smooth, simple, and closed and we integrate over each component of the boundary in the direction that keeps R on our immediate left as we go along (Figure 16.37).

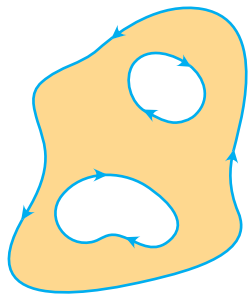


FIGURE 16.37 Green's Theorem holds for regions with more than one hole (Exercise 35).

- a. Let $f(x, y) = \ln(x^2 + y^2)$ and let C be the circle $x^2 + y^2 = a^2$. Evaluate the flux integral

$$\oint_C \nabla f \cdot \mathbf{n} \, ds.$$

- b. Let K be an arbitrary smooth simple closed curve in the plane

that does not pass through $(0, 0)$. Use Green's Theorem to show that

$$\oint_K \nabla f \cdot \mathbf{n} \, ds$$

has two possible values, depending on whether $(0, 0)$ lies inside K or outside K .

- 36. Bendixson's criterion** The *streamlines* of a planar fluid flow are the smooth curves traced by the fluid's individual particles. The vectors $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ of the flow's velocity field are the tangent vectors of the streamlines. Show that if the flow takes place over a simply connected region R (no holes or missing points) and that if $M_x + N_y \neq 0$ throughout R , then none of the streamlines in R is closed. In other words, no particle of fluid ever has a closed trajectory in R . The criterion $M_x + N_y \neq 0$ is called **Bendixson's criterion** for the nonexistence of closed trajectories.
- 37.** Establish Equation (7) to finish the proof of the special case of Green's Theorem.
- 38.** Establish Equation (10) to complete the argument for the extension of Green's Theorem.
- 39. Curl component of conservative fields** Can anything be said about the curl component of a conservative two-dimensional vector field? Give reasons for your answer.
- 40. Circulation of conservative fields** Does Green's Theorem give any information about the circulation of a conservative field? Does this agree with anything else you know? Give reasons for your answer.

COMPUTER EXPLORATIONS

Finding Circulation

In Exercises 41–44, use a CAS and Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F}$ around the simple closed curve C . Perform the following CAS steps.

- Plot C in the xy -plane.
 - Determine the integrand $(\partial N / \partial x) - (\partial M / \partial y)$ for the curl form of Green's Theorem.
 - Determine the (double integral) limits of integration from your plot in part (a) and evaluate the curl integral for the circulation.
- 41.** $\mathbf{F} = (2x - y)\mathbf{i} + (x + 3y)\mathbf{j}$, C : The ellipse $x^2 + 4y^2 = 4$
- 42.** $\mathbf{F} = (2x^3 - y^3)\mathbf{i} + (x^3 + y^3)\mathbf{j}$, C : The ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$
- 43.** $\mathbf{F} = x^{-1}e^y\mathbf{i} + (e^y \ln x + 2x)\mathbf{j}$,
 C : The boundary of the region defined by $y = 1 + x^4$ (below) and $y = 2$ (above)
- 44.** $\mathbf{F} = xe^y\mathbf{i} + 4x^2 \ln y \mathbf{j}$,
 C : The triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$

16.5 Surface Area and Surface Integrals

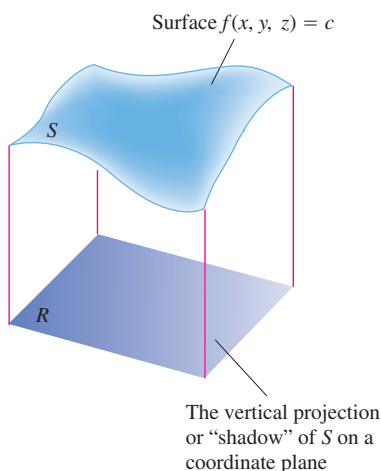


FIGURE 16.38 As we soon see, the integral of a function $g(x, y, z)$ over a surface S in space can be calculated by evaluating a related double integral over the vertical projection or “shadow” of S on a coordinate plane.

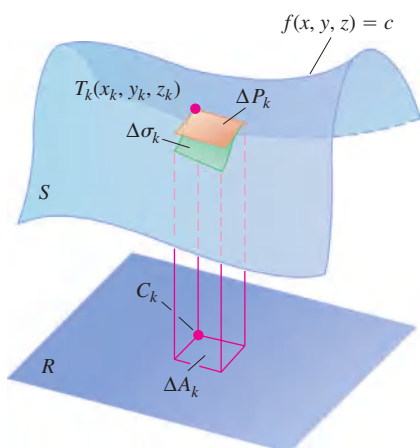


FIGURE 16.39 A surface S and its vertical projection onto a plane beneath it. You can think of R as the shadow of S on the plane. The tangent plane ΔP_k approximates the surface patch $\Delta \sigma_k$ above ΔA_k .

We know how to integrate a function over a flat region in a plane, but what if the function is defined over a curved surface? To evaluate one of these so-called surface integrals, we rewrite it as a double integral over a region in a coordinate plane beneath the surface (Figure 16.38). Surface integrals are used to compute quantities such as the flow of liquid across a membrane or the upward force on a falling parachute.

Surface Area

Figure 16.39 shows a surface S lying above its “shadow” region R in a plane beneath it. The surface is defined by the equation $f(x, y, z) = c$. If the surface is **smooth** (∇f is continuous and never vanishes on S), we can define and calculate its area as a double integral over R . We assume that this projection of the surface onto its shadow R is one-to-one. That is, each point in R corresponds to exactly one point (x, y, z) satisfying $f(x, y, z) = c$.

The first step in defining the area of S is to partition the region R into small rectangles ΔA_k of the kind we would use if we were defining an integral over R . Directly above each ΔA_k lies a patch of surface $\Delta \sigma_k$ that we may approximate by a parallelogram ΔP_k in the tangent plane to S at a point $T_k(x_k, y_k, z_k)$ in $\Delta \sigma_k$. This parallelogram in the tangent plane projects directly onto ΔA_k . To be specific, we choose the point $T_k(x_k, y_k, z_k)$ lying directly above the back corner C_k of ΔA_k , as shown in Figure 16.39. If the tangent plane is parallel to R , then ΔP_k will be congruent to ΔA_k . Otherwise, it will be a parallelogram whose area is somewhat larger than the area of ΔA_k .

Figure 16.40 gives a magnified view of $\Delta \sigma_k$ and ΔP_k , showing the gradient vector $\nabla f(x_k, y_k, z_k)$ at T_k and a unit vector $\mathbf{p}$ that is normal to R . The figure also shows the angle γ_k between ∇f and $\mathbf{p}$. The other vectors in the picture, $\mathbf{u}_k$ and $\mathbf{v}_k$, lie along the edges of the patch ΔP_k in the tangent plane. Thus, both $\mathbf{u}_k \times \mathbf{v}_k$ and ∇f are normal to the tangent plane.

We now need to know from advanced vector geometry that $|(\mathbf{u}_k \times \mathbf{v}_k) \cdot \mathbf{p}|$ is the area of the projection of the parallelogram determined by $\mathbf{u}_k$ and $\mathbf{v}_k$ onto any plane whose normal is $\mathbf{p}$. (A proof is given in Appendix 8.) In our case, this translates into the statement

$$|(\mathbf{u}_k \times \mathbf{v}_k) \cdot \mathbf{p}| = \Delta A_k.$$

To simplify the notation in the derivation that follows, we are now denoting the *area* of the small rectangular region by ΔA_k as well. Likewise, ΔP_k will also denote the area of the portion of the tangent plane directly above this small region.

Now, $|\mathbf{u}_k \times \mathbf{v}_k|$ itself is the area ΔP_k (standard fact about cross products) so this last equation becomes

$$\underbrace{|\mathbf{u}_k \times \mathbf{v}_k|}_{\Delta P_k} \underbrace{|\mathbf{p}|}_1 \underbrace{|\cos(\text{angle between } \mathbf{u}_k \times \mathbf{v}_k \text{ and } \mathbf{p})|}_{\text{Same as } |\cos \gamma_k| \text{ because } \nabla f \text{ and } \mathbf{u}_k \times \mathbf{v}_k \text{ are both normal to the tangent plane}} = \Delta A_k$$

or

$$\Delta P_k |\cos \gamma_k| = \Delta A_k$$

or

$$\Delta P_k = \frac{\Delta A_k}{|\cos \gamma_k|},$$

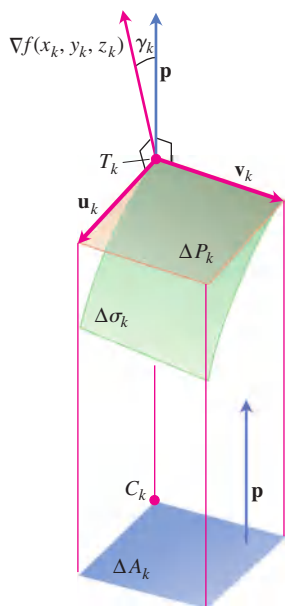


FIGURE 16.40 Magnified view from the preceding figure. The vector $\mathbf{u}_k \times \mathbf{v}_k$ (not shown) is parallel to the vector ∇f because both vectors are normal to the plane of ΔP_k .

provided $\cos \gamma_k \neq 0$. We will have $\cos \gamma_k \neq 0$ as long as ∇f is not parallel to the ground plane and $\nabla f \cdot \mathbf{p} \neq 0$.

Since the patches ΔP_k approximate the surface patches $\Delta \sigma_k$ that fit together to make S , the sum

$$\sum \Delta P_k = \sum \frac{\Delta A_k}{|\cos \gamma_k|} \quad (1)$$

looks like an approximation of what we might like to call the surface area of S . It also looks as if the approximation would improve if we refined the partition of R . In fact, the sums on the right-hand side of Equation (1) are approximating sums for the double integral

$$\iint_R \frac{1}{|\cos \gamma|} dA. \quad (2)$$

We therefore define the **area** of S to be the value of this integral whenever it exists. For any surface $f(x, y, z) = c$, we have $|\nabla f \cdot \mathbf{p}| = |\nabla f| |\mathbf{p}| |\cos \gamma|$, so

$$\frac{1}{|\cos \gamma|} = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}.$$

This combines with Equation (2) to give a practical formula for surface area.

Formula for Surface Area

The area of the surface $f(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA, \quad (3)$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla f \cdot \mathbf{p} \neq 0$.

Thus, the area is the double integral over R of the magnitude of ∇f divided by the magnitude of the scalar component of ∇f normal to R .

We reached Equation (3) under the assumption that $\nabla f \cdot \mathbf{p} \neq 0$ throughout R and that ∇f is continuous. Whenever the integral exists, however, we define its value to be the area of the portion of the surface $f(x, y, z) = c$ that lies over R . (Recall that the projection is assumed to be one-to-one.)

In the exercises (see Equation 11), we show how Equation (3) simplifies if the surface is defined by $z = f(x, y)$.



EXAMPLE 1 Finding Surface Area

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

Solution We sketch the surface S and the region R below it in the xy -plane (Figure 16.41). The surface S is part of the level surface $f(x, y, z) = x^2 + y^2 - z = 0$, and R is the disk $x^2 + y^2 \leq 4$ in the xy -plane. To get a unit vector normal to the plane of R , we can take $\mathbf{p} = \mathbf{k}$.

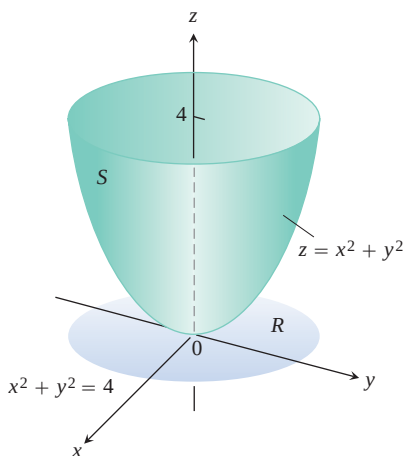


FIGURE 16.41 The area of this parabolic surface is calculated in Example 1.

At any point (x, y, z) on the surface, we have

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 - z \\ \nabla f &= 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \\ |\nabla f| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla f \cdot \mathbf{p}| &= |\nabla f \cdot \mathbf{k}| = |-1| = 1. \end{aligned}$$

In the region R , $dA = dx dy$. Therefore,

$$\begin{aligned} \text{Surface area} &= \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA && \text{Equation (3)} \\ &= \iint_{x^2+y^2 \leq 4} \sqrt{4x^2 + 4y^2 + 1} dx dy \\ &= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1} r dr d\theta && \text{Polar coordinates} \\ &= \int_0^{2\pi} \left[\frac{1}{12} (4r^2 + 1)^{3/2} \right]_0^2 d\theta \\ &= \int_0^{2\pi} \frac{1}{12} (17^{3/2} - 1) d\theta = \frac{\pi}{6} (17\sqrt{17} - 1). \end{aligned}$$

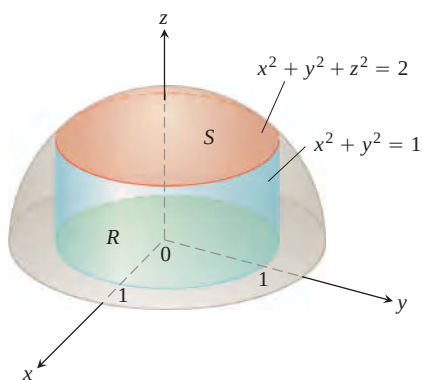


FIGURE 16.42 The cap cut from the hemisphere by the cylinder projects vertically onto the disk $R: x^2 + y^2 \leq 1$ in the xy -plane (Example 2).

EXAMPLE 2 Finding Surface Area

Find the area of the cap cut from the hemisphere $x^2 + y^2 + z^2 = 2$, $z \geq 0$, by the cylinder $x^2 + y^2 = 1$ (Figure 16.42).

Solution The cap S is part of the level surface $f(x, y, z) = x^2 + y^2 + z^2 = 2$. It projects one-to-one onto the disk $R: x^2 + y^2 \leq 1$ in the xy -plane. The unit vector $\mathbf{p} = \mathbf{k}$ is normal to the plane of R .

At any point on the surface,

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 + z^2 \\ \nabla f &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} \\ |\nabla f| &= 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{2} && \text{Because } x^2 + y^2 + z^2 = 2 \text{ at points of } S \\ |\nabla f \cdot \mathbf{p}| &= |\nabla f \cdot \mathbf{k}| = |2z| = 2z. \end{aligned}$$

Therefore,

$$\text{Surface area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \iint_R \frac{2\sqrt{2}}{2z} dA = \sqrt{2} \iint_R \frac{dA}{z}. \quad (4)$$

What do we do about the z ?

Since z is the z -coordinate of a point on the sphere, we can express it in terms of x and y as

$$z = \sqrt{2 - x^2 - y^2}.$$



We continue the work of Equation (4) with this substitution:

$$\begin{aligned}
 \text{Surface area} &= \sqrt{2} \iint_R \frac{dA}{z} = \sqrt{2} \iint_{x^2+y^2 \leq 1} \frac{dA}{\sqrt{2-x^2-y^2}} \\
 &= \sqrt{2} \int_0^{2\pi} \int_0^1 \frac{r \, dr \, d\theta}{\sqrt{2-r^2}} \quad \text{Polar coordinates} \\
 &= \sqrt{2} \int_0^{2\pi} \left[-(2-r^2)^{1/2} \right]_{r=0}^{r=1} d\theta \\
 &= \sqrt{2} \int_0^{2\pi} (\sqrt{2}-1) \, d\theta = 2\pi(2-\sqrt{2}). \quad \blacksquare
 \end{aligned}$$

Surface Integrals

We now show how to integrate a function over a surface, using the ideas just developed for calculating surface area.

Suppose, for example, that we have an electrical charge distributed over a surface $f(x, y, z) = c$ like the one shown in Figure 16.43 and that the function $g(x, y, z)$ gives the charge per unit area (charge density) at each point on S . Then we may calculate the total charge on S as an integral in the following way.

We partition the shadow region R on the ground plane beneath the surface into small rectangles of the kind we would use if we were defining the surface area of S . Then directly above each ΔA_k lies a patch of surface $\Delta \sigma_k$ that we approximate with a parallelogram-shaped portion of tangent plane, ΔP_k . (See Figure 16.43.)

Up to this point the construction proceeds as in the definition of surface area, but now we take an additional step: We evaluate g at (x_k, y_k, z_k) and approximate the total charge on the surface patch $\Delta \sigma_k$ by the product $g(x_k, y_k, z_k) \Delta P_k$. The rationale is that when the partition of R is sufficiently fine, the value of g throughout $\Delta \sigma_k$ is nearly constant and ΔP_k is nearly the same as $\Delta \sigma_k$. The total charge over S is then approximated by the sum

$$\text{Total charge} \approx \sum g(x_k, y_k, z_k) \Delta P_k = \sum g(x_k, y_k, z_k) \frac{\Delta A_k}{|\cos \gamma_k|}.$$

If f , the function defining the surface S , and its first partial derivatives are continuous, and if g is continuous over S , then the sums on the right-hand side of the last equation approach the limit

$$\iint_R g(x, y, z) \frac{dA}{|\cos \gamma|} = \iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA \quad (5)$$

as the partition of R is refined in the usual way. This limit is called the integral of g over the surface S and is calculated as a double integral over R . The value of the integral is the total charge on the surface S .

As you might expect, the formula in Equation (5) defines the integral of *any* function g over the surface S as long as the integral exists.

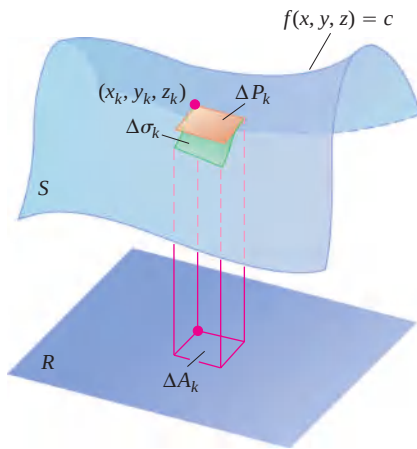


FIGURE 16.43 If we know how an electrical charge $g(x, y, z)$ is distributed over a surface, we can find the total charge with a suitably modified surface integral.

DEFINITION Surface Integral

If R is the shadow region of a surface S defined by the equation $f(x, y, z) = c$, and g is a continuous function defined at the points of S , then the **integral of g over S** is the integral

$$\iint_R g(x, y, z) \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA, \quad (6)$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla f \cdot \mathbf{p} \neq 0$. The integral itself is called a **surface integral**.

The integral in Equation (6) takes on different meanings in different applications. If g has the constant value 1, the integral gives the area of S . If g gives the mass density of a thin shell of material modeled by S , the integral gives the mass of the shell.

We can abbreviate the integral in Equation (6) by writing $d\sigma$ for $(|\nabla f|/|\nabla f \cdot \mathbf{p}|) dA$.

The Surface Area Differential and the Differential Form for Surface Integrals

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA \quad \iint_S g d\sigma \quad (7)$$

Surface area differential
Differential formula for surface integrals

Surface integrals behave like other double integrals, the integral of the sum of two functions being the sum of their integrals and so on. The domain Additivity Property takes the form

$$\iint_S g d\sigma = \iint_{S_1} g d\sigma + \iint_{S_2} g d\sigma + \cdots + \iint_{S_n} g d\sigma.$$

The idea is that if S is partitioned by smooth curves into a finite number of nonoverlapping smooth patches (i.e., if S is **piecewise smooth**), then the integral over S is the sum of the integrals over the patches. Thus, the integral of a function over the surface of a cube is the sum of the integrals over the faces of the cube. We integrate over a turtle shell of welded plates by integrating one plate at a time and adding the results.

EXAMPLE 3 Integrating Over a Surface

Integrate $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ (Figure 16.44).

