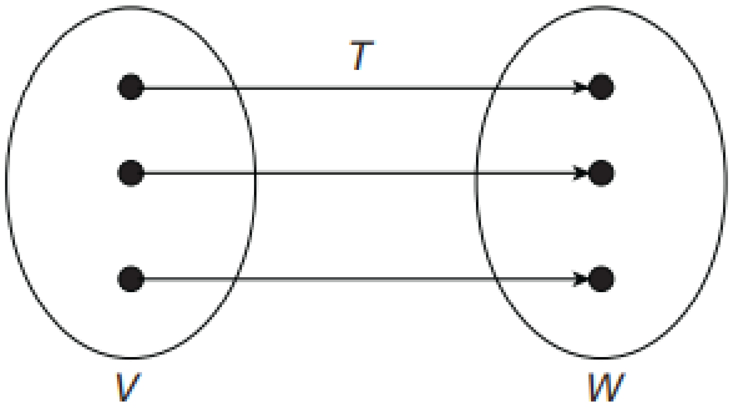
One-to-one and On-to Transformations

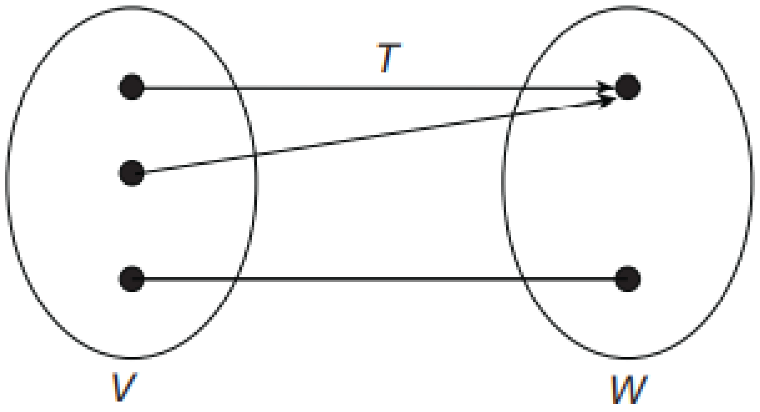
One-to-one Transformation

Let V and W be two vector spaces. A linear transformation $T : V \to W$ is one-to-one if T maps distinct vectors in V to distinct vectors in W.

A one-to-one transformation is also called injective transformation.



(i) T is one-to-one



(ii) T is not one-to-one

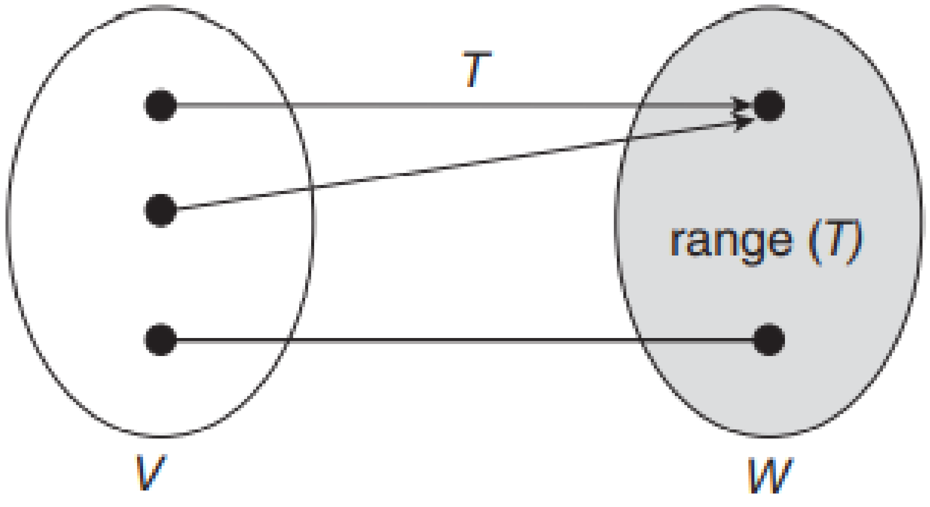
Theorem : A linear transformation $T : V \to W$ is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

Theorem : A linear transformation $T : V \to W$ is one-to-one if and only if $\dim(\ker(T)) = 0$, i.e., $\text{nullity}(T) = 0$.

