Milne Thomson Method for constructing analytic function $f(z) = u + iv$

by this method $f(z)$ is directly constructed without finding v (when u is given) or viceversa.

Case Ist:- when u is given:

we have $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{(using C-R eq}^n\text{s)}$$

say $\frac{\partial u}{\partial x} = \phi_1(x,y)$, $\frac{\partial u}{\partial y} = \phi_2(x,y)$

i.e $f'(z) = \phi_1(x,y) - i\,\phi_2(x,y)$

replace x by z, and y by 0 in ϕ_1 & ϕ_2

$$f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$$

on integrating

$$\int f'(z)\,dz = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$\boxed{f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + C}$$

case IInd. when v is given:-

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \text{(C-R eq}^n\text{)}$$

say $v_x = \psi_1(x,y)$, $v_y = \psi_2(x,y)$

$$f'(z) = \psi_1(x,y) + i\,\psi_2(x,y)$$

replace x by z, y by 0

$$f'(z) = \psi_1(z,0) + i\,\psi_2(z,0)$$

$$\int f'(z)\,dz = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz$$

$$\boxed{f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + C}$$

Case IIIrd :- when $u-v$ is given.

we have $f(z) = u+iv$

$if(z) = iu - v$

adding $(1+i)f(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

Now $F'(z) = \dfrac{\partial U}{\partial x} + i\dfrac{\partial V}{\partial x}$

$U = u - v$

$$F'(z) = \phi_1(z,0) + i\,\phi_2(z,0)$$

integrate & solve then $\boxed{f(z) = \dfrac{F(z)}{1+i}}$

Case IVth :- when $u+v$ is given

we have $f(z) = u+iv$,

$if(z) = iu - v$

$(1+i) f(z) = (u-v) + i(u+v)$

$F(z) = U + iV$

$V = u + v$ is given i.e

$F'(z) = U_x + i V_x$

$F'(z) = V_y + i V_x$ (C-R eqn)

$F'(z) = \phi_1(x, y) + i \phi_2(x, y)$

$F'(z) = \phi_1(z, 0) + i \phi_2(z, 0)$

Now integrale.

once $F(z)$ is Known

$$\boxed{f(z) = \frac{F(z)}{1+i}}$$

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-1. Show that $e^x(x\cos y - y\sin y)$ is harmonic function. Find analytic function for which $e^x(x\cos y - y\sin y)$ is imaginary part.

Soln:- given $V = e^x(x\cos y - y\sin y)$

$$\frac{\partial v}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$\frac{\partial^2 v}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x\cos y + e^x\cos y$$
$$= e^x(x\cos y - y\sin y) + 2e^x\cos y \quad \text{--- (1)}$$

$$\frac{\partial v}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^x(-x\cos y + y\sin y - 2\cos y) \quad \text{--- (2)}$$

adding (1) & (2)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\Rightarrow$ v is harmonic function.

Now

Let $f(z) = u + iv$

$$f'(z) = u_x + iv_x$$

$$f'(z) = \cancel{v}_y + iv_x \quad \text{--- (3)}$$

$$\varphi_1(x, y) = v_y = e^x(-x\sin y - y\cos y - \sin y)$$

$$\varphi_1(z, 0) = 0$$

$$\varphi_2(x, y) = v_x = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$\phi_2(z,0) = e^z(z) + e^z$$

from (iii)

$$f'(z) = \phi_1(z,0) + i\,\phi_2(z,0)$$

$$f'(z) = 0 + i(ze^z + e^z)$$

$$\int f'(z)\,dz = \int i(ze^z + e^z)\,dz + c$$

$$f(z) = i(ze^z \cancel{-e^z} + \cancel{e^z}) + c$$

$$\boxed{f(z) = ize^z + c}$$

required analytic function.

Q:- If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, find $f(z)$.

Hint:- $\phi_1(z,0) = \dfrac{2\cos 2z + 2}{(1+\cos 2z)^2}$

$$\phi_2(z,0) = 0$$

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$$

$$f(z) = 2\int \frac{1}{1+\cos 2z}\,dz + c$$

$$f(z) = \int \sec^2 z\,dz + c, \quad f(z) = \tan z + c$$