Solution We integrate xyz over each of the six sides and add the results. Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to

$$\iint_{\text{Cube surface}} xyz d\sigma = \iint_{\text{Side A}} xyz d\sigma + \iint_{\text{Side B}} xyz d\sigma + \iint_{\text{Side C}} xyz d\sigma.$$

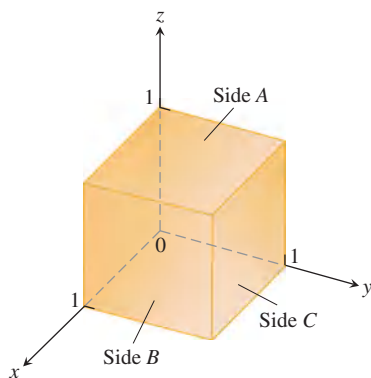


FIGURE 16.44 The cube in Example 3.



Side A is the surface $f(x, y, z) = z = 1$ over the square region $R_{xy}: 0 \leq x \leq 1, 0 \leq y \leq 1$, in the xy -plane. For this surface and region,

$$\mathbf{p} = \mathbf{k}, \quad \nabla f = \mathbf{k}, \quad |\nabla f| = 1, \quad |\nabla f \cdot \mathbf{p}| = |\mathbf{k} \cdot \mathbf{k}| = 1$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{1}{1} dx dy = dx dy$$

$$xyz = xy(1) = xy$$

and

$$\iint_{\text{Side } A} xyz \, d\sigma = \iint_{R_{xy}} xy \, dx \, dy = \int_0^1 \int_0^1 xy \, dx \, dy = \int_0^1 \frac{y}{2} dy = \frac{1}{4}.$$

Symmetry tells us that the integrals of xyz over sides B and C are also $1/4$. Hence,

$$\iint_{\text{Cube surface}} xyz \, d\sigma = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

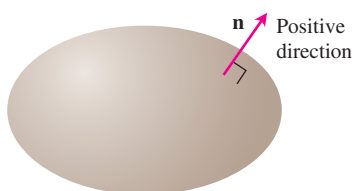


FIGURE 16.45 Smooth closed surfaces in space are orientable. The outward unit normal vector defines the positive direction at each point.

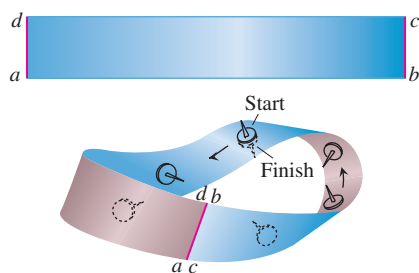


FIGURE 16.46 To make a Möbius band, take a rectangular strip of paper $abcd$, give the end bc a single twist, and paste the ends of the strip together to match a with c and b with d . The Möbius band is a nonorientable or one-sided surface.

Orientation

We call a smooth surface S **orientable** or **two-sided** if it is possible to define a field $\mathbf{n}$ of unit normal vectors on S that varies continuously with position. Any patch or subportion of an orientable surface is orientable. Spheres and other smooth closed surfaces in space (smooth surfaces that enclose solids) are orientable. By convention, we choose $\mathbf{n}$ on a closed surface to point outward.

Once $\mathbf{n}$ has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface**. The vector $\mathbf{n}$ at any point is called the **positive direction** at that point (Figure 16.45).

The Möbius band in Figure 16.46 is not orientable. No matter where you start to construct a continuous-unit normal field (shown as the shaft of a thumbtack in the figure), moving the vector continuously around the surface in the manner shown will return it to the starting point with a direction opposite to the one it had when it started out. The vector at that point cannot point both ways and yet it must if the field is to be continuous. We conclude that no such field exists.

Surface Integral for Flux

Suppose that $\mathbf{F}$ is a continuous vector field defined over an oriented surface S and that $\mathbf{n}$ is the chosen unit normal field on the surface. We call the integral of $\mathbf{F} \cdot \mathbf{n}$ over S the **flux** of $\mathbf{F}$ across S in the positive direction. Thus, the flux is the integral over S of the scalar component of $\mathbf{F}$ in the direction of $\mathbf{n}$.

DEFINITION Flux

The **flux** of a three-dimensional vector field $\mathbf{F}$ across an oriented surface S in the direction of $\mathbf{n}$ is

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma. \quad (8)$$

The definition is analogous to the flux of a two-dimensional field $\mathbf{F}$ across a plane curve C . In the plane (Section 16.2), the flux is

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

the integral of the scalar component of $\mathbf{F}$ normal to the curve.

If $\mathbf{F}$ is the velocity field of a three-dimensional fluid flow, the flux of $\mathbf{F}$ across S is the net rate at which fluid is crossing S in the chosen positive direction. We discuss such flows in more detail in Section 16.7.

If S is part of a level surface $g(x, y, z) = c$, then $\mathbf{n}$ may be taken to be one of the two fields

$$\mathbf{n} = \pm \frac{\nabla g}{|\nabla g|}, \quad (9)$$

depending on which one gives the preferred direction. The corresponding flux is

$$\begin{aligned} \text{Flux} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma \\ &= \iint_R \left(\mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g|} \right) \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} \, dA \quad \text{Equations (9) and (7)} \end{aligned} \quad (8)$$

$$= \iint_R \mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot \mathbf{p}|} \, dA. \quad (10)$$



EXAMPLE 4 Finding Flux

Find the flux of $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1$, $z \geq 0$, by the planes $x = 0$ and $x = 1$.

Solution The outward normal field on S (Figure 16.47) may be calculated from the gradient of $g(x, y, z) = y^2 + z^2$ to be

$$\mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{1}} = y\mathbf{j} + z\mathbf{k}.$$

With $\mathbf{p} = \mathbf{k}$, we also have

$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{k}|} \, dA = \frac{2}{|2z|} \, dA = \frac{1}{z} \, dA.$$

We can drop the absolute value bars because $z \geq 0$ on S .

The value of $\mathbf{F} \cdot \mathbf{n}$ on the surface is

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= (yz\mathbf{j} + z^2\mathbf{k}) \cdot (y\mathbf{j} + z\mathbf{k}) \\ &= y^2z + z^3 = z(y^2 + z^2) \\ &= z. \end{aligned} \quad y^2 + z^2 = 1 \text{ on } S$$

Therefore, the flux of $\mathbf{F}$ outward through S is

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S (z) \left(\frac{1}{z} \, dA \right) = \iint_{R_{xy}} dA = \text{area}(R_{xy}) = 2. \quad \blacksquare$$

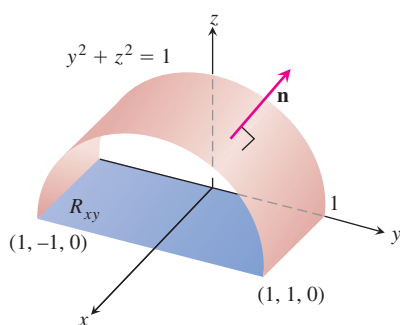


FIGURE 16.47 Calculating the flux of a vector field outward through this surface. The area of the shadow region R_{xy} is 2 (Example 4).

Moments and Masses of Thin Shells

Thin shells of material like bowls, metal drums, and domes are modeled with surfaces. Their moments and masses are calculated with the formulas in Table 16.3.

TABLE 16.3 Mass and moment formulas for very thin shells

Mass: $M = \iint_S \delta(x, y, z) \, d\sigma$ ($\delta(x, y, z)$ = density at (x, y, z) , mass per unit area)

First moments about the coordinate planes:

$$M_{yz} = \iint_S x \delta \, d\sigma, \quad M_{xz} = \iint_S y \delta \, d\sigma, \quad M_{xy} = \iint_S z \delta \, d\sigma$$

Coordinates of center of mass:

$$\bar{x} = M_{yz}/M, \quad \bar{y} = M_{xz}/M, \quad \bar{z} = M_{xy}/M$$

Moments of inertia about coordinate axes:

$$I_x = \iint_S (y^2 + z^2) \delta \, d\sigma, \quad I_y = \iint_S (x^2 + z^2) \delta \, d\sigma,$$

$$I_z = \iint_S (x^2 + y^2) \delta \, d\sigma, \quad I_L = \iint_S r^2 \delta \, d\sigma,$$

$$r(x, y, z) = \text{distance from point } (x, y, z) \text{ to line } L$$

Radius of gyration about a line L : $R_L = \sqrt{I_L/M}$

EXAMPLE 5 Finding Center of Mass

Find the center of mass of a thin hemispherical shell of radius a and constant density δ .

Solution We model the shell with the hemisphere

$$f(x, y, z) = x^2 + y^2 + z^2 = a^2, \quad z \geq 0$$

(Figure 16.48). The symmetry of the surface about the z -axis tells us that $\bar{x} = \bar{y} = 0$. It remains only to find $\bar{z}$ from the formula $\bar{z} = M_{xy}/M$.

The mass of the shell is

$$M = \iint_S \delta \, d\sigma = \delta \iint_S d\sigma = (\delta)(\text{area of } S) = 2\pi a^2 \delta.$$

To evaluate the integral for M_{xy} , we take $\mathbf{p} = \mathbf{k}$ and calculate

$$|\nabla f| = |2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}| = 2\sqrt{x^2 + y^2 + z^2} = 2a$$

$$|\nabla f \cdot \mathbf{p}| = |\nabla f \cdot \mathbf{k}| = |2z| = 2z$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{a}{z} dA.$$

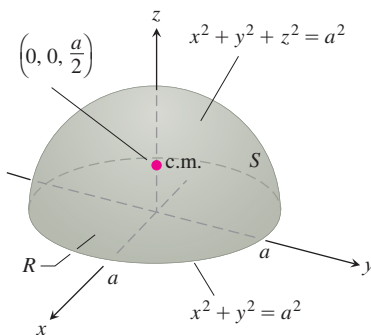


FIGURE 16.48 The center of mass of a thin hemispherical shell of constant density lies on the axis of symmetry halfway from the base to the top (Example 5).

Then

$$M_{xy} = \iint_S z \delta \, d\sigma = \delta \iint_R z \frac{a}{z} \, dA = \delta a \iint_R dA = \delta a(\pi a^2) = \delta \pi a^3$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\pi a^3 \delta}{2\pi a^2 \delta} = \frac{a}{2}.$$

The shell's center of mass is the point $(0, 0, a/2)$. ■

EXERCISES 16.5

Surface Area

- Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.
- Find the area of the band cut from the paraboloid $x^2 + y^2 - z = 0$ by the planes $z = 2$ and $z = 6$.
- Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$.
- Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}$, $y = 0$, and $y = x$ in the xy -plane.
- Find the area of the surface $x^2 - 2y - 2z = 0$ that lies above the triangle bounded by the lines $x = 2$, $y = 0$, and $y = 3x$ in the xy -plane.
- Find the area of the cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.
- Find the area of the ellipse cut from the plane $z = cx$ (c a constant) by the cylinder $x^2 + y^2 = 1$.
- Find the area of the upper portion of the cylinder $x^2 + z^2 = 1$ that lies between the planes $x = \pm 1/2$ and $y = \pm 1/2$.
- Find the area of the portion of the paraboloid $x = 4 - y^2 - z^2$ that lies above the ring $1 \leq y^2 + z^2 \leq 4$ in the yz -plane.
- Find the area of the surface cut from the paraboloid $x^2 + y + z^2 = 0$ by the plane $y = 0$.
- Find the area of the surface $x^2 - 2 \ln x + \sqrt{15}y - z = 0$ above the square $R: 1 \leq x \leq 2, 0 \leq y \leq 1$, in the xy -plane.
- Find the area of the surface $2x^{3/2} + 2y^{3/2} - 3z = 0$ above the square $R: 0 \leq x \leq 1, 0 \leq y \leq 1$, in the xy -plane.

Surface Integrals

- Integrate $g(x, y, z) = x + y + z$ over the surface of the cube cut from the first octant by the planes $x = a$, $y = a$, $z = a$.

- Integrate $g(x, y, z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.
- Integrate $g(x, y, z) = xyz$ over the surface of the rectangular solid cut from the first octant by the planes $x = a$, $y = b$, and $z = c$.
- Integrate $g(x, y, z) = xyz$ over the surface of the rectangular solid bounded by the planes $x = \pm a$, $y = \pm b$, and $z = \pm c$.
- Integrate $g(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.
- Integrate $g(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$.

Flux Across a Surface

In Exercises 19 and 20, find the flux of the field $\mathbf{F}$ across the portion of the given surface in the specified direction.

- $\mathbf{F}(x, y, z) = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

S : rectangular surface $z = 0$, $0 \leq x \leq 2$, $0 \leq y \leq 3$, direction $\mathbf{k}$

- $\mathbf{F}(x, y, z) = yx^2\mathbf{i} - 2\mathbf{j} + xz\mathbf{k}$

S : rectangular surface $y = 0$, $-1 \leq x \leq 2$, $2 \leq z \leq 7$, direction $-\mathbf{j}$

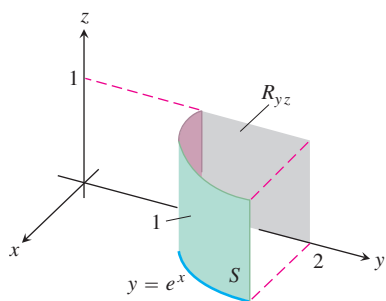
In Exercises 21–26, find the flux of the field $\mathbf{F}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

- $\mathbf{F}(x, y, z) = z\mathbf{k}$
- $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$
- $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$
- $\mathbf{F}(x, y, z) = zx\mathbf{i} + zy\mathbf{j} + z^2\mathbf{k}$
- $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- $\mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$





27. Find the flux of the field $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$.
28. Find the flux of the field $\mathbf{F}(x, y, z) = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (away from the z -axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.
29. Let S be the portion of the cylinder $y = e^x$ in the first octant that projects parallel to the x -axis onto the rectangle R_{yz} : $1 \leq y \leq 2$, $0 \leq z \leq 1$ in the yz -plane (see the accompanying figure). Let $\mathbf{n}$ be the unit vector normal to S that points away from the yz -plane. Find the flux of the field $\mathbf{F}(x, y, z) = -2\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$ across S in the direction of $\mathbf{n}$.

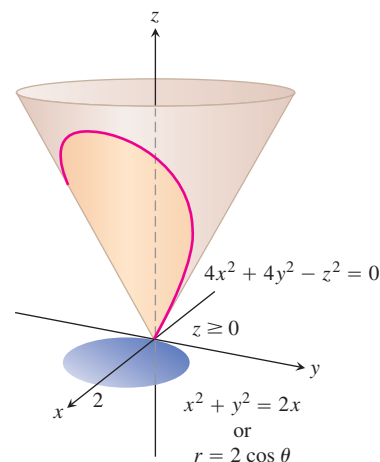


30. Let S be the portion of the cylinder $y = \ln x$ in the first octant whose projection parallel to the y -axis onto the xz -plane is the rectangle R_{xz} : $1 \leq x \leq e$, $0 \leq z \leq 1$. Let $\mathbf{n}$ be the unit vector normal to S that points away from the xz -plane. Find the flux of $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$ through S in the direction of $\mathbf{n}$.
31. Find the outward flux of the field $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the surface of the cube cut from the first octant by the planes $x = a$, $y = a$, $z = a$.
32. Find the outward flux of the field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \leq 25$ by the plane $z = 3$.

Moments and Masses



33. **Centroid** Find the centroid of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.
34. **Centroid** Find the centroid of the surface cut from the cylinder $y^2 + z^2 = 9$, $z \geq 0$, by the planes $x = 0$ and $x = 3$ (resembles the surface in Example 4).
35. **Thin shell of constant density** Find the center of mass and the moment of inertia and radius of gyration about the z -axis of a thin shell of constant density δ cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$.
36. **Conical surface of constant density** Find the moment of inertia about the z -axis of a thin shell of constant density δ cut from the cone $4x^2 + 4y^2 - z^2 = 0$, $z \geq 0$, by the circular cylinder $x^2 + y^2 = 2x$ (see the accompanying figure).



37. Spherical shells

- Find the moment of inertia about a diameter of a thin spherical shell of radius a and constant density δ . (Work with a hemispherical shell and double the result.)
- Use the Parallel Axis Theorem (Exercises 15.5) and the result in part (a) to find the moment of inertia about a line tangent to the shell.

38. **a. Cones with and without ice cream** Find the centroid of the lateral surface of a solid cone of base radius a and height h (cone surface minus the base).
- Use Pappus's formula (Exercises 15.5) and the result in part (a) to find the centroid of the complete surface of a solid cone (side plus base).
 - A cone of radius a and height h is joined to a hemisphere of radius a to make a surface S that resembles an ice cream cone. Use Pappus's formula and the results in part (a) and Example 5 to find the centroid of S . How high does the cone have to be to place the centroid in the plane shared by the bases of the hemisphere and cone?

Special Formulas for Surface Area

If S is the surface defined by a function $z = f(x, y)$ that has continuous first partial derivatives throughout a region R_{xy} in the xy -plane (Figure 16.49), then S is also the level surface $F(x, y, z) = 0$ of the function $F(x, y, z) = f(x, y) - z$. Taking the unit normal to R_{xy} to be $\mathbf{p} = \mathbf{k}$ then gives

$$|\nabla F| = |f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}| = \sqrt{f_x^2 + f_y^2 + 1}$$

$$|\nabla F \cdot \mathbf{p}| = |(f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}) \cdot \mathbf{k}| = |-1| = 1$$

and

$$\iint_{R_{xy}} \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA = \iint_{R_{xy}} \sqrt{f_x^2 + f_y^2 + 1} dx dy, \quad (11)$$

Similarly, the area of a smooth surface $x = f(y, z)$ over a region R_{yz} in the yz -plane is

$$A = \iint_{R_{yz}} \sqrt{f_y^2 + f_z^2 + 1} \, dy \, dz, \quad (12)$$

and the area of a smooth $y = f(x, z)$ over a region R_{xz} in the xz -plane is

$$A = \iint_{R_{xz}} \sqrt{f_x^2 + f_z^2 + 1} \, dx \, dz. \quad (13)$$

Use Equations (11)–(13) to find the area of the surfaces in Exercises 39–44.



39. The surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 3$
40. The surface cut from the “nose” of the paraboloid $x = 1 - y^2 - z^2$ by the yz -plane
41. The portion of the cone $z = \sqrt{x^2 + y^2}$ that lies over the region between the circle $x^2 + y^2 = 1$ and the ellipse $9x^2 + 4y^2 = 36$ in the xy -plane. (*Hint:* Use formulas from geometry to find the area of the region.)
42. The triangle cut from the plane $2x + 6y + 3z = 6$ by the bounding planes of the first octant. Calculate the area three ways, once with each area formula

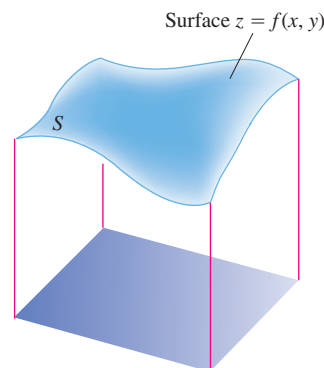


FIGURE 16.49 For a surface $z = f(x, y)$, the surface area formula in Equation (3) takes the form

$$A = \iint_{R_{xy}} \sqrt{f_x^2 + f_y^2 + 1} \, dx \, dy.$$

43. The surface in the first octant cut from the cylinder $y = (2/3)z^{3/2}$ by the planes $x = 1$ and $y = 16/3$
44. The portion of the plane $y + z = 4$ that lies above the region cut from the first quadrant of the xz -plane by the parabola $x = 4 - z^2$

16.6

Parametrized Surfaces

We have defined curves in the plane in three different ways:

Explicit form: $y = f(x)$

Implicit form: $F(x, y) = 0$

Parametric vector form: $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \quad a \leq t \leq b.$

We have analogous definitions of surfaces in space:

Explicit form: $z = f(x, y)$

Implicit form: $F(x, y, z) = 0.$

There is also a parametric form that gives the position of a point on the surface as a vector function of two variables. The present section extends the investigation of surface area and surface integrals to surfaces described parametrically.

Parametrizations of Surfaces

Let

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k} \quad (1)$$

be a continuous vector function that is defined on a region R in the uv -plane and one-to-one on the interior of R (Figure 16.50). We call the range of $\mathbf{r}$ the **surface** S defined or traced by $\mathbf{r}$. Equation (1) together with the domain R constitute a **parametrization** of the surface. The variables u and v are the **parameters**, and R is the **parameter domain**.