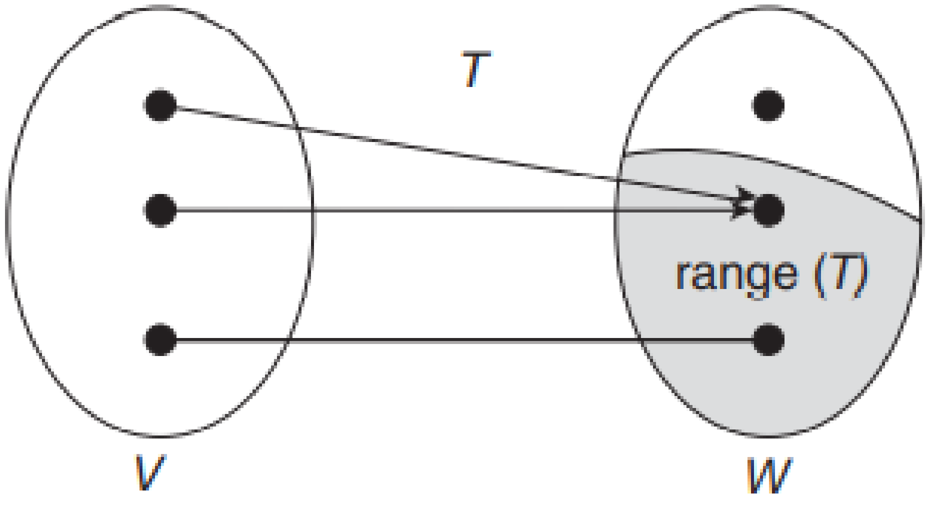
Theorem: A linear transformation $T : V \to W$ is one-to-one if and only if $\text{rank}(T) = \dim V$.

Onto Transformation

Let V and W be two vector spaces. A linear transformation $T : V \to W$ is onto if the range of T is W, i.e., T is onto if and only if for every $\mathbf{w}$ in W, there is a $\mathbf{v}$ in V such that $T(\mathbf{v}) = W$. An onto transformation is also called surjective transformation.



(i) *T* is onto



(ii) *T* is not onto

Theorem : A linear transformation $T : V \to W$ is onto if and only if rank $(T) = \dim W$.

Theorem : If A is an $m \times n$ matrix and $T_A : R^n \to R^m$ is multiplication by A then T_A is onto if and only if rank $(A) = m$.

Theorem : Let $T : V \to W$ be a linear transformation and let $\dim V = \dim W$

(i) If T is one-to-one, then it is onto.

(ii) If T is onto, then it is one-to-one.

Bijective Transformation

If a transformation $T : V \rightarrow W$ is both one-to-one and onto then it is called bijective transformation.

Example:

In each case, determine whether the linear transformation is one-to-one, onto, both or neither.

(i) $T: P_2 \to P_2$, where $T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_1) + (a_2 + 2a_1)x$
(ii) $T: P_2 \to P_2$, where $T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1 (x+1) + a_2 (x+1)^2$
(iii) $T: R^2 \to P_1$, where $T(a, b) = a + (a+b)x$
(iv) $T: P_2 \to R^3$, where $T(a + bx + cx^2) = \begin{bmatrix} 2a - b \\ a + b - 3c \\ c - a \end{bmatrix}$

Solution: (i) (a) Let $T(a_0 + a_1x + a_2x^2) = \mathbf{0}$

$$(a_0 + a_1) + (a_2 + 2a_1)x = 0$$

$$a_0 + a_1 = 0$$

$$a_2 + 2a_1 = 0$$

Let

$$a_2 = t$$

$$a_1 = -\frac{1}{2}t$$

$$a_0 = \frac{1}{2}t$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one to one.

(b) $\dim(\ker(T)) = 1 = \text{nullity}(T)$

From the dimension theorem,

$$\begin{aligned} \text{rank}(T) &= \dim P_2 - \text{nullity}(T) \\ &= 3 - 1 \\ &= 2 \end{aligned}$$

Dimension of $W(P_2) = 3$.

$$\text{rank}(T) \neq \dim W$$

Hence, T is not onto.

(ii) (a) Let $$T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1(x+1) + a_2(x+1)^2 = \mathbf{0}$$

i.e.,
$$a_0 + a_1 x + a_1 + a_2 x^2 + 2a_2 x + a_2 = 0$$
$$(a_0 + a_1 + a_2) + (a_1 + 2a_2)x + a_2 x^2 = 0$$
$$\begin{aligned} a_0 + a_1 + a_2 &= 0 \\ a_1 + 2a_2 &= 0 \\ a_2 &= 0 \end{aligned}$$

Solving these equations,
$$\begin{aligned} a_0 &= 0 \\ a_1 &= 0 \\ a_2 &= 0 \end{aligned}$$
$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\ker(T) = \{\mathbf{0}\}$$

Hence, T is one-to-one.