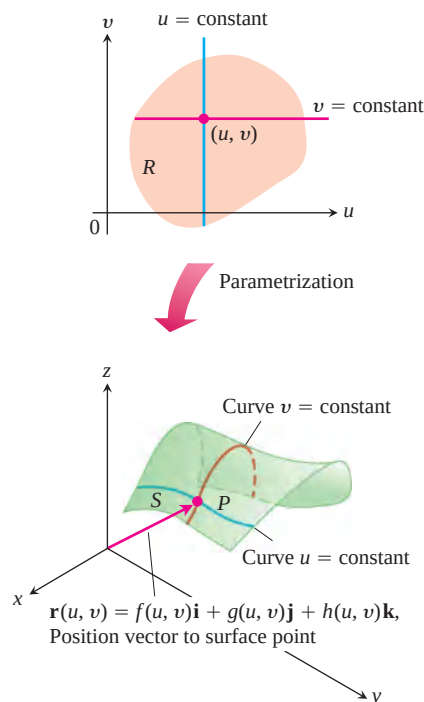


FIGURE 16.50 A parametrized surface S expressed as a vector function of two variables defined on a region R .

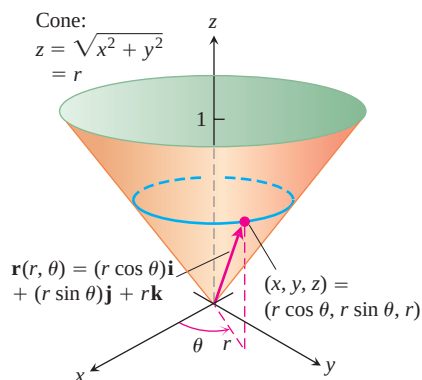


FIGURE 16.51 The cone in Example 1 can be parametrized using cylindrical coordinates.

To simplify our discussion, we take R to be a rectangle defined by inequalities of the form $a \leq u \leq b$, $c \leq v \leq d$. The requirement that $\mathbf{r}$ be one-to-one on the interior of R ensures that S does not cross itself. Notice that Equation (1) is the vector equivalent of *three* parametric equations:

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v).$$

EXAMPLE 1 Parametrizing a Cone

Find a parametrization of the cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \leq z \leq 1.$$

Solution Here, cylindrical coordinates provide everything we need. A typical point (x, y, z) on the cone (Figure 16.51) has $x = r \cos \theta$, $y = r \sin \theta$, and $z = \sqrt{x^2 + y^2} = r$, with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Taking $u = r$ and $v = \theta$ in Equation (1) gives the parametrization

$$\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

EXAMPLE 2 Parametrizing a Sphere

Find a parametrization of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution Spherical coordinates provide what we need. A typical point (x, y, z) on the sphere (Figure 16.52) has $x = a \sin \phi \cos \theta$, $y = a \sin \phi \sin \theta$, and $z = a \cos \phi$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$. Taking $u = \phi$ and $v = \theta$ in Equation (1) gives the parametrization

$$\mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k},$$

$$0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.$$

EXAMPLE 3 Parametrizing a Cylinder

Find a parametrization of the cylinder

$$x^2 + (y - 3)^2 = 9, \quad 0 \leq z \leq 5.$$

Solution In cylindrical coordinates, a point (x, y, z) has $x = r \cos \theta$, $y = r \sin \theta$, and $z = z$. For points on the cylinder $x^2 + (y - 3)^2 = 9$ (Figure 16.53), the equation is the same as the polar equation for the cylinder's base in the xy -plane:

$$x^2 + (y^2 - 6y + 9) = 9$$

$$r^2 - 6r \sin \theta = 0$$

or

$$r = 6 \sin \theta, \quad 0 \leq \theta \leq \pi.$$

A typical point on the cylinder therefore has

$$x = r \cos \theta = 6 \sin \theta \cos \theta = 3 \sin 2\theta$$

$$y = r \sin \theta = 6 \sin^2 \theta$$

$$z = z.$$



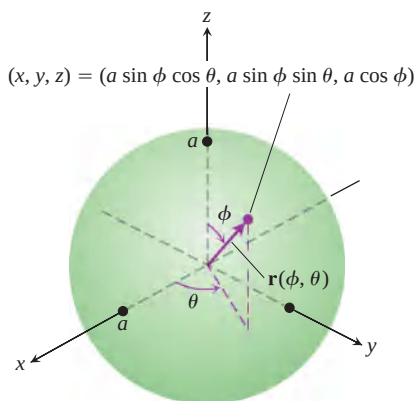


FIGURE 16.52 The sphere in Example 2 can be parametrized using spherical coordinates.

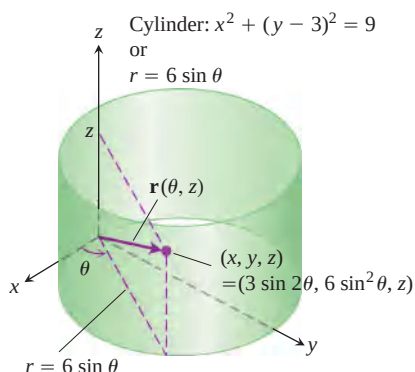


FIGURE 16.53 The cylinder in Example 3 can be parametrized using cylindrical coordinates.

Taking $u = \theta$ and $v = z$ in Equation (1) gives the parametrization

$$\mathbf{r}(\theta, z) = (3 \sin 2\theta)\mathbf{i} + (6 \sin^2 \theta)\mathbf{j} + z\mathbf{k}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq z \leq 5.$$

Surface Area

Our goal is to find a double integral for calculating the area of a curved surface S based on the parametrization

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \quad a \leq u \leq b, \quad c \leq v \leq d.$$

We need S to be smooth for the construction we are about to carry out. The definition of smoothness involves the partial derivatives of $\mathbf{r}$ with respect to u and v :

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

DEFINITION Smooth Parametrized Surface

A parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the parameter domain.

The condition that $\mathbf{r}_u \times \mathbf{r}_v$ is never the zero vector in the definition of smoothness means that the two vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and never lie along the same line, so they always determine a plane tangent to the surface.

Now consider a small rectangle ΔA_{uv} in R with sides on the lines $u = u_0$, $u = u_0 + \Delta u$, $v = v_0$ and $v = v_0 + \Delta v$ (Figure 16.54). Each side of ΔA_{uv} maps to a curve on the surface S , and together these four curves bound a “curved area element” $\Delta \sigma_{uv}$. In the notation of the figure, the side $v = v_0$ maps to curve C_1 , the side $u = u_0$ maps to C_2 , and their common vertex (u_0, v_0) maps to P_0 .

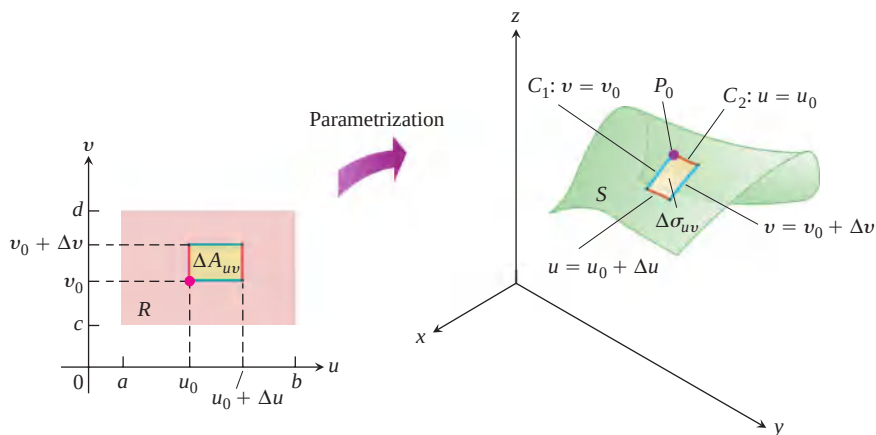


FIGURE 16.54 A rectangular area element ΔA_{uv} in the uv -plane maps onto a curved area element $\Delta \sigma_{uv}$ on S .

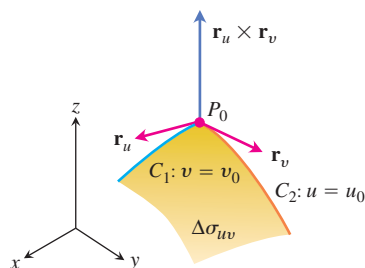


FIGURE 16.55 A magnified view of a surface area element $\Delta\sigma_{uv}$.

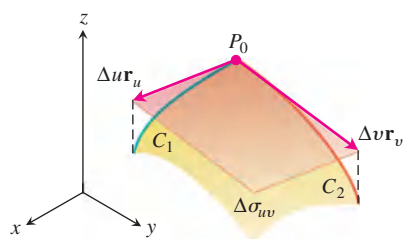


FIGURE 16.56 The parallelogram determined by the vectors $\Delta u \mathbf{r}_u$ and $\Delta v \mathbf{r}_v$ approximates the surface area element $\Delta\sigma_{uv}$.

Figure 16.55 shows an enlarged view of $\Delta\sigma_{uv}$. The vector $\mathbf{r}_u(u_0, v_0)$ is tangent to C_1 at P_0 . Likewise, $\mathbf{r}_v(u_0, v_0)$ is tangent to C_2 at P_0 . The cross product $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface at P_0 . (Here is where we begin to use the assumption that S is smooth. We want to be sure that $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$.)

We next approximate the surface element $\Delta\sigma_{uv}$ by the parallelogram on the tangent plane whose sides are determined by the vectors $\Delta u \mathbf{r}_u$ and $\Delta v \mathbf{r}_v$ (Figure 16.56). The area of this parallelogram is

$$|\Delta u \mathbf{r}_u \times \Delta v \mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v. \quad (2)$$

A partition of the region R in the uv -plane by rectangular regions ΔA_{uv} generates a partition of the surface S into surface area elements $\Delta\sigma_{uv}$. We approximate the area of each surface element $\Delta\sigma_{uv}$ by the parallelogram area in Equation (2) and sum these areas together to obtain an approximation of the area of S :

$$\sum_u \sum_v |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v. \quad (3)$$

As Δu and Δv approach zero independently, the continuity of $\mathbf{r}_u$ and $\mathbf{r}_v$ guarantees that the sum in Equation (3) approaches the double integral $\int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v| du dv$. This double integral defines the area of the surface S and agrees with previous definitions of area, though it is more general.

DEFINITION Area of a Smooth Surface

The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \quad a \leq u \leq b, \quad c \leq v \leq d$$

is

$$A = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v| du dv. \quad (4)$$

As in Section 16.5, we can abbreviate the integral in Equation (4) by writing $d\sigma$ for $|\mathbf{r}_u \times \mathbf{r}_v| du dv$.

Surface Area Differential and Differential Formula for Surface Area

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v| du dv \quad \iint_S d\sigma \quad (5)$$

Surface area
differential

Differential formula
for surface area



EXAMPLE 4 Finding Surface Area (Cone)

Find the surface area of the cone in Example 1 (Figure 16.51).

Solution In Example 1, we found the parametrization

$$\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

To apply Equation (4), we first find $\mathbf{r}_r \times \mathbf{r}_\theta$:

$$\begin{aligned}\mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\ &= -(r \cos \theta)\mathbf{i} - (r \sin \theta)\mathbf{j} + \underbrace{(r \cos^2 \theta + r \sin^2 \theta)}_r \mathbf{k}.\end{aligned}$$

Thus, $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} = \sqrt{2r^2} = \sqrt{2}r$. The area of the cone is

$$\begin{aligned}A &= \int_0^{2\pi} \int_0^1 |\mathbf{r}_r \times \mathbf{r}_\theta| \, dr \, d\theta \quad \text{Equation (4) with } u = r, v = \theta \\ &= \int_0^{2\pi} \int_0^1 \sqrt{2} r \, dr \, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2} d\theta = \frac{\sqrt{2}}{2} (2\pi) = \pi\sqrt{2} \text{ units squared.} \quad \blacksquare\end{aligned}$$

EXAMPLE 5 Finding Surface Area (Sphere)

Find the surface area of a sphere of radius a .

Solution We use the parametrization from Example 2:

$$\begin{aligned}\mathbf{r}(\phi, \theta) &= (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k}, \\ 0 &\leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi.\end{aligned}$$

For $\mathbf{r}_\phi \times \mathbf{r}_\theta$, we get

$$\begin{aligned}\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} \\ &= (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k}.\end{aligned}$$

Thus,

$$\begin{aligned}|\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \sin^2 \phi \cos^2 \phi} \\ &= \sqrt{a^4 \sin^4 \phi + a^4 \sin^2 \phi \cos^2 \phi} = \sqrt{a^4 \sin^2 \phi (\sin^2 \phi + \cos^2 \phi)} \\ &= a^2 \sqrt{\sin^2 \phi} = a^2 \sin \phi,\end{aligned}$$

since $\sin \phi \geq 0$ for $0 \leq \phi \leq \pi$. Therefore, the area of the sphere is

$$\begin{aligned}A &= \int_0^{2\pi} \int_0^\pi a^2 \sin \phi \, d\phi \, d\theta \\ &= \int_0^{2\pi} \left[-a^2 \cos \phi \right]_0^\pi d\theta = \int_0^{2\pi} 2a^2 \, d\theta = 4\pi a^2 \text{ units squared.}\end{aligned}$$

This agrees with the well-known formula for the surface area of a sphere. ■

Surface Integrals

Having found a formula for calculating the area of a parametrized surface, we can now integrate a function over the surface using the parametrized form.

DEFINITION Parametric Surface Integral

If S is a smooth surface defined parametrically as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $a \leq u \leq b$, $c \leq v \leq d$, and $G(x, y, z)$ is a continuous function defined on S , then the **integral of G over S** is

$$\iint_S G(x, y, z) d\sigma = \int_c^d \int_a^b G(f(u, v), g(u, v), h(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| du dv.$$

**EXAMPLE 6** Integrating Over a Surface Defined Parametrically

Integrate $G(x, y, z) = x^2$ over the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$.

Solution Continuing the work in Examples 1 and 4, we have $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r$ and

$$\begin{aligned} \iint_S x^2 d\sigma &= \int_0^{2\pi} \int_0^1 (r^2 \cos^2 \theta)(\sqrt{2}r) dr d\theta && x = r \cos \theta \\ &= \sqrt{2} \int_0^{2\pi} \int_0^1 r^3 \cos^2 \theta dr d\theta \\ &= \frac{\sqrt{2}}{4} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{\sqrt{2}}{4} \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{\pi\sqrt{2}}{4}. \end{aligned}$$

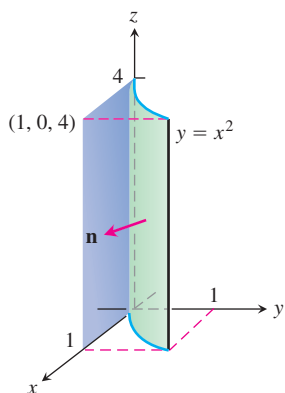


FIGURE 16.57 Finding the flux through the surface of a parabolic cylinder (Example 7).

**EXAMPLE 7** Finding Flux

Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ outward through the parabolic cylinder $y = x^2$, $0 \leq x \leq 1$, $0 \leq z \leq 4$ (Figure 16.57).

Solution On the surface we have $x = x$, $y = x^2$, and $z = z$, so we automatically have the parametrization $\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $0 \leq x \leq 1$, $0 \leq z \leq 4$. The cross product of tangent vectors is

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j}.$$

The unit normal pointing outward from the surface is

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} = \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2 + 1}}.$$

On the surface, $y = x^2$, so the vector field there is

$$\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}.$$

Thus,

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= \frac{1}{\sqrt{4x^2 + 1}} ((x^2z)(2x) + (x)(-1) + (-z^2)(0)) \\ &= \frac{2x^3z - x}{\sqrt{4x^2 + 1}}. \end{aligned}$$

The flux of $\mathbf{F}$ outward through the surface is

$$\begin{aligned}
 \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \int_0^4 \int_0^1 \frac{2x^3z - x}{\sqrt{4x^2 + 1}} |\mathbf{r}_x \times \mathbf{r}_z| \, dx \, dz \\
 &= \int_0^4 \int_0^1 \frac{2x^3z - x}{\sqrt{4x^2 + 1}} \sqrt{4x^2 + 1} \, dx \, dz \\
 &= \int_0^4 \int_0^1 (2x^3z - x) \, dx \, dz = \int_0^4 \left[\frac{1}{2}x^4z - \frac{1}{2}x^2 \right]_{x=0}^{x=1} dz \\
 &= \int_0^4 \frac{1}{2}(z - 1) \, dz = \frac{1}{4}(z - 1)^2 \Big|_0^4 \\
 &= \frac{1}{4}(9) - \frac{1}{4}(1) = 2.
 \end{aligned}$$

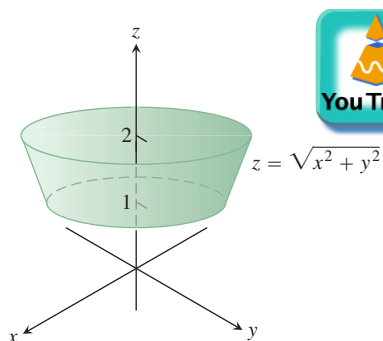


FIGURE 16.58 The cone frustum formed when the cone $z = \sqrt{x^2 + y^2}$ is cut by the planes $z = 1$ and $z = 2$ (Example 8).



EXAMPLE 8 Finding a Center of Mass

Find the center of mass of a thin shell of constant density δ cut from the cone $z = \sqrt{x^2 + y^2}$ by the planes $z = 1$ and $z = 2$ (Figure 16.58).

Solution The symmetry of the surface about the z -axis tells us that $\bar{x} = \bar{y} = 0$. We find $\bar{z} = M_{xy}/M$. Working as in Examples 1 and 4, we have

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r \mathbf{k}, \quad 1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi,$$

and

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r.$$

Therefore,

$$\begin{aligned}
 M &= \iint_S \delta \, d\sigma = \int_0^{2\pi} \int_1^2 \delta \sqrt{2}r \, dr \, d\theta \\
 &= \delta \sqrt{2} \int_0^{2\pi} \left[\frac{r^2}{2} \right]_1^2 d\theta = \delta \sqrt{2} \int_0^{2\pi} \left(2 - \frac{1}{2} \right) d\theta \\
 &= \delta \sqrt{2} \left[\frac{3\theta}{2} \right]_0^{2\pi} = 3\pi\delta\sqrt{2} \\
 M_{xy} &= \iint_S \delta z \, d\sigma = \int_0^{2\pi} \int_1^2 \delta r \sqrt{2}r \, dr \, d\theta \\
 &= \delta \sqrt{2} \int_0^{2\pi} \int_1^2 r^2 \, dr \, d\theta = \delta \sqrt{2} \int_0^{2\pi} \left[\frac{r^3}{3} \right]_1^2 d\theta \\
 &= \delta \sqrt{2} \int_0^{2\pi} \frac{7}{3} d\theta = \frac{14}{3} \pi \delta \sqrt{2} \\
 \bar{z} &= \frac{M_{xy}}{M} = \frac{14\pi\delta\sqrt{2}}{3(3\pi\delta\sqrt{2})} = \frac{14}{9}.
 \end{aligned}$$

The shell's center of mass is the point $(0, 0, 14/9)$.

EXERCISES 16.6

Finding Parametrizations for Surfaces

In Exercises 1–16, find a parametrization of the surface. (There are many correct ways to do these, so your answers may not be the same as those in the back of the book.)

1. The paraboloid $z = x^2 + y^2, z \leq 4$
2. The paraboloid $z = 9 - x^2 - y^2, z \geq 0$
3. **Cone frustum** The first-octant portion of the cone $z = \sqrt{x^2 + y^2}/2$ between the planes $z = 0$ and $z = 3$
4. **Cone frustum** The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$
5. **Spherical cap** The cap cut from the sphere $x^2 + y^2 + z^2 = 9$ by the cone $z = \sqrt{x^2 + y^2}$
6. **Spherical cap** The portion of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant between the xy -plane and the cone $z = \sqrt{x^2 + y^2}$
7. **Spherical band** The portion of the sphere $x^2 + y^2 + z^2 = 3$ between the planes $z = \sqrt{3}/2$ and $z = -\sqrt{3}/2$
8. **Spherical cap** The upper portion cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $z = -2$
9. **Parabolic cylinder between planes** The surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0, x = 2$, and $z = 0$
10. **Parabolic cylinder between planes** The surface cut from the parabolic cylinder $y = x^2$ by the planes $z = 0, z = 3$ and $y = 2$
11. **Circular cylinder band** The portion of the cylinder $y^2 + z^2 = 9$ between the planes $x = 0$ and $x = 3$
12. **Circular cylinder band** The portion of the cylinder $x^2 + z^2 = 4$ above the xy -plane between the planes $y = -2$ and $y = 2$
13. **Tilted plane inside cylinder** The portion of the plane $x + y + z = 1$
 - a. Inside the cylinder $x^2 + y^2 = 9$
 - b. Inside the cylinder $y^2 + z^2 = 9$
14. **Tilted plane inside cylinder** The portion of the plane $x - y + 2z = 2$
 - a. Inside the cylinder $x^2 + z^2 = 3$
 - b. Inside the cylinder $y^2 + z^2 = 2$
15. **Circular cylinder band** The portion of the cylinder $(x - 2)^2 + z^2 = 4$ between the planes $y = 0$ and $y = 3$
16. **Circular cylinder band** The portion of the cylinder $y^2 + (z - 5)^2 = 25$ between the planes $x = 0$ and $x = 10$

Areas of Parametrized Surfaces

In Exercises 17–26, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral. (There are

many correct ways to set up the integrals, so your integrals may not be the same as those in the back of the book. They should have the same values, however.)

17. **Titled plane inside cylinder** The portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$
18. **Plane inside cylinder** The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$
19. **Cone frustum** The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$
20. **Cone frustum** The portion of the cone $z = \sqrt{x^2 + y^2}/3$ between the planes $z = 1$ and $z = 4/3$
21. **Circular cylinder band** The portion of the cylinder $x^2 + y^2 = 1$ between the planes $z = 1$ and $z = 4$
22. **Circular cylinder band** The portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$
23. **Parabolic cap** The cap cut from the paraboloid $z = 2 - x^2 - y^2$ by the cone $z = \sqrt{x^2 + y^2}$
24. **Parabolic band** The portion of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$
25. **Sawed-off sphere** The lower portion cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$
26. **Spherical band** The portion of the sphere $x^2 + y^2 + z^2 = 4$ between the planes $z = -1$ and $z = \sqrt{3}$

Integrals Over Parametrized Surfaces

In Exercises 27–34, integrate the given function over the given surface.

27. **Parabolic cylinder** $G(x, y, z) = x$, over the parabolic cylinder $y = x^2, 0 \leq x \leq 2, 0 \leq z \leq 3$
28. **Circular cylinder** $G(x, y, z) = z$, over the cylindrical surface $y^2 + z^2 = 4, z \geq 0, 1 \leq x \leq 4$
29. **Sphere** $G(x, y, z) = x^2$, over the unit sphere $x^2 + y^2 + z^2 = 1$
30. **Hemisphere** $G(x, y, z) = z^2$, over the hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$
31. **Portion of plane** $F(x, y, z) = z$, over the portion of the plane $x + y + z = 4$ that lies above the square $0 \leq x \leq 1, 0 \leq y \leq 1$, in the xy -plane
32. **Cone** $F(x, y, z) = z - x$, over the cone $z = \sqrt{x^2 + y^2}, 0 \leq z \leq 1$
33. **Parabolic dome** $H(x, y, z) = x^2\sqrt{5 - 4z}$, over the parabolic dome $z = 1 - x^2 - y^2, z \geq 0$
34. **Spherical cap** $H(x, y, z) = yz$, over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$



Flux Across Parametrized Surfaces

In Exercises 35–44, use a parametrization to find the flux $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$ across the surface in the given direction.

- 35. Parabolic cylinder** $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x -axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$
- 36. Parabolic cylinder** $\mathbf{F} = x^2\mathbf{j} - xz\mathbf{k}$ outward (normal away from the yz -plane) through the surface cut from the parabolic cylinder $y = x^2$, $-1 \leq x \leq 1$, by the planes $z = 0$ and $z = 2$
- 37. Sphere** $\mathbf{F} = z\mathbf{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin
- 38. Sphere** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$ in the direction away from the origin
- 39. Plane** $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ upward across the portion of the plane $x + y + z = 2a$ that lies above the square $0 \leq x \leq a$, $0 \leq y \leq a$, in the xy -plane
- 40. Cylinder** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through the portion of the cylinder $x^2 + y^2 = 1$ cut by the planes $z = 0$ and $z = a$
- 41. Cone** $\mathbf{F} = xy\mathbf{i} - z\mathbf{k}$ outward (normal away from the z -axis) through the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$
- 42. Cone** $\mathbf{F} = y^2\mathbf{i} + xz\mathbf{j} - \mathbf{k}$ outward (normal away from the z -axis) through the cone $z = 2\sqrt{x^2 + y^2}$, $0 \leq z \leq 2$
- 43. Cone frustum** $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$ outward (normal away from the z -axis) through the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$
- 44. Paraboloid** $\mathbf{F} = 4x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k}$ outward (normal way from the z -axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$

Moments and Masses

- 45.** Find the centroid of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.
- 46.** Find the center of mass and the moment of inertia and radius of gyration about the z -axis of a thin shell of constant density δ cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$.
- 47.** Find the moment of inertia about the z -axis of a thin spherical shell $x^2 + y^2 + z^2 = a^2$ of constant density δ .
- 48.** Find the moment of inertia about the z -axis of a thin conical shell $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$, of constant density δ .

Planes Tangent to Parametrized Surfaces

The tangent plane at a point $P_0(f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ on a parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is the plane through P_0 normal to the vector $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$, the cross product of the tangent vectors $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ at P_0 . In Exercises 49–52, find an equation for the plane tangent to the surface at P_0 . Then find a Cartesian equation for the surface and sketch the surface and tangent plane together.

- 49. Cone** The cone $\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}$, $r \geq 0$, $0 \leq \theta \leq 2\pi$ at the point $P_0(\sqrt{2}, \sqrt{2}, 2)$ corresponding to $(r, \theta) = (2, \pi/4)$
- 50. Hemisphere** The hemisphere surface $\mathbf{r}(\phi, \theta) = (4 \sin \phi \cos \theta)\mathbf{i} + (4 \sin \phi \sin \theta)\mathbf{j} + (4 \cos \phi)\mathbf{k}$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$, at the point $P_0(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$ corresponding to $(\phi, \theta) = (\pi/6, \pi/4)$
- 51. Circular cylinder** The circular cylinder $\mathbf{r}(\theta, z) = (3 \sin 2\theta)\mathbf{i} + (6 \sin^2 \theta)\mathbf{j} + z\mathbf{k}$, $0 \leq \theta \leq \pi$, at the point $P_0(3\sqrt{3}/2, 9/2, 0)$ corresponding to $(\theta, z) = (\pi/3, 0)$ (See Example 3.)
- 52. Parabolic cylinder** The parabolic cylinder surface $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$, $-\infty < x < \infty$, $-\infty < y < \infty$, at the point $P_0(1, 2, -1)$ corresponding to $(x, y) = (1, 2)$

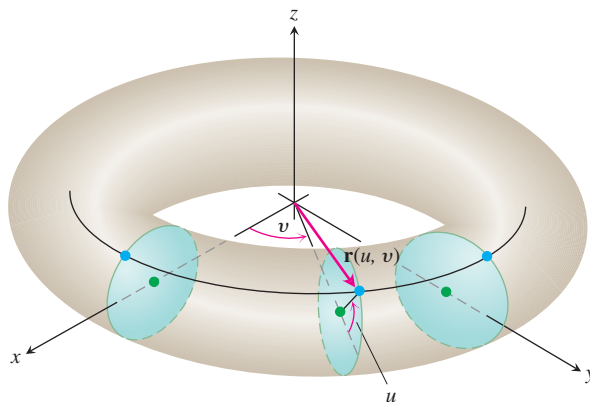
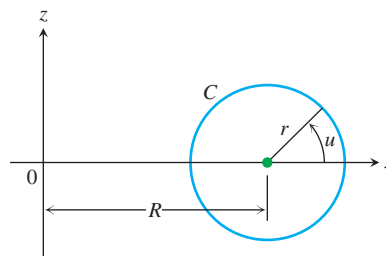
Further Examples of Parametrizations

- 53. a.** A *torus of revolution* (doughnut) is obtained by rotating a circle C in the xz -plane about the z -axis in space. (See the accompanying figure.) If C has radius $r > 0$ and center $(R, 0, 0)$, show that a parametrization of the torus is

$$\mathbf{r}(u, v) = ((R + r \cos u) \cos v)\mathbf{i} + ((R + r \cos u) \sin v)\mathbf{j} + (r \sin u)\mathbf{k},$$

where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$ are the angles in the figure.

- b.** Show that the surface area of the torus is $A = 4\pi^2 Rr$.

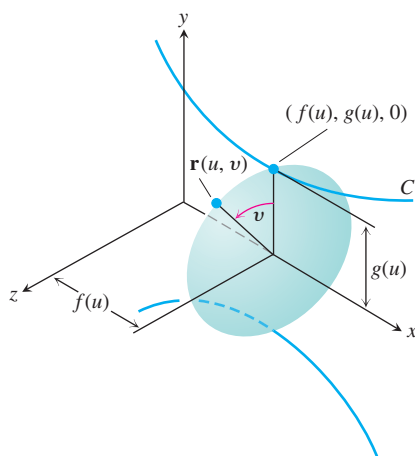


54. Parametrization of a surface of revolution Suppose that the parametrized curve $C: (f(u), g(u))$ is revolved about the x -axis, where $g(u) > 0$ for $a \leq u \leq b$.

a. Show that

$$\mathbf{r}(u, v) = f(u)\mathbf{i} + (g(u)\cos v)\mathbf{j} + (g(u)\sin v)\mathbf{k}$$

is a parametrization of the resulting surface of revolution, where $0 \leq v \leq 2\pi$ is the angle from the xy -plane to the point $\mathbf{r}(u, v)$ on the surface. (See the accompanying figure.) Notice that $f(u)$ measures distance *along* the axis of revolution and $g(u)$ measures distance *from* the axis of revolution.



b. Find a parametrization for the surface obtained by revolving the curve $x = y^2, y \geq 0$, about the x -axis.

55. a. Parametrization of an ellipsoid Recall the parametrization $x = a \cos \theta, y = b \sin \theta, 0 \leq \theta \leq 2\pi$ for the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ (Section 3.5, Example 13). Using the angles θ and ϕ in spherical coordinates, show that

$$\mathbf{r}(\theta, \phi) = (a \cos \theta \cos \phi)\mathbf{i} + (b \sin \theta \cos \phi)\mathbf{j} + (c \sin \phi)\mathbf{k}$$

is a parametrization of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.

b. Write an integral for the surface area of the ellipsoid, but do not evaluate the integral.

56. Hyperboloid of one sheet

a. Find a parametrization for the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ in terms of the angle θ associated with the circle $x^2 + y^2 = r^2$ and the hyperbolic parameter u associated with the hyperbolic function $r^2 - z^2 = 1$. (See Section 7.8, Exercise 84.)

b. Generalize the result in part (a) to the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.

57. (Continuation of Exercise 56.) Find a Cartesian equation for the plane tangent to the hyperboloid $x^2 + y^2 - z^2 = 25$ at the point $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$.

58. Hyperboloid of two sheets Find a parametrization of the hyperboloid of two sheets $(z^2/c^2) - (x^2/a^2) - (y^2/b^2) = 1$.

16.7

Stokes' Theorem

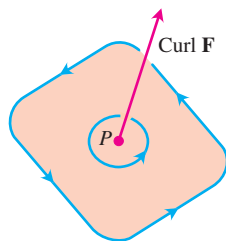


FIGURE 16.59 The circulation vector at a point P in a plane in a three-dimensional fluid flow. Notice its right-hand relation to the circulation line.

As we saw in Section 16.4, the circulation density or curl component of a two-dimensional field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at a point (x, y) is described by the scalar quantity $(\partial N/\partial x - \partial M/\partial y)$. In three dimensions, the circulation around a point P in a plane is described with a vector. This vector is normal to the plane of the circulation (Figure 16.59) and points in the direction that gives it a right-hand relation to the circulation line. The length of the vector gives the rate of the fluid's rotation, which usually varies as the circulation plane is tilted about P . It turns out that the vector of greatest circulation in a flow with velocity field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the **curl vector**

$$\text{curl } \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}. \quad (1)$$

We get this information from Stokes' Theorem, the generalization of the circulation-curl form of Green's Theorem to space.

Notice that $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = (\partial N/\partial x - \partial M/\partial y)$ is consistent with our definition in Section 16.4 when $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. The formula for $\text{curl } \mathbf{F}$ in Equation (1) is often written using the symbolic operator

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}. \quad (2)$$

(The symbol ∇ is pronounced “del.”) The curl of $\mathbf{F}$ is $\nabla \times \mathbf{F}$:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \\ &= \text{curl } \mathbf{F}.\end{aligned}$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} \quad (3)$$

EXAMPLE 1 Finding Curl $\mathbf{F}$

Find the curl of $\mathbf{F} = (x^2 - y)\mathbf{i} + 4z\mathbf{j} + x^2\mathbf{k}$.

Solution

$$\begin{aligned}\text{curl } \mathbf{F} &= \nabla \times \mathbf{F} && \text{Equation (3)} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y & 4z & x^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(4z) \right) \mathbf{i} - \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(x^2 - y) \right) \mathbf{j} \\ &\quad + \left(\frac{\partial}{\partial x}(4z) - \frac{\partial}{\partial y}(x^2 - y) \right) \mathbf{k} \\ &= (0 - 4)\mathbf{i} - (2x - 0)\mathbf{j} + (0 + 1)\mathbf{k} \\ &= -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k}\end{aligned}$$

As we will see, the operator ∇ has a number of other applications. For instance, when applied to a scalar function $f(x, y, z)$, it gives the gradient of f :

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$

This may now be read as “del f ” as well as “grad f .”

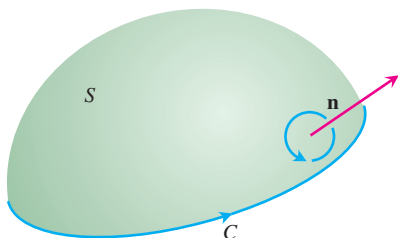


FIGURE 16.60 The orientation of the bounding curve C gives it a right-handed relation to the normal field $\mathbf{n}$.

Stokes' Theorem

Stokes' Theorem says that, under conditions normally met in practice, the circulation of a vector field around the boundary of an oriented surface in space in the direction counter-clockwise with respect to the surface's unit normal vector field $\mathbf{n}$ (Figure 16.60) equals the integral of the normal component of the curl of the field over the surface.

THEOREM 5 Stokes' Theorem

The circulation of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ around the boundary C of an oriented surface S in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S .

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma \quad (4)$$

Counterclockwise
circulation Curl integral

Notice from Equation (4) that if two different oriented surfaces S_1 and S_2 have the same boundary C , their curl integrals are equal:

$$\iint_{S_1} \nabla \times \mathbf{F} \cdot \mathbf{n}_1 \, d\sigma = \iint_{S_2} \nabla \times \mathbf{F} \cdot \mathbf{n}_2 \, d\sigma.$$

Both curl integrals equal the counterclockwise circulation integral on the left side of Equation (4) as long as the unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ correctly orient the surfaces.

Naturally, we need some mathematical restrictions on $\mathbf{F}$, C , and S to ensure the existence of the integrals in Stokes' equation. The usual restrictions are that all functions, vector fields, and their derivatives be continuous.

If C is a curve in the xy -plane, oriented counterclockwise, and R is the region in the xy -plane bounded by C , then $d\sigma = dx \, dy$ and

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = (\nabla \times \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right).$$

Under these conditions, Stokes' equation becomes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy,$$

which is the circulation-curl form of the equation in Green's Theorem. Conversely, by reversing these steps we can rewrite the circulation-curl form of Green's Theorem for two-dimensional fields in del notation as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA. \quad (5)$$

See Figure 16.61.

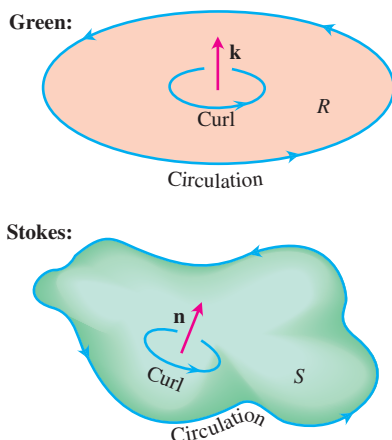


FIGURE 16.61 Comparison of Green's Theorem and Stokes' Theorem.

**EXAMPLE 2** Verifying Stokes' Equation for a Hemisphere

Evaluate Equation (4) for the hemisphere $S: x^2 + y^2 + z^2 = 9, z \geq 0$, its bounding circle $C: x^2 + y^2 = 9, z = 0$, and the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

Solution We calculate the counterclockwise circulation around C (as viewed from above) using the parametrization $\mathbf{r}(\theta) = (3 \cos \theta)\mathbf{i} + (3 \sin \theta)\mathbf{j}$, $0 \leq \theta \leq 2\pi$:

$$d\mathbf{r} = (-3 \sin \theta d\theta)\mathbf{i} + (3 \cos \theta d\theta)\mathbf{j}$$

$$\mathbf{F} = y\mathbf{i} - x\mathbf{j} = (3 \sin \theta)\mathbf{i} - (3 \cos \theta)\mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = -9 \sin^2 \theta d\theta - 9 \cos^2 \theta d\theta = -9 d\theta$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -9 d\theta = -18\pi.$$

For the curl integral of $\mathbf{F}$, we have

$$\nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k}$$

$$= (0 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (-1 - 1)\mathbf{k} = -2\mathbf{k}$$

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{3} \quad \text{Outer unit normal}$$

$$d\sigma = \frac{3}{z} dA$$

Section 16.5, Example 5,
with $a = 3$

$$\nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = -\frac{2z}{3} \frac{3}{z} dA = -2 dA$$

and

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{x^2+y^2 \leq 9} -2 dA = -18\pi.$$

The circulation around the circle equals the integral of the curl over the hemisphere, as it should. ■



EXAMPLE 3 Finding Circulation

Find the circulation of the field $\mathbf{F} = (x^2 - y)\mathbf{i} + 4z\mathbf{j} + x^2\mathbf{k}$ around the curve C in which the plane $z = 2$ meets the cone $z = \sqrt{x^2 + y^2}$, counterclockwise as viewed from above (Figure 16.62).

Solution Stokes' Theorem enables us to find the circulation by integrating over the surface of the cone. Traversing C in the counterclockwise direction viewed from above corresponds to taking the *inner* normal $\mathbf{n}$ to the cone, the normal with a positive z -component.