(b) $\dim(\ker(T)) = 0 = \text{nullity}(T)$

From the dimension theorem,

$$\begin{aligned}\text{rank}(T) &= \dim P_2 - \text{nullity}(T)\\ &= 3 - 0\\ &= 3\end{aligned}$$

Dimension of $W(P_2) = 3$.

$$\therefore \text{rank}(T) = \dim W$$

Hence, T is onto.

(iii) (a) Let $T(a, b) = a + (a + b)\,x = \mathbf{0}$

$$a = 0$$

$$a + b = 0$$

Solving these equations,

$$a = 0$$

$$b = 0$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\ker(T) = \{\mathbf{0}\}$$

Hence, T is one-to-one.

(b) $\dim(\ker(T)) = 0 = \text{nullity}(T)$

From the dimension theorem,

$$\begin{aligned}\text{rank}(T) &= \dim R^2 - \text{nullity}(T)\\ &= 2-0\\ &= 2\end{aligned}$$

Dimension of $W(P_1) = 2$.

$$\text{rank}(T) = \dim W$$

Hence, T is onto.

(iv) (a) Let $T(a + bx + cx^2) = \mathbf{0}$

$$\begin{bmatrix} 2a-b \\ a+b-3c \\ c-a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2a-b &= 0 \\ a+b-3c &= 0 \\ c-a &= 0 \end{aligned}$$

$$\therefore a = \frac{b}{2} = c$$

Let

$$\begin{aligned} c &= t \\ b &= 2t \\ a &= t \end{aligned}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one-to-one.

(b) $\dim(\ker(T)) - 1 - \text{nullity}(T)$

From the dimension theorem,

$$\begin{aligned}\text{rank}(T) &= \dim P_2 - \text{nullity}(T)\\ &= 3-1\\ &= 2\end{aligned}$$

Dimension of $W(R^3) = 3$.

$$\text{rank}(T) \neq \dim \mathrm{W}$$

Hence, T is not onto.

Example : In each case, determine whether linear transformation is one-to-one, onto, both or neither.

(i) $T: M_{22} \to M_{22}$, where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(ii) $T: M_{22} \to M_{22}$, where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2d & 0 \\ 0 & 0 \end{bmatrix}$

(iii) $T: M_{22} \to R^3$, where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix}$

Solution: (i) (a) Let $$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathbf{0}$$

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$d = 0$$

$$-b = 0 \Rightarrow b = 0$$

$$-c = 0 \Rightarrow c = 0$$

$$a = 0$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\ker(T) = \{\mathbf{0}\}$$

Hence, T is one-to-one.

(b) $\dim(\ker(T)) = 0 = \text{nullity}(T)$

From the dimension theorem,

$$\begin{aligned}\text{rank}(T) &= \dim M_{22} - \text{nullity}(T)\\ &= 4-0\\ &= 4\end{aligned}$$

Dimension of $W(M_{22}) = 4$

$$\therefore \text{rank}(T) = \dim W$$

Hence T, is onto.

(ii) (a) Let $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathbf{0}$

$$\begin{bmatrix} 2d & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2d = 0$$

$$\therefore d = 0$$

Let

$$a = t_1$$
$$b = t_2$$
$$c = t_3$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one-to-one.

(b) $\dim(\ker(T)) = 3 = \text{nullity}(T)$

From the dimension theorem,

$$\begin{aligned}\text{rank}(T) &= \dim M_{22} - \text{nullity}(T)\\ &= 4-3\\ &= 1\end{aligned}$$

Dimension of $W(M_{22}) = 4$.

$$\therefore \text{rank}(T) \neq \dim W$$

Hence, T is not onto.

(iii) (a) Let
$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \mathbf{0}$$

$$\begin{bmatrix} a+b \\ b+c \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
a+b \quad\quad\quad &= 0 \\
b+c \quad\quad &= 0 \\
c+d &= 0
\end{aligned}
$$

Let

$$
\begin{aligned}
d &= t \\
c &= -t \\
b &= t \\
a &= -t
\end{aligned}
$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\ker(T) \neq \{\mathbf{0}\}$$

Hence, T is not one to one.