We parametrize the cone as

$$\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}, \quad 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi.$$

We then have

$$\begin{aligned} \mathbf{n} &= \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|} = \frac{-(r \cos \theta)\mathbf{i} - (r \sin \theta)\mathbf{j} + r\mathbf{k}}{r\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \left(-(\cos \theta)\mathbf{i} - (\sin \theta)\mathbf{j} + \mathbf{k} \right) \end{aligned} \quad \begin{array}{l} \text{Section 16.6,} \\ \text{Example 4} \end{array}$$

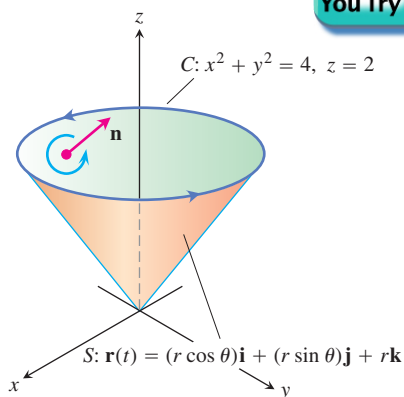


FIGURE 16.62 The curve C and cone S in Example 3.

$$\begin{aligned}
 d\sigma &= r\sqrt{2} \, dr \, d\theta && \text{Section 16.6, Example 4} \\
 \nabla \times \mathbf{F} &= -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k} && \text{Example 1} \\
 &= -4\mathbf{i} - 2r \cos \theta \mathbf{j} + \mathbf{k}. && x = r \cos \theta
 \end{aligned}$$

Accordingly,

$$\begin{aligned}
 \nabla \times \mathbf{F} \cdot \mathbf{n} &= \frac{1}{\sqrt{2}} \left(4 \cos \theta + 2r \cos \theta \sin \theta + 1 \right) \\
 &= \frac{1}{\sqrt{2}} \left(4 \cos \theta + r \sin 2\theta + 1 \right)
 \end{aligned}$$

and the circulation is

$$\begin{aligned}
 \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma && \text{Stokes' Theorem, Equation (4)} \\
 &= \int_0^{2\pi} \int_0^2 \frac{1}{\sqrt{2}} \left(4 \cos \theta + r \sin 2\theta + 1 \right) (r\sqrt{2} \, dr \, d\theta) = 4\pi. \quad \blacksquare
 \end{aligned}$$

Paddle Wheel Interpretation of $\nabla \times \mathbf{F}$

Suppose that $\mathbf{v}(x, y, z)$ is the velocity of a moving fluid whose density at (x, y, z) is $\delta(x, y, z)$ and let $\mathbf{F} = \delta \mathbf{v}$. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is the circulation of the fluid around the closed curve C . By Stokes' Theorem, the circulation is equal to the flux of $\nabla \times \mathbf{F}$ through a surface S spanning C :

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

Suppose we fix a point Q in the domain of $\mathbf{F}$ and a direction $\mathbf{u}$ at Q . Let C be a circle of radius ρ , with center at Q , whose plane is normal to $\mathbf{u}$. If $\nabla \times \mathbf{F}$ is continuous at Q , the average value of the $\mathbf{u}$ -component of $\nabla \times \mathbf{F}$ over the circular disk S bounded by C approaches the $\mathbf{u}$ -component of $\nabla \times \mathbf{F}$ at Q as $\rho \rightarrow 0$:

$$(\nabla \times \mathbf{F} \cdot \mathbf{u})_Q = \lim_{\rho \rightarrow 0} \frac{1}{\pi \rho^2} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{u} \, d\sigma.$$

If we replace the surface integral in this last equation by the circulation, we get

$$(\nabla \times \mathbf{F} \cdot \mathbf{u})_Q = \lim_{\rho \rightarrow 0} \frac{1}{\pi \rho^2} \oint_C \mathbf{F} \cdot d\mathbf{r}. \quad (6)$$

The left-hand side of Equation (6) has its maximum value when $\mathbf{u}$ is the direction of $\nabla \times \mathbf{F}$. When ρ is small, the limit on the right-hand side of Equation (6) is approximately

$$\frac{1}{\pi \rho^2} \oint_C \mathbf{F} \cdot d\mathbf{r},$$

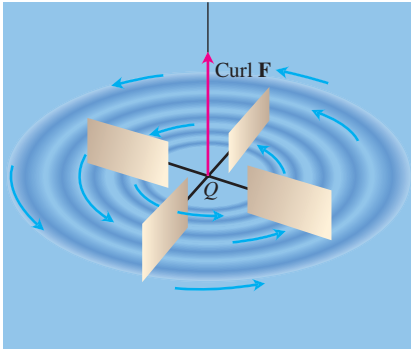


FIGURE 16.63 The paddle wheel interpretation of $\text{curl } \mathbf{F}$.

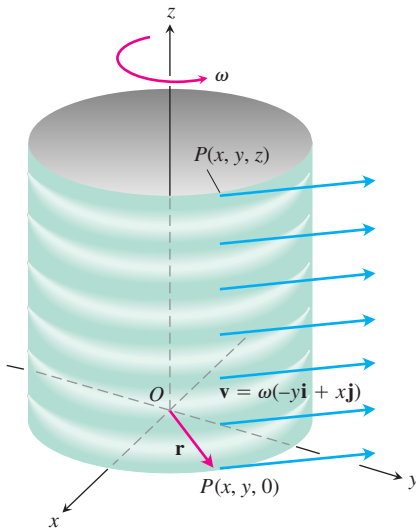


FIGURE 16.64 A steady rotational flow parallel to the xy -plane, with constant angular velocity ω in the positive (counterclockwise) direction (Example 4).

which is the circulation around C divided by the area of the disk (circulation density). Suppose that a small paddle wheel of radius ρ is introduced into the fluid at Q , with its axle directed along $\mathbf{u}$. The circulation of the fluid around C will affect the rate of spin of the paddle wheel. The wheel will spin fastest when the circulation integral is maximized; therefore it will spin fastest when the axle of the paddle wheel points in the direction of $\nabla \times \mathbf{F}$ (Figure 16.63).

EXAMPLE 4 Relating $\nabla \times \mathbf{F}$ to Circulation Density

A fluid of constant density rotates around the z -axis with velocity $\mathbf{v} = \omega(-y\mathbf{i} + x\mathbf{j})$, where ω is a positive constant called the *angular velocity* of the rotation (Figure 16.64). If $\mathbf{F} = \mathbf{v}$, find $\nabla \times \mathbf{F}$ and relate it to the circulation density.

Solution With $\mathbf{F} = \mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$,

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) \mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right) \mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \mathbf{k} \\ &= (0 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (\omega - (-\omega))\mathbf{k} = 2\omega\mathbf{k}.\end{aligned}$$

By Stokes' Theorem, the circulation of $\mathbf{F}$ around a circle C of radius ρ bounding a disk S in a plane normal to $\nabla \times \mathbf{F}$, say the xy -plane, is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S 2\omega\mathbf{k} \cdot \mathbf{k} \, dx \, dy = (2\omega)(\pi\rho^2).$$

Thus,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = 2\omega = \frac{1}{\pi\rho^2} \oint_C \mathbf{F} \cdot d\mathbf{r},$$

consistent with Equation (6) when $\mathbf{u} = \mathbf{k}$. ■

EXAMPLE 5 Applying Stokes' Theorem

Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F} = xz\mathbf{i} + xy\mathbf{j} + 3xz\mathbf{k}$ and C is the boundary of the portion of the plane $2x + y + z = 2$ in the first octant, traversed counterclockwise as viewed from above (Figure 16.65).

Solution The plane is the level surface $f(x, y, z) = 2$ of the function $f(x, y, z) = 2x + y + z$. The unit normal vector

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{(2\mathbf{i} + \mathbf{j} + \mathbf{k})}{|2\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

is consistent with the counterclockwise motion around C . To apply Stokes' Theorem, we find

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & 3xz \end{vmatrix} = (x - 3z)\mathbf{j} + y\mathbf{k}.$$

On the plane, z equals $2 - 2x - y$, so

$$\nabla \times \mathbf{F} = (x - 3(2 - 2x - y))\mathbf{j} + y\mathbf{k} = (7x + 3y - 6)\mathbf{j} + y\mathbf{k}$$



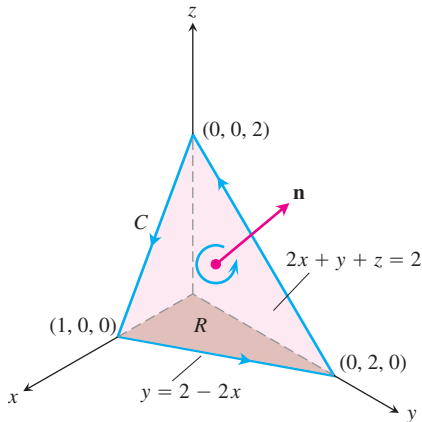


FIGURE 16.65 The planar surface in Example 5.

and

$$\nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{6}} (7x + 3y - 6 + y) = \frac{1}{\sqrt{6}} (7x + 4y - 6).$$

The surface area element is

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|} dA = \frac{\sqrt{6}}{1} dx dy.$$

The circulation is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma && \text{Stokes' Theorem, Equation (4)} \\ &= \int_0^1 \int_0^{2-2x} \frac{1}{\sqrt{6}} (7x + 4y - 6) \sqrt{6} dy dx \\ &= \int_0^1 \int_0^{2-2x} (7x + 4y - 6) dy dx = -1. \end{aligned}$$

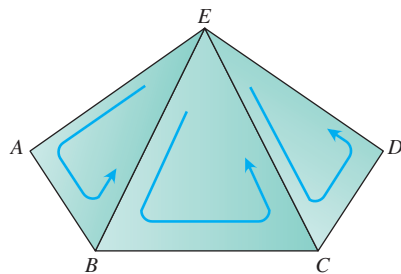


FIGURE 16.66 Part of a polyhedral surface.

Proof of Stokes' Theorem for Polyhedral Surfaces

Let S be a polyhedral surface consisting of a finite number of plane regions. (See Figure 16.66 for an example.) We apply Green's Theorem to each separate panel of S . There are two types of panels:

1. Those that are surrounded on all sides by other panels
2. Those that have one or more edges that are not adjacent to other panels.

The boundary Δ of S consists of those edges of the type 2 panels that are not adjacent to other panels. In Figure 16.66, the triangles EAB , BCE , and CDE represent a part of S , with $ABCDE$ part of the boundary Δ . Applying Green's Theorem to the three triangles in turn and adding the results, we get

$$\left(\oint_{EAB} + \oint_{BCE} + \oint_{CDE} \right) \mathbf{F} \cdot d\mathbf{r} = \left(\iint_{EAB} + \iint_{BCE} + \iint_{CDE} \right) \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma. \quad (7)$$

The three line integrals on the left-hand side of Equation (7) combine into a single line integral taken around the periphery $ABCDE$ because the integrals along interior segments cancel in pairs. For example, the integral along segment BE in triangle ABE is opposite in sign to the integral along the same segment in triangle EBC . The same holds for segment CE . Hence, Equation (7) reduces to

$$\oint_{ABCDE} \mathbf{F} \cdot d\mathbf{r} = \iint_{ABCDE} \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma.$$

When we apply Green's Theorem to all the panels and add the results, we get

$$\oint_{\Delta} \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma.$$

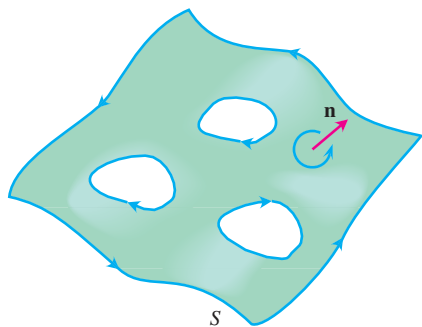


FIGURE 16.67 Stokes' Theorem also holds for oriented surfaces with holes.

This is Stokes' Theorem for a polyhedral surface S . You can find proofs for more general surfaces in advanced calculus texts.

Stokes' Theorem for Surfaces with Holes

Stokes' Theorem can be extended to an oriented surface S that has one or more holes (Figure 16.67), in a way analogous to the extension of Green's Theorem: The surface integral over S of the normal component of $\nabla \times \mathbf{F}$ equals the sum of the line integrals around all the boundary curves of the tangential component of $\mathbf{F}$, where the curves are to be traced in the direction induced by the orientation of S .

An Important Identity

The following identity arises frequently in mathematics and the physical sciences.

$$\text{curl grad } f = \mathbf{0} \quad \text{or} \quad \nabla \times \nabla f = \mathbf{0} \quad (8)$$

This identity holds for any function $f(x, y, z)$ whose second partial derivatives are continuous. The proof goes like this:

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = (f_{zy} - f_{yz})\mathbf{i} - (f_{zx} - f_{xz})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k}.$$

If the second partial derivatives are continuous, the mixed second derivatives in parentheses are equal (Theorem 2, Section 14.3) and the vector is zero.

Conservative Fields and Stokes' Theorem

In Section 16.3, we found that a field $\mathbf{F}$ is conservative in an open region D in space is equivalent to the integral of $\mathbf{F}$ around every closed loop in D being zero. This, in turn, is equivalent in *simply connected* open regions to saying that $\nabla \times \mathbf{F} = \mathbf{0}$.

THEOREM 6 Curl $\mathbf{F} = \mathbf{0}$ Related to the Closed-Loop Property

If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D ,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Sketch of a Proof Theorem 6 is usually proved in two steps. The first step is for simple closed curves. A theorem from topology, a branch of advanced mathematics, states that

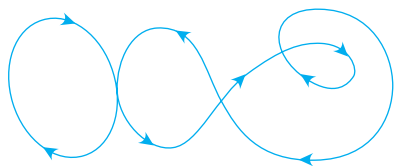


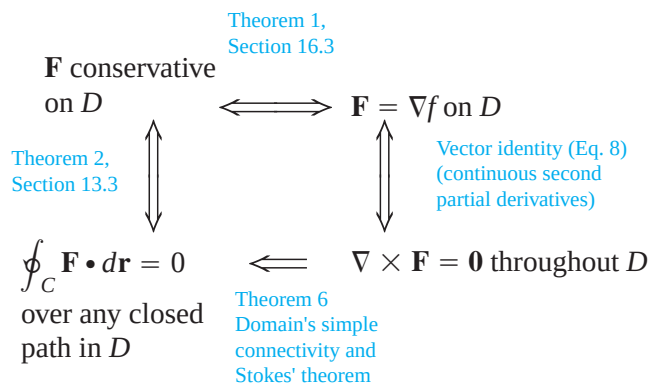
FIGURE 16.68 In a simply connected open region in space, differentiable curves that cross themselves can be divided into loops to which Stokes' Theorem applies.

every differentiable simple closed curve C in a simply connected open region D is the boundary of a smooth two-sided surface S that also lies in D . Hence, by Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma = 0.$$

The second step is for curves that cross themselves, like the one in Figure 16.68. The idea is to break these into simple loops spanned by orientable surfaces, apply Stokes' Theorem one loop at a time, and add the results. ■

The following diagram summarizes the results for conservative fields defined on connected, simply connected open regions.



EXERCISES 16.7

Using Stokes' Theorem to Calculate Circulation

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

1. $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$

C : The ellipse $4x^2 + y^2 = 4$ in the xy -plane, counterclockwise when viewed from above

2. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$

C : The circle $x^2 + y^2 = 9$ in the xy -plane, counterclockwise when viewed from above

3. $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$

C : The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

4. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

C : The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

5. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

C : The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy -plane, counterclockwise when viewed from above

6. $\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$

C : The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16, z \geq 0$, counterclockwise when viewed from above.

Flux of the Curl

7. Let $\mathbf{n}$ be the outer unit normal of the elliptical shell

$$S: 4x^2 + 9y^2 + 36z^2 = 36, \quad z \geq 0,$$

and let

$$\mathbf{F} = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2} \sin e^{\sqrt{xyz}} \mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

(Hint: One parametrization of the ellipse at the base of the shell is $x = 3 \cos t, y = 2 \sin t, 0 \leq t \leq 2\pi$.)

8. Let $\mathbf{n}$ be the outer unit normal (normal away from the origin) of the parabolic shell

$$S: 4x^2 + y + z^2 = 4, \quad y \geq 0,$$





and let

$$\mathbf{F} = \left(-z + \frac{1}{2+x}\right)\mathbf{i} + (\tan^{-1}y)\mathbf{j} + \left(x + \frac{1}{4+z}\right)\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

9. Let S be the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$, together with its top, $x^2 + y^2 \leq a^2$, $z = h$. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through S .

10. Evaluate

$$\iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n} \, d\sigma,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

11. **Flux of curl $\mathbf{F}$** Show that

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

has the same value for all oriented surfaces S that span C and that induce the same positive direction on C .

12. Let $\mathbf{F}$ be a differentiable vector field defined on a region containing a smooth closed oriented surface S and its interior. Let $\mathbf{n}$ be the unit normal vector field on S . Suppose that S is the union of two surfaces S_1 and S_2 joined along a smooth simple closed curve C . Can anything be said about

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma?$$

Give reasons for your answer.

Stokes' Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.



13. $\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$

$$S: \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (4 - r^2)\mathbf{k}, \\ 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

14. $\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x + z)\mathbf{k}$

$$S: \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (9 - r^2)\mathbf{k}, \\ 0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi$$

15. $\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3zk\mathbf{k}$

$$S: \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}, \\ 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

16. $\mathbf{F} = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$

$$S: \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (5 - r)\mathbf{k}, \\ 0 \leq r \leq 5, \quad 0 \leq \theta \leq 2\pi$$

17. $\mathbf{F} = 3y\mathbf{i} + (5 - 2x)\mathbf{j} + (z^2 - 2)\mathbf{k}$

$$S: \mathbf{r}(\phi, \theta) = (\sqrt{3} \sin \phi \cos \theta)\mathbf{i} + (\sqrt{3} \sin \phi \sin \theta)\mathbf{j} + (\sqrt{3} \cos \phi)\mathbf{k}, \\ 0 \leq \phi \leq \pi/2, \quad 0 \leq \theta \leq 2\pi$$

18. $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$

$$S: \mathbf{r}(\phi, \theta) = (2 \sin \phi \cos \theta)\mathbf{i} + (2 \sin \phi \sin \theta)\mathbf{j} + (2 \cos \phi)\mathbf{k}, \\ 0 \leq \phi \leq \pi/2, \quad 0 \leq \theta \leq 2\pi$$



Theory and Examples

19. **Zero circulation** Use the identity $\nabla \times \nabla f = \mathbf{0}$ (Equation (8) in the text) and Stokes' Theorem to show that the circulations of the following fields around the boundary of any smooth orientable surface in space are zero.

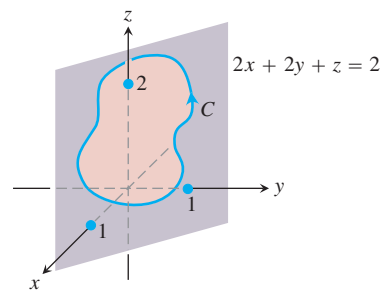
- $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$
- $\mathbf{F} = \nabla(xy^2z^3)$
- $\mathbf{F} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$
- $\mathbf{F} = \nabla f$

20. **Zero circulation** Let $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$. Show that the clockwise circulation of the field $\mathbf{F} = \nabla f$ around the circle $x^2 + y^2 = a^2$ in the xy -plane is zero

- by taking $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$, and integrating $\mathbf{F} \cdot d\mathbf{r}$ over the circle.
- by applying Stokes' Theorem.

21. Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented as shown here. Show that

$$\oint_C 2y \, dx + 3z \, dy - x \, dz$$



depends only on the area of the region enclosed by C and not on the position or shape of C .

- Show that if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\nabla \times \mathbf{F} = \mathbf{0}$.
- Find a vector field with twice-differentiable components whose curl is $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or prove that no such field exists.
- Does Stokes' Theorem say anything special about circulation in a field whose curl is zero? Give reasons for your answer.
- Let R be a region in the xy -plane that is bounded by a piecewise-smooth simple closed curve C and suppose that the moments of

inertia of R about the x - and y -axes are known to be I_x and I_y . Evaluate the integral

$$\oint_C \nabla(r^4) \cdot \mathbf{n} \, ds,$$

where $r = \sqrt{x^2 + y^2}$, in terms of I_x and I_y .

26. Zero curl, yet field not conservative Show that the curl of

$$\mathbf{F} = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j} + z\mathbf{k}$$

is zero but that

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is not zero if C is the circle $x^2 + y^2 = 1$ in the xy -plane. (Theorem 6 does not apply here because the domain of $\mathbf{F}$ is not simply connected. The field $\mathbf{F}$ is not defined along the z -axis so there is no way to contract C to a point without leaving the domain of $\mathbf{F}$.)

16.8

The Divergence Theorem and a Unified Theory



The divergence form of Green's Theorem in the plane states that the net outward flux of a vector field across a simple closed curve can be calculated by integrating the divergence of the field over the region enclosed by the curve. The corresponding theorem in three dimensions, called the Divergence Theorem, states that the net outward flux of a vector field across a closed surface in space can be calculated by integrating the divergence of the field over the region enclosed by the surface. In this section, we prove the Divergence Theorem and show how it simplifies the calculation of flux. We also derive Gauss's law for flux in an electric field and the continuity equation of hydrodynamics. Finally, we unify the chapter's vector integral theorems into a single fundamental theorem.

Divergence in Three Dimensions

The **divergence** of a vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is the scalar function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \quad (1)$$

The symbol “ $\operatorname{div} \mathbf{F}$ ” is read as “divergence of $\mathbf{F}$ ” or “ $\operatorname{div} \mathbf{F}$.” The notation $\nabla \cdot \mathbf{F}$ is read “del dot $\mathbf{F}$.”

$\operatorname{Div} \mathbf{F}$ has the same physical interpretation in three dimensions that it does in two. If $\mathbf{F}$ is the velocity field of a fluid flow, the value of $\operatorname{div} \mathbf{F}$ at a point (x, y, z) is the rate at which fluid is being piped in or drained away at (x, y, z) . The divergence is the flux per unit volume or flux density at the point.

**EXAMPLE 1** Finding Divergence

Find the divergence of $\mathbf{F} = 2xz\mathbf{i} - xy\mathbf{j} - z\mathbf{k}$.

Solution The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(-xy) + \frac{\partial}{\partial z}(-z) = 2z - x - 1. \quad \blacksquare$$

Divergence Theorem

The Divergence Theorem says that under suitable conditions, the outward flux of a vector field across a closed surface (oriented outward) equals the triple integral of the divergence of the field over the region enclosed by the surface.

THEOREM 7 Divergence Theorem

The flux of a vector field $\mathbf{F}$ across a closed oriented surface S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the integral of $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma}_{\text{Outward flux}} = \underbrace{\iiint_D \nabla \cdot \mathbf{F} \, dV}_{\text{Divergence integral}}. \quad (2)$$

EXAMPLE 2 Supporting the Divergence Theorem

Evaluate both sides of Equation (2) for the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$.

Solution The outer unit normal to S , calculated from the gradient of $f(x, y, z) = x^2 + y^2 + z^2 - a^2$, is

$$\mathbf{n} = \frac{2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\sqrt{4(x^2 + y^2 + z^2)}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}.$$

Hence,

$$\mathbf{F} \cdot \mathbf{n} \, d\sigma = \frac{x^2 + y^2 + z^2}{a} \, d\sigma = \frac{a^2}{a} \, d\sigma = a \, d\sigma$$

because $x^2 + y^2 + z^2 = a^2$ on the surface. Therefore,

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_S a \, d\sigma = a \iint_S d\sigma = a(4\pi a^2) = 4\pi a^3.$$

The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

so

$$\iiint_D \nabla \cdot \mathbf{F} \, dV = \iiint_D 3 \, dV = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3. \quad \blacksquare$$



EXAMPLE 3 Finding Flux

Find the flux of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

Solution Instead of calculating the flux as a sum of six separate integrals, one for each face of the cube, we can calculate the flux by integrating the divergence

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

over the cube's interior:

$$\begin{aligned} \text{Flux} &= \iint_{\text{Cube surface}} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_{\text{Cube interior}} \nabla \cdot \mathbf{F} \, dV && \text{The Divergence Theorem} \\ &= \int_0^1 \int_0^1 \int_0^1 (x + y + z) \, dx \, dy \, dz = \frac{3}{2}. && \text{Routine integration} \end{aligned}$$

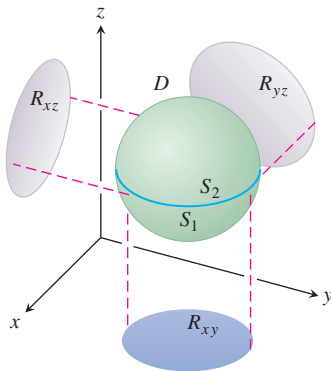


FIGURE 16.69 We first prove the Divergence Theorem for the kind of three-dimensional region shown here. We then extend the theorem to other regions.

Proof of the Divergence Theorem for Special Regions

To prove the Divergence Theorem, we assume that the components of $\mathbf{F}$ have continuous first partial derivatives. We also assume that D is a convex region with no holes or bubbles, such as a solid sphere, cube, or ellipsoid, and that S is a piecewise smooth surface. In addition, we assume that any line perpendicular to the xy -plane at an interior point of the region R_{xy} that is the projection of D on the xy -plane intersects the surface S in exactly two points, producing surfaces

$$\begin{aligned} S_1: \quad z &= f_1(x, y), & (x, y) \text{ in } R_{xy} \\ S_2: \quad z &= f_2(x, y), & (x, y) \text{ in } R_{xy}, \end{aligned}$$

with $f_1 \leq f_2$. We make similar assumptions about the projection of D onto the other coordinate planes. See Figure 16.69.

The components of the unit normal vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ are the cosines of the angles α , β , and γ that $\mathbf{n}$ makes with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (Figure 16.70). This is true because all the vectors involved are unit vectors. We have

$$\begin{aligned} n_1 &= \mathbf{n} \cdot \mathbf{i} = |\mathbf{n}| |\mathbf{i}| \cos \alpha = \cos \alpha \\ n_2 &= \mathbf{n} \cdot \mathbf{j} = |\mathbf{n}| |\mathbf{j}| \cos \beta = \cos \beta \\ n_3 &= \mathbf{n} \cdot \mathbf{k} = |\mathbf{n}| |\mathbf{k}| \cos \gamma = \cos \gamma \end{aligned}$$

Thus,

$$\mathbf{n} = (\cos \alpha)\mathbf{i} + (\cos \beta)\mathbf{j} + (\cos \gamma)\mathbf{k}$$

and

$$\mathbf{F} \cdot \mathbf{n} = M \cos \alpha + N \cos \beta + P \cos \gamma.$$

In component form, the Divergence Theorem states that

$$\iint_S (M \cos \alpha + N \cos \beta + P \cos \gamma) \, d\sigma = \iiint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) \, dx \, dy \, dz.$$

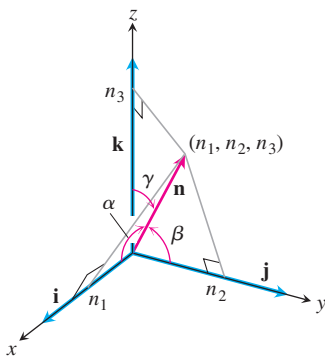


FIGURE 16.70 The scalar components of the unit normal vector $\mathbf{n}$ are the cosines of the angles α , β , and γ that it makes with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

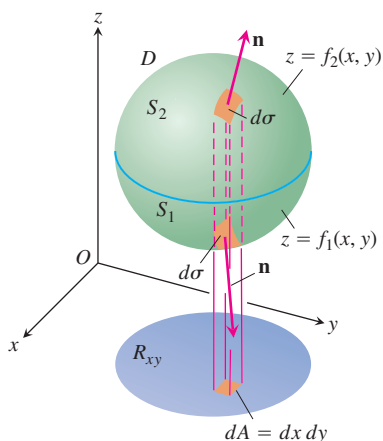


FIGURE 16.71 The three-dimensional region D enclosed by the surfaces S_1 and S_2 shown here projects vertically onto a two-dimensional region R_{xy} in the xy -plane.

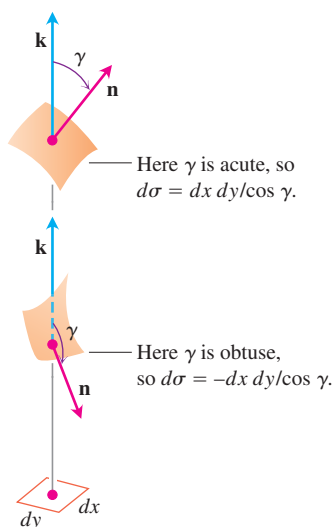


FIGURE 16.72 An enlarged view of the area patches in Figure 16.71. The relations $d\sigma = \pm dx dy / \cos \gamma$ are derived in Section 16.5.

We prove the theorem by proving the three following equalities:

$$\iint_S M \cos \alpha \, d\sigma = \iiint_D \frac{\partial M}{\partial x} \, dx \, dy \, dz \quad (3)$$

$$\iint_S N \cos \beta \, d\sigma = \iiint_D \frac{\partial N}{\partial y} \, dx \, dy \, dz \quad (4)$$

$$\iint_S P \cos \gamma \, d\sigma = \iiint_D \frac{\partial P}{\partial z} \, dx \, dy \, dz \quad (5)$$

Proof of Equation (5) We prove Equation (5) by converting the surface integral on the left to a double integral over the projection R_{xy} of D on the xy -plane (Figure 16.71). The surface S consists of an upper part S_2 whose equation is $z = f_2(x, y)$ and a lower part S_1 whose equation is $z = f_1(x, y)$. On S_2 , the outer normal $\mathbf{n}$ has a positive $\mathbf{k}$ -component and

$$\cos \gamma \, d\sigma = dx \, dy \quad \text{because} \quad d\sigma = \frac{dA}{|\cos \gamma|} = \frac{dx \, dy}{\cos \gamma}.$$

See Figure 16.72. On S_1 , the outer normal $\mathbf{n}$ has a negative $\mathbf{k}$ -component and

$$\cos \gamma \, d\sigma = -dx \, dy.$$

Therefore,

$$\begin{aligned} \iint_S P \cos \gamma \, d\sigma &= \iint_{S_2} P \cos \gamma \, d\sigma + \iint_{S_1} P \cos \gamma \, d\sigma \\ &= \iint_{R_{xy}} P(x, y, f_2(x, y)) \, dx \, dy - \iint_{R_{xy}} P(x, y, f_1(x, y)) \, dx \, dy \\ &= \iint_{R_{xy}} [P(x, y, f_2(x, y)) - P(x, y, f_1(x, y))] \, dx \, dy \\ &= \iint_{R_{xy}} \left[\int_{f_1(x, y)}^{f_2(x, y)} \frac{\partial P}{\partial z} \, dz \right] \, dx \, dy = \iiint_D \frac{\partial P}{\partial z} \, dz \, dx \, dy. \end{aligned}$$

This proves Equation (5). ■

The proofs for Equations (3) and (4) follow the same pattern; or just permute x, y, z ; M, N, P ; α, β, γ , in order, and get those results from Equation (5).

Divergence Theorem for Other Regions

The Divergence Theorem can be extended to regions that can be partitioned into a finite number of simple regions of the type just discussed and to regions that can be defined as limits of simpler regions in certain ways. For example, suppose that D is the region between two concentric spheres and that $\mathbf{F}$ has continuously differentiable components throughout D and on the bounding surfaces. Split D by an equatorial plane and apply the

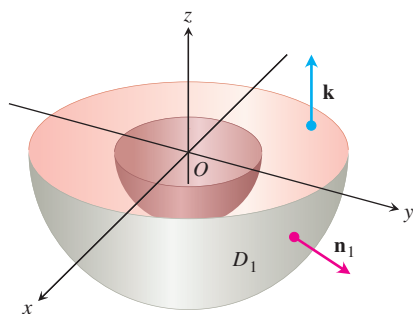


FIGURE 16.73 The lower half of the solid region between two concentric spheres.

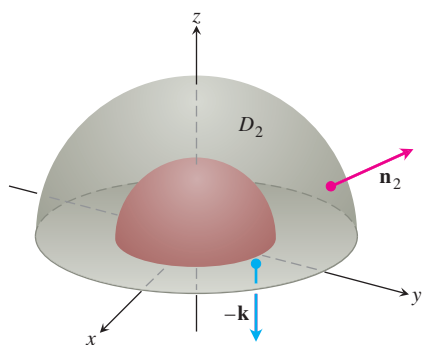


FIGURE 16.74 The upper half of the solid region between two concentric spheres.

Divergence Theorem to each half separately. The bottom half, D_1 , is shown in Figure 16.73. The surface S_1 that bounds D_1 consists of an outer hemisphere, a plane washer-shaped base, and an inner hemisphere. The Divergence Theorem says that

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 \, d\sigma_1 = \iiint_{D_1} \nabla \cdot \mathbf{F} \, dV_1. \quad (6)$$

The unit normal $\mathbf{n}_1$ that points outward from D_1 points away from the origin along the outer surface, equals $\mathbf{k}$ along the flat base, and points toward the origin along the inner surface. Next apply the Divergence Theorem to D_2 , and its surface S_2 (Figure 16.74):

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, d\sigma_2 = \iiint_{D_2} \nabla \cdot \mathbf{F} \, dV_2. \quad (7)$$

As we follow $\mathbf{n}_2$ over S_2 , pointing outward from D_2 , we see that $\mathbf{n}_2$ equals $-\mathbf{k}$ along the washer-shaped base in the xy -plane, points away from the origin on the outer sphere, and points toward the origin on the inner sphere. When we add Equations (6) and (7), the integrals over the flat base cancel because of the opposite signs of $\mathbf{n}_1$ and $\mathbf{n}_2$. We thus arrive at the result

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV,$$

with D the region between the spheres, S the boundary of D consisting of two spheres, and $\mathbf{n}$ the unit normal to S directed outward from D .

EXAMPLE 4 Finding Outward Flux

Find the net outward flux of the field

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3}, \quad \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region D : $0 < a^2 \leq x^2 + y^2 + z^2 \leq b^2$.

Solution The flux can be calculated by integrating $\nabla \cdot \mathbf{F}$ over D . We have

$$\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\rho}$$

and

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(x\rho^{-3}) = \rho^{-3} - 3x\rho^{-4}\frac{\partial \rho}{\partial x} = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}.$$

Similarly,

$$\frac{\partial N}{\partial y} = \frac{1}{\rho^3} - \frac{3y^2}{\rho^5} \quad \text{and} \quad \frac{\partial P}{\partial z} = \frac{1}{\rho^3} - \frac{3z^2}{\rho^5}.$$

Hence,

$$\operatorname{div} \mathbf{F} = \frac{3}{\rho^3} - \frac{3}{\rho^5}(x^2 + y^2 + z^2) = \frac{3}{\rho^3} - \frac{3\rho^2}{\rho^5} = 0$$



and

$$\iiint_D \nabla \cdot \mathbf{F} \, dV = 0.$$

So the integral of $\nabla \cdot \mathbf{F}$ over D is zero and the net outward flux across the boundary of D is zero. There is more to learn from this example, though. The flux leaving D across the inner sphere S_a is the negative of the flux leaving D across the outer sphere S_b (because the sum of these fluxes is zero). Hence, the flux of $\mathbf{F}$ across S_a in the direction away from the origin equals the flux of $\mathbf{F}$ across S_b in the direction away from the origin. Thus, the flux of $\mathbf{F}$ across a sphere centered at the origin is independent of the radius of the sphere. What is this flux?

To find it, we evaluate the flux integral directly. The outward unit normal on the sphere of radius a is

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}.$$

Hence, on the sphere,

$$\mathbf{F} \cdot \mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a^3} \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} = \frac{x^2 + y^2 + z^2}{a^4} = \frac{a^2}{a^4} = \frac{1}{a^2}$$

and

$$\iint_{S_a} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \frac{1}{a^2} \iint_{S_a} d\sigma = \frac{1}{a^2} (4\pi a^2) = 4\pi.$$

The outward flux of $\mathbf{F}$ across any sphere centered at the origin is 4π . ■

Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

There is still more to be learned from Example 4. In electromagnetic theory, the electric field created by a point charge q located at the origin is

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r}|^2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{q}{4\pi\epsilon_0} \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3},$$

where ϵ_0 is a physical constant, $\mathbf{r}$ is the position vector of the point (x, y, z) , and $\rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. In the notation of Example 4,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \mathbf{F}.$$

The calculations in Example 4 show that the outward flux of $\mathbf{E}$ across any sphere centered at the origin is q/ϵ_0 , but this result is not confined to spheres. The outward flux of $\mathbf{E}$ across any closed surface S that encloses the origin (and to which the Divergence Theorem applies) is also q/ϵ_0 . To see why, we have only to imagine a large sphere S_a centered at the origin and enclosing the surface S . Since

$$\nabla \cdot \mathbf{E} = \nabla \cdot \frac{q}{4\pi\epsilon_0} \mathbf{F} = \frac{q}{4\pi\epsilon_0} \nabla \cdot \mathbf{F} = 0$$

when $\rho > 0$, the integral of $\nabla \cdot \mathbf{E}$ over the region D between S and S_a is zero. Hence, by the Divergence Theorem,

$$\iint_{\text{Boundary of } D} \mathbf{E} \cdot \mathbf{n} \, d\sigma = 0,$$

and the flux of $\mathbf{E}$ across S in the direction away from the origin must be the same as the flux of $\mathbf{E}$ across S_a in the direction away from the origin, which is q/ϵ_0 . This statement, called *Gauss's Law*, also applies to charge distributions that are more general than the one assumed here, as you will see in nearly any physics text.

$$\text{Gauss's law: } \iint_S \mathbf{E} \cdot \mathbf{n} \, d\sigma = \frac{q}{\epsilon_0}$$

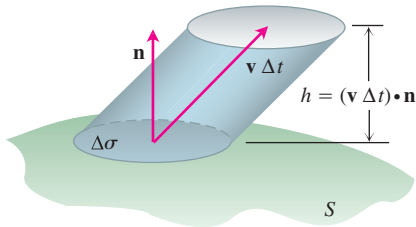


FIGURE 16.75 The fluid that flows upward through the patch $\Delta\sigma$ in a short time Δt fills a “cylinder” whose volume is approximately base $\times$ height = $\mathbf{v} \cdot \mathbf{n} \, \Delta\sigma \, \Delta t$.

Continuity Equation of Hydrodynamics

Let D be a region in space bounded by a closed oriented surface S . If $\mathbf{v}(x, y, z)$ is the velocity field of a fluid flowing smoothly through D , $\delta = \delta(t, x, y, z)$ is the fluid's density at (x, y, z) at time t , and $\mathbf{F} = \delta\mathbf{v}$, then the **continuity equation** of hydrodynamics states that

$$\nabla \cdot \mathbf{F} + \frac{\partial \delta}{\partial t} = 0.$$

If the functions involved have continuous first partial derivatives, the equation evolves naturally from the Divergence Theorem, as we now see.

First, the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

is the rate at which mass leaves D across S (leaves because $\mathbf{n}$ is the outer normal). To see why, consider a patch of area $\Delta\sigma$ on the surface (Figure 16.75). In a short time interval Δt , the volume ΔV of fluid that flows across the patch is approximately equal to the volume of a cylinder with base area $\Delta\sigma$ and height $(\mathbf{v}\Delta t) \cdot \mathbf{n}$, where $\mathbf{v}$ is a velocity vector rooted at a point of the patch:

$$\Delta V \approx \mathbf{v} \cdot \mathbf{n} \, \Delta\sigma \, \Delta t.$$

The mass of this volume of fluid is about

$$\Delta m \approx \delta \mathbf{v} \cdot \mathbf{n} \, \Delta\sigma \, \Delta t,$$

so the rate at which mass is flowing out of D across the patch is about

$$\frac{\Delta m}{\Delta t} \approx \delta \mathbf{v} \cdot \mathbf{n} \, \Delta\sigma.$$

This leads to the approximation

$$\frac{\sum \Delta m}{\Delta t} \approx \sum \delta \mathbf{v} \cdot \mathbf{n} \, \Delta\sigma$$

as an estimate of the average rate at which mass flows across S . Finally, letting $\Delta\sigma \rightarrow 0$ and $\Delta t \rightarrow 0$ gives the instantaneous rate at which mass leaves D across S as

$$\frac{dm}{dt} = \iint_S \delta \mathbf{v} \cdot \mathbf{n} \, d\sigma,$$

which for our particular flow is

$$\frac{dm}{dt} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

Now let B be a solid sphere centered at a point Q in the flow. The average value of $\nabla \cdot \mathbf{F}$ over B is

$$\frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F} \, dV.$$

It is a consequence of the continuity of the divergence that $\nabla \cdot \mathbf{F}$ actually takes on this value at some point P in B . Thus,

$$\begin{aligned} (\nabla \cdot \mathbf{F})_P &= \frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F} \, dV = \frac{\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma}{\text{volume of } B} \\ &= \frac{\text{rate at which mass leaves } B \text{ across its surface } S}{\text{volume of } B} \end{aligned} \quad (8)$$

The fraction on the right describes decrease in mass per unit volume.