(b) $\dim(\ker(T)) = 1 = \text{nullity}(T)$
From the dimension theorem,

$$\begin{aligned} \text{rank}(T) &= \dim M_{22} - \text{nullity}(T) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

Dimension of $W(R^3) = 3$.

$$\therefore \text{rank}(T) = \dim W$$

Hence, T is onto.

Example 1: In each case, determine whether the linear transformation is one-to-one, onto, or both or neither.

(i) $T : R^2 \to R^2$, where $T(x, y) = (x + y, x - y)$

(ii) $T : R^2 \to R^3$, where $T(x, y) = (x - y, y - x, 2x - 2y)$

(iii) $T : R^3 \to R^2$, where $T(x, y, z) = (x + y + z, x - y - z)$

(iv) $T : R^3 \to R^3$, where $T(x, y, z) = (x + 3y, y, z + 2x)$

INVERSE LINEAR TRANSFORMATIONS

If $T : V \to W$ is a linear transformation then the range of T is the subs consisting of all images of vectors in V under T. If T is one-to-one then eac in $R(T)$ is the image of a unique vector **u** in V. Hence, inverse linear tran $T^{-1}: W \to V$ maps **w** back into **v**.

Example 1: Let $T : R^3 \to R^3$ be the linear operator defined by the formula

$$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3,\ 2x_2 - x_3,\ 2x_1 + 3x_2)$$

Determine whether T is one-to-one. If so, find $T^{-1}(x_1, x_2, x_3)$.

Example 7. *Let T be a linear operator on R^3 defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. Show that T is invertible and find T^{-1}.*

Solution : (*i*) Let W be the null space of T.

So W is the set of all (x, y, z) such that

$$T(x, y, z) = (0, 0, 0)$$

i.e. $(2x, 4x - y, 2x + 3y - z) = (0, 0, 0)$

$\Rightarrow$ W is the solution space of

$$2x = 0,\ 4x - y = 0,\ 2x + 3y - z = 0,$$

which has $(0, 0, 0)$ as trivial solution.

Thus $W = \{0\}$.

Hence T is non-singular and hence is invertible.

(*ii*) Let $T(x, y, z) = (r, s, t)$

$\therefore T^{-1}(r, s, t) = (x, y, z)$.

Now $T(x, y, z) = (2x, 4x - y, 2x + 3y - z) = (r, s, t)$

$\Rightarrow 2x = r,\ 4x - y = s,\ 2x + 3y - z = t$

$\Rightarrow x = \frac{1}{2}r, y = 2r - s, z = 7r - 3s - t.$

Hence T^{-1} is defined by

$$T^{-1}(r, s, t) = \left(\frac{1}{2}r, 2r - s, 7r - 3s - t\right).$$

... T_2 are linear transformations

Example 4. Let $T : V_3(R) \rightarrow V_3(R)$ be defined as

$$T(a, b, c). = (3a, a - b, 2a + b + c).$$

Prove that T is invertible and find T^{-1}

Solution : (*i*) We know that T is invertible if T is one-one and onto.

Let $x_1 = (a_1, b_1, c_1), x_2 = (a_2, b_2, c_2) \in V_3(R)$,

then $T(x_1) = T(x_2) \Rightarrow (3a_1, a_1 - b_1, 2a_1 + b_1 + c_1)$

$$= (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$$

$\Rightarrow 3a_1 = 3a_2, a_1 - b_1 = a_2 - b_2, 2a_1 + b_1 + c_1 = 2a_2 + b_2 + c_2$

$\Rightarrow \quad a_1 = a_2, b_1 = b_2, c_1 = c_2$

$\Rightarrow \quad x_1 = x_2.$

Thus T is one-one.

Let $(r, s, t) \in V_3(R)$. We shall show that $\exists$ a vector in $V_3(R)$ whose T image is (r, s, t).

Let that vector be (a, b, c) so that

$$T(a, b, c) = (r, s, t) = (3a, a - b, 2a + b + c)$$

$$\Rightarrow r = 3a, s = a - b, t = 2a + b + c$$

$$\Rightarrow a = \frac{r}{3}, b = \frac{r}{3} - s, c = t - r + s.$$

Since $r, s, t \in R$, $\therefore a, b, c$ also $\in R$ so that $(a, b, c) \in V_3(R)$.

Thus T is onto.

Hence T is invertible.

$$\therefore \quad T(a, b, c) = (r, s, t)$$

$$\Rightarrow T^{-1}(r, s, t) = (a, b, c)$$

Hence $T^{-1}(r, s, t) = \left(\frac{r}{3}, \frac{r}{3} - s, t - r + s\right).$