Now let the radius of B approach zero while the center Q stays fixed. The left side of Equation (8) converges to $(\nabla \cdot \mathbf{F})_Q$, the right side to $(-\partial\delta/\partial t)_Q$. The equality of these two limits is the continuity equation

$$\nabla \cdot \mathbf{F} = -\frac{\partial\delta}{\partial t}.$$

The continuity equation “explains” $\nabla \cdot \mathbf{F}$: The divergence of $\mathbf{F}$ at a point is the rate at which the density of the fluid is decreasing there.

The Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_D \nabla \cdot \mathbf{F} \, dV$$

now says that the net decrease in density of the fluid in region D is accounted for by the mass transported across the surface S . So, the theorem is a statement about conservation of mass (Exercise 31).

Unifying the Integral Theorems

If we think of a two-dimensional field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ as a three-dimensional field whose $\mathbf{k}$ -component is zero, then $\nabla \cdot \mathbf{F} = (\partial M/\partial x) + (\partial N/\partial y)$ and the normal form of Green’s Theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy = \iint_R \nabla \cdot \mathbf{F} \, dA.$$

Similarly, $\nabla \times \mathbf{F} \cdot \mathbf{k} = (\partial N/\partial x) - (\partial M/\partial y)$, so the tangential form of Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} dA.$$

With the equations of Green's Theorem now in del notation, we can see their relationships to the equations in Stokes' Theorem and the Divergence Theorem.

Green's Theorem and Its Generalization to Three Dimensions

Normal form of Green's Theorem: $\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R \nabla \cdot \mathbf{F} dA$

Divergence Theorem: $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_D \nabla \cdot \mathbf{F} dV$

Tangential form of Green's Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k} dA$

Stokes' Theorem: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma$

Notice how Stokes' Theorem generalizes the tangential (curl) form of Green's Theorem from a flat surface in the plane to a surface in three-dimensional space. In each case, the integral of the normal component of curl $\mathbf{F}$ over the interior of the surface equals the circulation of $\mathbf{F}$ around the boundary.

Likewise, the Divergence Theorem generalizes the normal (flux) form of Green's Theorem from a two-dimensional region in the plane to a three-dimensional region in space. In each case, the integral of $\nabla \cdot \mathbf{F}$ over the interior of the region equals the total flux of the field across the boundary.

There is still more to be learned here. All these results can be thought of as forms of a *single fundamental theorem*. Think back to the Fundamental Theorem of Calculus in Section 5.3. It says that if $f(x)$ is differentiable on (a, b) and continuous on $[a, b]$, then

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a).$$

If we let $\mathbf{F} = f(x)\mathbf{i}$ throughout $[a, b]$, then $(df/dx) = \nabla \cdot \mathbf{F}$. If we define the unit vector field $\mathbf{n}$ normal to the boundary of $[a, b]$ to be $\mathbf{i}$ at b and $-\mathbf{i}$ at a (Figure 16.76), then

$$\begin{aligned} f(b) - f(a) &= f(b)\mathbf{i} \cdot (\mathbf{i}) + f(a)\mathbf{i} \cdot (-\mathbf{i}) \\ &= \mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} \\ &= \text{total outward flux of } \mathbf{F} \text{ across the boundary of } [a, b]. \end{aligned}$$

The Fundamental Theorem now says that

$$\mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} = \int_{[a,b]} \nabla \cdot \mathbf{F} dx.$$



FIGURE 16.76 The outward unit normals at the boundary of $[a, b]$ in one-dimensional space.

The Fundamental Theorem of Calculus, the normal form of Green's Theorem, and the Divergence Theorem all say that the integral of the differential operator $\nabla \cdot$ operating on a field $\mathbf{F}$ over a region equals the sum of the normal field components over the boundary of the region. (Here we are interpreting the line integral in Green's Theorem and the surface integral in the Divergence Theorem as "sums" over the boundary.)

Stokes' Theorem and the tangential form of Green's Theorem say that, when things are properly oriented, the integral of the normal component of the curl operating on a field equals the sum of the tangential field components on the boundary of the surface.

The beauty of these interpretations is the observance of a single unifying principle, which we might state as follows.

The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.

EXERCISES 16.8

Calculating Divergence

In Exercises 1–4, find the divergence of the field.

1. The spin field in Figure 16.14.
2. The radial field in Figure 16.13.
3. The gravitational field in Figure 16.9.
4. The velocity field in Figure 16.12.

Using the Divergence Theorem to Calculate Outward Flux

In Exercises 5–16, use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D .

5. **Cube** $\mathbf{F} = (y - x)\mathbf{i} + (z - y)\mathbf{j} + (y - x)\mathbf{k}$
 D : The cube bounded by the planes $x = \pm 1, y = \pm 1$, and $z = \pm 1$
6. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$
 - a. **Cube** D : The cube cut from the first octant by the planes $x = 1, y = 1$, and $z = 1$
 - b. **Cube** D : The cube bounded by the planes $x = \pm 1, y = \pm 1$, and $z = \pm 1$
 - c. **Cylindrical can** D : The region cut from the solid cylinder $x^2 + y^2 \leq 4$ by the planes $z = 0$ and $z = 1$
7. **Cylinder and paraboloid** $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$
 D : The region inside the solid cylinder $x^2 + y^2 \leq 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$

8. **Sphere** $\mathbf{F} = x^2\mathbf{i} + xz\mathbf{j} + 3z\mathbf{k}$

D : The solid sphere $x^2 + y^2 + z^2 \leq 4$

9. **Portion of sphere** $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$

D : The region cut from the first octant by the sphere $x^2 + y^2 + z^2 = 4$

10. **Cylindrical can** $\mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2z)\mathbf{j} + 4x^2y^3\mathbf{k}$

D : The region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$

11. **Wedge** $\mathbf{F} = 2xz\mathbf{i} - xy\mathbf{j} - z^2\mathbf{k}$

D : The wedge cut from the first octant by the plane $y + z = 4$ and the elliptical cylinder $4x^2 + y^2 = 16$

12. **Sphere** $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$

D : The solid sphere $x^2 + y^2 + z^2 \leq a^2$

13. **Thick sphere** $\mathbf{F} = \sqrt{x^2 + y^2 + z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

D : The region $1 \leq x^2 + y^2 + z^2 \leq 2$

14. **Thick sphere** $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2 + y^2 + z^2}$

D : The region $1 \leq x^2 + y^2 + z^2 \leq 4$

15. **Thick sphere** $\mathbf{F} = (5x^3 + 12xy^2)\mathbf{i} + (y^3 + e^y \sin z)\mathbf{j} + (5z^3 + e^y \cos z)\mathbf{k}$

D : The solid region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$

16. **Thick cylinder** $\mathbf{F} = \ln(x^2 + y^2)\mathbf{i} - \left(\frac{2z}{x} \tan^{-1} \frac{y}{x}\right)\mathbf{j} + z\sqrt{x^2 + y^2}\mathbf{k}$

D : The thick-walled cylinder $1 \leq x^2 + y^2 \leq 2, -1 \leq z \leq 2$

Properties of Curl and Divergence

17. $\operatorname{div}(\operatorname{curl} \mathbf{G})$ is zero

- Show that if the necessary partial derivatives of the components of the field $\mathbf{G} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ are continuous, then $\nabla \cdot \nabla \times \mathbf{G} = 0$.
- What, if anything, can you conclude about the flux of the field $\nabla \times \mathbf{G}$ across a closed surface? Give reasons for your answer.

18. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be differentiable vector fields and let a and b be arbitrary real constants. Verify the following identities.

- $\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$
- $\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$
- $\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

19. Let $\mathbf{F}$ be a differentiable vector field and let $g(x, y, z)$ be a differentiable scalar function. Verify the following identities.

- $\nabla \cdot (g\mathbf{F}) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$
- $\nabla \times (g\mathbf{F}) = g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$

20. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a differentiable vector field, we define the notation $\mathbf{F} \cdot \nabla$ to mean

$$M \frac{\partial}{\partial x} + N \frac{\partial}{\partial y} + P \frac{\partial}{\partial z}.$$

For differentiable vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$, verify the following identities.

- $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$
- $\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$

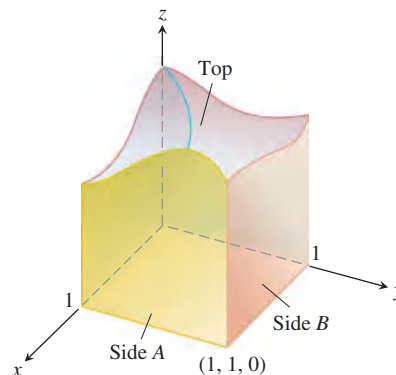
Theory and Examples

21. Let $\mathbf{F}$ be a field whose components have continuous first partial derivatives throughout a portion of space containing a region D bounded by a smooth closed surface S . If $|\mathbf{F}| \leq 1$, can any bound be placed on the size of

$$\iiint_D \nabla \cdot \mathbf{F} \, dV?$$

Give reasons for your answer.

22. The base of the closed cubelike surface shown here is the unit square in the xy -plane. The four sides lie in the planes $x = 0$, $x = 1$, $y = 0$, and $y = 1$. The top is an arbitrary smooth surface whose identity is unknown. Let $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + (z + 3)\mathbf{k}$ and suppose the outward flux of $\mathbf{F}$ through side A is 1 and through side B is -3 . Can you conclude anything about the outward flux through the top? Give reasons for your answer.



- Show that the flux of the position vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through a smooth closed surface S is three times the volume of the region enclosed by the surface.
 - Let $\mathbf{n}$ be the outward unit normal vector field on S . Show that it is not possible for $\mathbf{F}$ to be orthogonal to $\mathbf{n}$ at every point of S .
24. **Maximum flux** Among all rectangular solids defined by the inequalities $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq 1$, find the one for which the total flux of $\mathbf{F} = (-x^2 - 4xy)\mathbf{i} - 6yz\mathbf{j} + 12z\mathbf{k}$ outward through the six sides is greatest. What is the greatest flux?
25. **Volume of a solid region** Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and suppose that the surface S and region D satisfy the hypotheses of the Divergence Theorem. Show that the volume of D is given by the formula

$$\text{Volume of } D = \frac{1}{3} \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

26. **Flux of a constant field** Show that the outward flux of a constant vector field $\mathbf{F} = \mathbf{C}$ across any closed surface to which the Divergence Theorem applies is zero.
27. **Harmonic functions** A function $f(x, y, z)$ is said to be *harmonic* in a region D in space if it satisfies the Laplace equation

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout D .

- Suppose that f is harmonic throughout a bounded region D enclosed by a smooth surface S and that $\mathbf{n}$ is the chosen unit normal vector on S . Show that the integral over S of $\nabla f \cdot \mathbf{n}$, the derivative of f in the direction of $\mathbf{n}$, is zero.
- Show that if f is harmonic on D , then

$$\iint_S f \nabla f \cdot \mathbf{n} \, d\sigma = \iiint_D |\nabla f|^2 \, dV.$$

- 28. Flux of a gradient field** Let S be the surface of the portion of the solid sphere $x^2 + y^2 + z^2 \leq a^2$ that lies in the first octant and let $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$. Calculate

$$\iint_S \nabla f \cdot \mathbf{n} \, d\sigma.$$

($\nabla f \cdot \mathbf{n}$ is the derivative of f in the direction of $\mathbf{n}$.)

- 29. Green's first formula** Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives throughout a region D that is bounded by a closed piecewise-smooth surface S . Show that

$$\iint_S f \nabla g \cdot \mathbf{n} \, d\sigma = \iiint_D (f \nabla^2 g + \nabla f \cdot \nabla g) \, dV. \quad (9)$$

Equation (9) is **Green's first formula**. (Hint: Apply the Divergence Theorem to the field $\mathbf{F} = f \nabla g$.)

- 30. Green's second formula** (Continuation of Exercise 29.) Interchange f and g in Equation (9) to obtain a similar formula. Then subtract this formula from Equation (9) to show that

$$\iint_S (f \nabla g - g \nabla f) \cdot \mathbf{n} \, d\sigma = \iiint_D (f \nabla^2 g - g \nabla^2 f) \, dV. \quad (10)$$

This equation is **Green's second formula**.

- 31. Conservation of mass** Let $\mathbf{v}(t, x, y, z)$ be a continuously differentiable vector field over the region D in space and let $p(t, x, y, z)$ be a continuously differentiable scalar function. The variable t represents the time domain. The Law of Conservation of Mass asserts that

$$\frac{d}{dt} \iiint_D p(t, x, y, z) \, dV = - \iint_S p \mathbf{v} \cdot \mathbf{n} \, d\sigma,$$

where S is the surface enclosing D .

- a. Give a physical interpretation of the conservation of mass law if $\mathbf{v}$ is a velocity flow field and p represents the density of the fluid at point (x, y, z) at time t .

- b. Use the Divergence Theorem and Leibniz's Rule,

$$\frac{d}{dt} \iiint_D p(t, x, y, z) \, dV = \iiint_D \frac{\partial p}{\partial t} \, dV,$$

to show that the Law of Conservation of Mass is equivalent to the continuity equation,

$$\nabla \cdot p \mathbf{v} + \frac{\partial p}{\partial t} = 0.$$

(In the first term $\nabla \cdot p \mathbf{v}$, the variable t is held fixed, and in the second term $\partial p / \partial t$, it is assumed that the point (x, y, z) in D is held fixed.)

- 32. The heat diffusion equation** Let $T(t, x, y, z)$ be a function with continuous second derivatives giving the temperature at time t at the point (x, y, z) of a solid occupying a region D in space. If the solid's heat capacity and mass density are denoted by the constants c and ρ , respectively, the quantity $c\rho T$ is called the solid's **heat energy per unit volume**.

- a. Explain why $-\nabla T$ points in the direction of heat flow.
- b. Let $-k\nabla T$ denote the **energy flux vector**. (Here the constant k is called the **conductivity**.) Assuming the Law of Conservation of Mass with $-k\nabla T = \mathbf{v}$ and $c\rho T = p$ in Exercise 31, derive the diffusion (heat) equation

$$\frac{\partial T}{\partial t} = K \nabla^2 T,$$

where $K = k/(c\rho) > 0$ is the *diffusivity* constant. (Notice that if $T(t, x)$ represents the temperature at time t at position x in a uniform conducting rod with perfectly insulated sides, then $\nabla^2 T = \partial^2 T / \partial x^2$ and the diffusion equation reduces to the one-dimensional heat equation in Chapter 14's Additional Exercises.)

Chapter 16

Questions to Guide Your Review

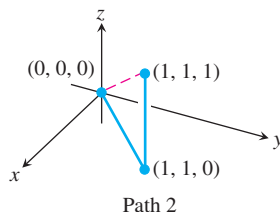
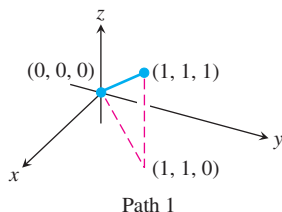
1. What are line integrals? How are they evaluated? Give examples.
2. How can you use line integrals to find the centers of mass of springs? Explain.
3. What is a vector field? A gradient field? Give examples.
4. How do you calculate the work done by a force in moving a particle along a curve? Give an example.
5. What are flow, circulation, and flux?
6. What is special about path independent fields?
7. How can you tell when a field is conservative?
8. What is a potential function? Show by example how to find a potential function for a conservative field.
9. What is a differential form? What does it mean for such a form to be exact? How do you test for exactness? Give examples.
10. What is the divergence of a vector field? How can you interpret it?
11. What is the curl of a vector field? How can you interpret it?
12. What is Green's theorem? How can you interpret it?
13. How do you calculate the area of a curved surface in space? Give an example.

14. What is an oriented surface? How do you calculate the flux of a three-dimensional vector field across an oriented surface? Give an example.
15. What are surface integrals? What can you calculate with them? Give an example.
16. What is a parametrized surface? How do you find the area of such a surface? Give examples.
17. How do you integrate a function over a parametrized surface? Give an example.
18. What is Stokes' Theorem? How can you interpret it?
19. Summarize the chapter's results on conservative fields.
20. What is the Divergence Theorem? How can you interpret it?
21. How does the Divergence Theorem generalize Green's Theorem?
22. How does Stokes' Theorem generalize Green's Theorem?
23. How can Green's Theorem, Stokes' Theorem, and the Divergence Theorem be thought of as forms of a single fundamental theorem?

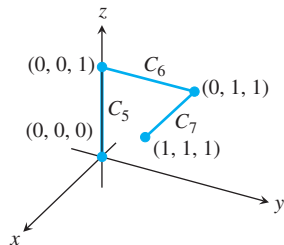
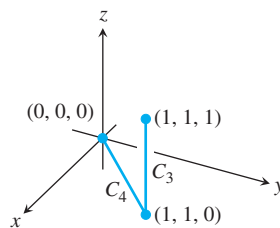
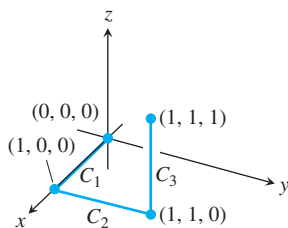
Chapter 16 Practice Exercises

Evaluating Line Integrals

1. The accompanying figure shows two polygonal paths in space joining the origin to the point $(1, 1, 1)$. Integrate $f(x, y, z) = 2x - 3y^2 - 2z + 3$ over each path.



2. The accompanying figure shows three polygonal paths joining the origin to the point $(1, 1, 1)$. Integrate $f(x, y, z) = x^2 + y - z$ over each path.



3. Integrate $f(x, y, z) = \sqrt{x^2 + z^2}$ over the circle
 $\mathbf{r}(t) = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k}, \quad 0 \leq t \leq 2\pi.$

4. Integrate $f(x, y, z) = \sqrt{x^2 + y^2}$ over the involute curve
 $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad 0 \leq t \leq \sqrt{3}.$

Evaluate the integrals in Exercises 5 and 6.

5. $\int_{(-1,1,1)}^{(4,-3,0)} \frac{dx + dy + dz}{\sqrt{x + y + z}}$

6. $\int_{(1,1,1)}^{(10,3,3)} dx - \sqrt{\frac{z}{y}} dy - \sqrt{\frac{y}{z}} dz$

7. Integrate $\mathbf{F} = -(y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$ around the circle cut from the sphere $x^2 + y^2 + z^2 = 5$ by the plane $z = -1$, clockwise as viewed from above.
8. Integrate $\mathbf{F} = 3x^2y\mathbf{i} + (x^3 + 1)\mathbf{j} + 9z^2\mathbf{k}$ around the circle cut from the sphere $x^2 + y^2 + z^2 = 9$ by the plane $x = 2$.

Evaluate the integrals in Exercises 9 and 10.

9. $\int_C 8x \sin y \, dx - 8y \cos x \, dy$

C is the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.

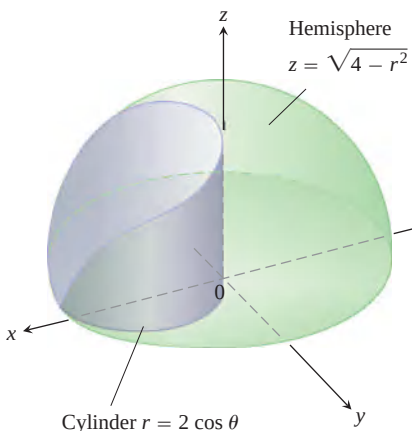
10. $\int_C y^2 \, dx + x^2 \, dy$

C is the circle $x^2 + y^2 = 4$.

Evaluating Surface Integrals

11. **Area of an elliptical region** Find the area of the elliptical region cut from the plane $x + y + z = 1$ by the cylinder $x^2 + y^2 = 1$.
12. **Area of a parabolic cap** Find the area of the cap cut from the paraboloid $y^2 + z^2 = 3x$ by the plane $x = 1$.
13. **Area of a spherical cap** Find the area of the cap cut from the top of the sphere $x^2 + y^2 + z^2 = 1$ by the plane $z = \sqrt{2}/2$.

- 14. a. Hemisphere cut by cylinder** Find the area of the surface cut from the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$, by the cylinder $x^2 + y^2 = 2x$.
- b.** Find the area of the portion of the cylinder that lies inside the hemisphere. (*Hint:* Project onto the xz -plane. Or evaluate the integral $\int h \, ds$, where h is the altitude of the cylinder and ds is the element of arc length on the circle $x^2 + y^2 = 2x$ in the xy -plane.)



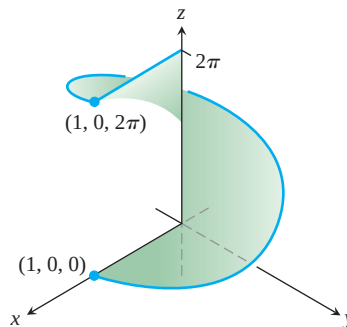
- 15. Area of a triangle** Find the area of the triangle in which the plane $(x/a) + (y/b) + (z/c) = 1$ ($a, b, c > 0$) intersects the first octant. Check your answer with an appropriate vector calculation.
- 16. Parabolic cylinder cut by planes** Integrate
- a.** $g(x, y, z) = \frac{yz}{\sqrt{4y^2 + 1}}$ **b.** $g(x, y, z) = \frac{z}{\sqrt{4y^2 + 1}}$
- over the surface cut from the parabolic cylinder $y^2 - z = 1$ by the planes $x = 0, x = 3$, and $z = 0$.
- 17. Circular cylinder cut by planes** Integrate $g(x, y, z) = x^4 y(y^2 + z^2)$ over the portion of the cylinder $y^2 + z^2 = 25$ that lies in the first octant between the planes $x = 0$ and $x = 1$ and above the plane $z = 0$.
- 18. Area of Wyoming** The state of Wyoming is bounded by the meridians $111^\circ 3'$ and $104^\circ 3'$ west longitude and by the circles 41° and 45° north latitude. Assuming that Earth is a sphere of radius $R = 3959$ mi, find the area of Wyoming.

Parametrized Surfaces

Find the parametrizations for the surfaces in Exercises 19–24. (There are many ways to do these, so your answers may not be the same as those in the back of the book.)

- 19. Spherical band** The portion of the sphere $x^2 + y^2 + z^2 = 36$ between the planes $z = -3$ and $z = 3\sqrt{3}$
- 20. Parabolic cap** The portion of the paraboloid $z = -(x^2 + y^2)/2$ above the plane $z = -2$

- 21. Cone** The cone $z = 1 + \sqrt{x^2 + y^2}, z \leq 3$
- 22. Plane above square** The portion of the plane $4x + 2y + 4z = 12$ that lies above the square $0 \leq x \leq 2, 0 \leq y \leq 2$ in the first quadrant
- 23. Portion of paraboloid** The portion of the paraboloid $y = 2(x^2 + z^2), y \leq 2$, that lies above the xy -plane
- 24. Portion of hemisphere** The portion of the hemisphere $x^2 + y^2 + z^2 = 10, y \geq 0$, in the first octant
- 25. Surface area** Find the area of the surface
- $$\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k},$$
- $$0 \leq u \leq 1, \quad 0 \leq v \leq 1.$$
- 26. Surface integral** Integrate $f(x, y, z) = xy - z^2$ over the surface in Exercise 25.
- 27. Area of a helicoid** Find the surface area of the helicoid
- $$\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \theta\mathbf{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 1,$$
- in the accompanying figure.



- 28. Surface integral** Evaluate the integral $\iint_S \sqrt{x^2 + y^2 + 1} \, d\sigma$, where S is the helicoid in Exercise 27.

Conservative Fields

Which of the fields in Exercises 29–32 are conservative, and which are not?

- 29.** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
- 30.** $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$
- 31.** $\mathbf{F} = xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$
- 32.** $\mathbf{F} = (\mathbf{i} + z\mathbf{j} + y\mathbf{k})/(x + yz)$

Find potential functions for the fields in Exercises 33 and 34.

- 33.** $\mathbf{F} = 2\mathbf{i} + (2y + z)\mathbf{j} + (y + 1)\mathbf{k}$
- 34.** $\mathbf{F} = (z \cos xz)\mathbf{i} + e^y\mathbf{j} + (x \cos xz)\mathbf{k}$

Work and Circulation

In Exercises 35 and 36, find the work done by each field along the paths from $(0, 0, 0)$ to $(1, 1, 1)$ in Exercise 1.

35. $\mathbf{F} = 2xy\mathbf{i} + \mathbf{j} + x^2\mathbf{k}$ 36. $\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j} + \mathbf{k}$

37. **Finding work in two ways** Find the work done by

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}}$$

over the plane curve $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}$ from the point $(1, 0)$ to the point $(e^{2\pi}, 0)$ in two ways:

- By using the parametrization of the curve to evaluate the work integral
- By evaluating a potential function for $\mathbf{F}$.

38. **Flow along different paths** Find the flow of the field $\mathbf{F} = \nabla(x^2ze^y)$

- Once around the ellipse C in which the plane $x + y + z = 1$ intersects the cylinder $x^2 + z^2 = 25$, clockwise as viewed from the positive y -axis
- Along the curved boundary of the helicoid in Exercise 27 from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

In Exercises 39 and 40, use the surface integral in Stokes' Theorem to find the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

39. **Circulation around an ellipse** $\mathbf{F} = y^2\mathbf{i} - y\mathbf{j} + 3z^2\mathbf{k}$

C : The ellipse in which the plane $2x + 6y - 3z = 6$ meets the cylinder $x^2 + y^2 = 1$, counterclockwise as viewed from above

40. **Circulation around a circle** $\mathbf{F} = (x^2 + y)\mathbf{i} + (x + y)\mathbf{j} + (4y^2 - z)\mathbf{k}$

C : The circle in which the plane $z = -y$ meets the sphere $x^2 + y^2 + z^2 = 4$, counterclockwise as viewed from above

Mass and Moments

41. **Wire with different densities** Find the mass of a thin wire lying along the curve $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}$, $0 \leq t \leq 1$, if the density at t is (a) $\delta = 3t$ and (b) $\delta = 1$.

42. **Wire with variable density** Find the center of mass of a thin wire lying along the curve $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + (2/3)t^{3/2}\mathbf{k}$, $0 \leq t \leq 2$, if the density at t is $\delta = 3\sqrt{5 + t}$.

43. **Wire with variable density** Find the center of mass and the moments of inertia and radii of gyration about the coordinate axes of a thin wire lying along the curve

$$\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}, \quad 0 \leq t \leq 2,$$

if the density at t is $\delta = 1/(t + 1)$.

44. **Center of mass of an arch** A slender metal arch lies along the semicircle $y = \sqrt{a^2 - x^2}$ in the xy -plane. The density at the point (x, y) on the arch is $\delta(x, y) = 2a - y$. Find the center of mass.

45. **Wire with constant density** A wire of constant density $\delta = 1$ lies along the curve $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + e^t\mathbf{k}$, $0 \leq t \leq \ln 2$. Find $\bar{x}$, I_x , and R_x .

46. **Helical wire with constant density** Find the mass and center of mass of a wire of constant density δ that lies along the helix $\mathbf{r}(t) = (2 \sin t)\mathbf{i} + (2 \cos t)\mathbf{j} + 3t\mathbf{k}$, $0 \leq t \leq 2\pi$.

47. **Inertia, radius of gyration, center of mass of a shell** Find I_x , R_x , and the center of mass of a thin shell of density $\delta(x, y, z) = z$ cut from the upper portion of the sphere $x^2 + y^2 + z^2 = 25$ by the plane $z = 3$.

48. **Moment of inertia of a cube** Find the moment of inertia about the z -axis of the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ if the density is $\delta = 1$.

Flux Across a Plane Curve or Surface

Use Green's Theorem to find the counterclockwise circulation and outward flux for the fields and curves in Exercises 49 and 50.

49. **Square** $\mathbf{F} = (2xy + x)\mathbf{i} + (xy - y)\mathbf{j}$

C : The square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$

50. **Triangle** $\mathbf{F} = (y - 6x^2)\mathbf{i} + (x + y^2)\mathbf{j}$

C : The triangle made by the lines $y = 0$, $y = x$, and $x = 1$

51. **Zero line integral** Show that

$$\oint_C \ln x \sin y \, dy - \frac{\cos y}{x} \, dx = 0$$

for any closed curve C to which Green's Theorem applies.

52. **a. Outward flux and area** Show that the outward flux of the position vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ across any closed curve to which Green's Theorem applies is twice the area of the region enclosed by the curve.

b. Let $\mathbf{n}$ be the outward unit normal vector to a closed curve to which Green's Theorem applies. Show that it is not possible for $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ to be orthogonal to $\mathbf{n}$ at every point of C .

In Exercises 53–56, find the outward flux of $\mathbf{F}$ across the boundary of D .

53. **Cube** $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$

D : The cube cut from the first octant by the planes $x = 1$, $y = 1$, $z = 1$

54. **Spherical cap** $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$

D : The entire surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \leq 25$ by the plane $z = 3$

55. **Spherical cap** $\mathbf{F} = -2x\mathbf{i} - 3y\mathbf{j} + z\mathbf{k}$

D : The upper region cut from the solid sphere $x^2 + y^2 + z^2 \leq 2$ by the paraboloid $z = x^2 + y^2$

56. **Cone and cylinder** $\mathbf{F} = (6x + y)\mathbf{i} - (x + z)\mathbf{j} + 4yz\mathbf{k}$

D : The region in the first octant bounded by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 1$, and the coordinate planes

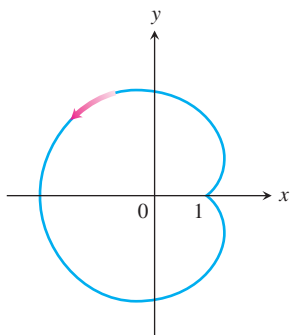
- 57. Hemisphere, cylinder, and plane** Let S be the surface that is bounded on the left by the hemisphere $x^2 + y^2 + z^2 = a^2, y \leq 0$, in the middle by the cylinder $x^2 + z^2 = a^2, 0 \leq y \leq a$, and on the right by the plane $y = a$. Find the flux of $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ outward across S .
- 58. Cylinder and planes** Find the outward flux of the field $\mathbf{F} = 3xz^2\mathbf{i} + y\mathbf{j} - z^3\mathbf{k}$ across the surface of the solid in the first octant that is bounded by the cylinder $x^2 + 4y^2 = 16$ and the planes $y = 2z, x = 0$, and $z = 0$.
- 59. Cylindrical can** Use the Divergence Theorem to find the flux of $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ outward through the surface of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 1$ and $z = -1$.
- 60. Hemisphere** Find the flux of $\mathbf{F} = (3z + 1)\mathbf{k}$ upward across the hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$ **(a)** with the Divergence Theorem and **(b)** by evaluating the flux integral directly.

Chapter 16 Additional and Advanced Exercises

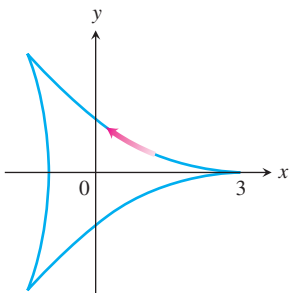
Finding Areas with Green's Theorem

Use the Green's Theorem area formula, Equation (13) in Exercises 16.4, to find the areas of the regions enclosed by the curves in Exercises 1–4.

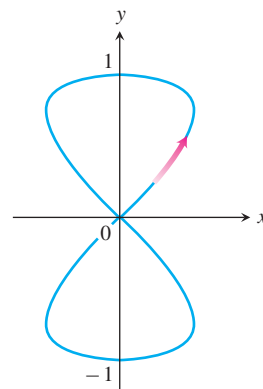
1. The limaçon $x = 2 \cos t - \cos 2t$, $y = 2 \sin t - \sin 2t$, $0 \leq t \leq 2\pi$



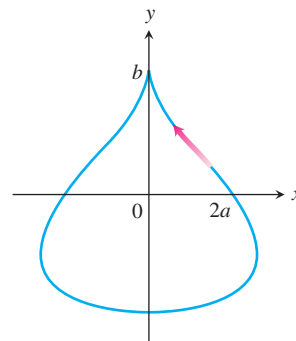
2. The deltoid $x = 2 \cos t + \cos 2t$, $y = 2 \sin t - \sin 2t$, $0 \leq t \leq 2\pi$



3. The eight curve $x = (1/2) \sin 2t$, $y = \sin t$, $0 \leq t \leq \pi$ (one loop)



4. The teardrop $x = 2a \cos t - a \sin 2t$, $y = b \sin t$, $0 \leq t \leq 2\pi$



Theory and Applications

5. a. Give an example of a vector field $\mathbf{F}(x, y, z)$ that has value $\mathbf{0}$ at only one point and such that $\text{curl } \mathbf{F}$ is nonzero everywhere. Be sure to identify the point and compute the curl.

- b. Give an example of a vector field $\mathbf{F}(x, y, z)$ that has value $\mathbf{0}$ on precisely one line and such that $\text{curl } \mathbf{F}$ is nonzero everywhere. Be sure to identify the line and compute the curl.
- c. Give an example of a vector field $\mathbf{F}(x, y, z)$ that has value $\mathbf{0}$ on a surface and such that $\text{curl } \mathbf{F}$ is nonzero everywhere. Be sure to identify the surface and compute the curl.
6. Find all points (a, b, c) on the sphere $x^2 + y^2 + z^2 = R^2$ where the vector field $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$ is normal to the surface and $\mathbf{F}(a, b, c) \neq \mathbf{0}$.
7. Find the mass of a spherical shell of radius R such that at each point (x, y, z) on the surface the mass density $\delta(x, y, z)$ is its distance to some fixed point (a, b, c) of the surface.
8. Find the mass of a helicoid
- $$\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + \theta\mathbf{k},$$
- $$0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, \text{ if the density function is } \delta(x, y, z) = 2\sqrt{x^2 + y^2}. \text{ See Practice Exercise 27 for a figure.}$$
9. Among all rectangular regions $0 \leq x \leq a, 0 \leq y \leq b$, find the one for which the total outward flux of $\mathbf{F} = (x^2 + 4xy)\mathbf{i} - 6y\mathbf{j}$ across the four sides is least. What is the least flux?
10. Find an equation for the plane through the origin such that the circulation of the flow field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ around the circle of intersection of the plane with the sphere $x^2 + y^2 + z^2 = 4$ is a maximum.
11. A string lies along the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$ in the first quadrant. The density of the string is $\rho(x, y) = xy$
- a. Partition the string into a finite number of subarcs to show that the work done by gravity to move the string straight down to the x -axis is given by

$$\text{Work} = \lim_{n \rightarrow \infty} \sum_{k=1}^n g x_k y_k^2 \Delta s_k = \int_C g xy^2 ds,$$

where g is the gravitational constant.

- b. Find the total work done by evaluating the line integral in part (a).
- c. Show that the total work done equals the work required to move the string's center of mass $(\bar{x}, \bar{y})$ straight down to the x -axis.
12. A thin sheet lies along the portion of the plane $x + y + z = 1$ in the first octant. The density of the sheet is $\delta(x, y, z) = xy$.
- a. Partition the sheet into a finite number of subpieces to show that the work done by gravity to move the sheet straight down to the xy -plane is given by

$$\text{Work} = \lim_{n \rightarrow \infty} \sum_{k=1}^n g x_k y_k z_k \Delta \sigma_k = \iint_S g xyz d\sigma,$$

where g is the gravitational constant.

- b. Find the total work done by evaluating the surface integral in part (a).

- c. Show that the total work done equals the work required to move the sheet's center of mass $(\bar{x}, \bar{y}, \bar{z})$ straight down to the xy -plane.

13. **Archimedes' principle** If an object such as a ball is placed in a liquid, it will either sink to the bottom, float, or sink a certain distance and remain suspended in the liquid. Suppose a fluid has constant weight density w and that the fluid's surface coincides with the plane $z = 4$. A spherical ball remains suspended in the fluid and occupies the region $x^2 + y^2 + (z - 2)^2 \leq 1$.

- a. Show that the surface integral giving the magnitude of the total force on the ball due to the fluid's pressure is

$$\text{Force} = \lim_{n \rightarrow \infty} \sum_{k=1}^n w(4 - z_k) \Delta \sigma_k = \iint_S w(4 - z) d\sigma.$$

- b. Since the ball is not moving, it is being held up by the buoyant force of the liquid. Show that the magnitude of the buoyant force on the sphere is

$$\text{Buoyant force} = \iint_S w(z - 4)\mathbf{k} \cdot \mathbf{n} d\sigma,$$

where $\mathbf{n}$ is the outer unit normal at (x, y, z) . This illustrates Archimedes' principle that the magnitude of the buoyant force on a submerged solid equals the weight of the displaced fluid.

- c. Use the Divergence Theorem to find the magnitude of the buoyant force in part (b).

14. **Fluid force on a curved surface** A cone in the shape of the surface $z = \sqrt{x^2 + y^2}, 0 \leq z \leq 2$ is filled with a liquid of constant weight density w . Assuming the xy -plane is "ground level," show that the total force on the portion of the cone from $z = 1$ to $z = 2$ due to liquid pressure is the surface integral

$$F = \iint_S w(2 - z) d\sigma.$$

Evaluate the integral.

15. **Faraday's Law** If $\mathbf{E}(t, x, y, z)$ and $\mathbf{B}(t, x, y, z)$ represent the electric and magnetic fields at point (x, y, z) at time t , a basic principle of electromagnetic theory says that $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$. In this expression $\nabla \times \mathbf{E}$ is computed with t held fixed and $\partial \mathbf{B} / \partial t$ is calculated with (x, y, z) fixed. Use Stokes' Theorem to derive Faraday's Law

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \mathbf{n} d\sigma,$$

where C represents a wire loop through which current flows counterclockwise with respect to the surface's unit normal $\mathbf{n}$, giving rise to the voltage

$$\oint_C \mathbf{E} \cdot d\mathbf{r}$$

around C . The surface integral on the right side of the equation is called the *magnetic flux*, and S is any oriented surface with boundary C .

16. Let

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^3} \mathbf{r}$$

be the gravitational force field defined for $\mathbf{r} \neq \mathbf{0}$. Use Gauss's Law in Section 16.8 to show that there is no continuously differentiable vector field $\mathbf{H}$ satisfying $\mathbf{F} = \nabla \times \mathbf{H}$.

17. If $f(x, y, z)$ and $g(x, y, z)$ are continuously differentiable scalar functions defined over the oriented surface S with boundary curve C , prove that

$$\iint_S (\nabla f \times \nabla g) \cdot \mathbf{n} \, d\sigma = \oint_C f \nabla g \cdot d\mathbf{r}.$$

18. Suppose that $\nabla \cdot \mathbf{F}_1 = \nabla \cdot \mathbf{F}_2$ and $\nabla \times \mathbf{F}_1 = \nabla \times \mathbf{F}_2$ over a region D enclosed by the oriented surface S with outward unit normal $\mathbf{n}$ and that $\mathbf{F}_1 \cdot \mathbf{n} = \mathbf{F}_2 \cdot \mathbf{n}$ on S . Prove that $\mathbf{F}_1 = \mathbf{F}_2$ throughout D .

19. Prove or disprove that if $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F} = \mathbf{0}$.

20. Let S be an oriented surface parametrized by $\mathbf{r}(u, v)$. Define the notation $d\boldsymbol{\sigma} = \mathbf{r}_u \, du \times \mathbf{r}_v \, dv$ so that $d\boldsymbol{\sigma}$ is a vector normal to the surface. Also, the magnitude $d\sigma = |d\boldsymbol{\sigma}|$ is the element of surface area (by Equation 5 in Section 16.6). Derive the identity

$$d\sigma = (EG - F^2)^{1/2} \, du \, dv$$

where

$$E = |\mathbf{r}_u|^2, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad \text{and} \quad G = |\mathbf{r}_v|^2.$$

21. Show that the volume V of a region D in space enclosed by the oriented surface S with outward normal $\mathbf{n}$ satisfies the identity

$$V = \frac{1}{3} \iint_S \mathbf{r} \cdot \mathbf{n} \, d\sigma,$$

where $\mathbf{r}$ is the position vector of the point (x, y, z) in D .