

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b ,

$$A = \int_a^b f(x) dx.$$

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$** , also called its **mean**, is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x) dx.$$

EXAMPLE Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution We recognize $f(x) = \sqrt{4 - x^2}$ as a function whose graph is the upper semi-circle of radius 2 centered at the origin

The area between the semicircle and the x -axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi(2)^2 = 2\pi.$$

Because f is nonnegative, the area is also the value of the integral of f from -2 to 2 ,

$$\int_{-2}^2 \sqrt{4 - x^2} dx = 2\pi.$$

Therefore, the average value of f is

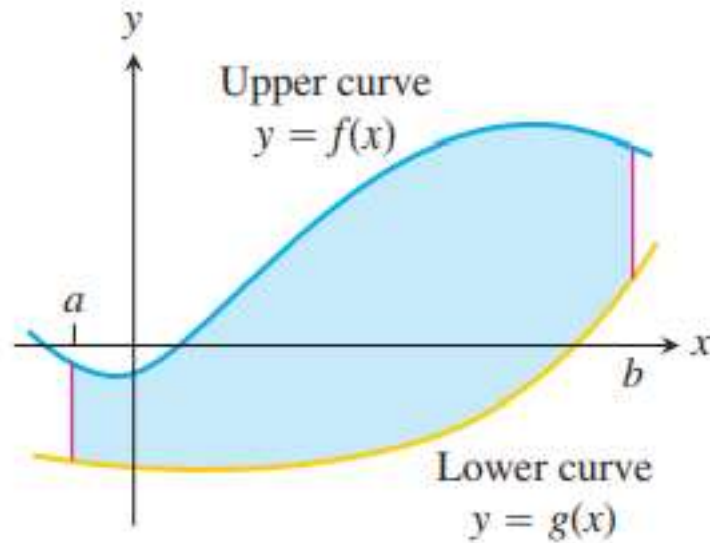
$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^2 \sqrt{4 - x^2} dx = \frac{1}{4} (2\pi) = \frac{\pi}{2}.$$

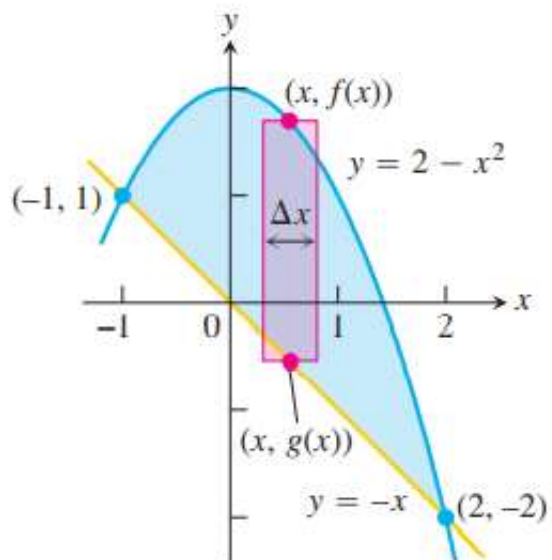
$$\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right)$$

Areas Between Curves

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b :

$$A = \int_a^b [f(x) - g(x)] dx.$$





EXAMPLE Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves. The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x .

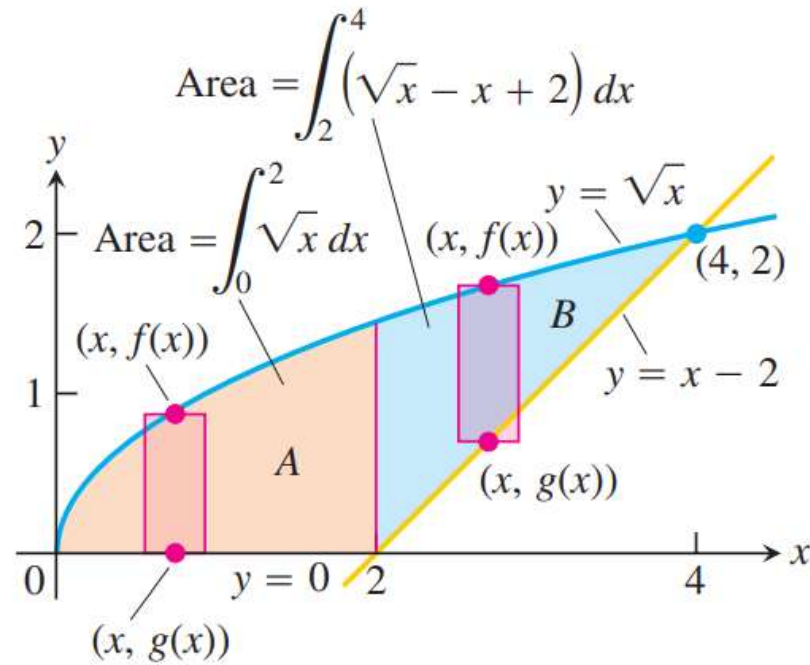
$$\begin{aligned}
 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\
 x^2 - x - 2 &= 0 && \text{Rewrite.} \\
 (x + 1)(x - 2) &= 0 && \text{Factor.} \\
 x = -1, \quad x = 2. &&& \text{Solve.}
 \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.
The area between the curves is

$$\begin{aligned}
 A &= \int_a^b [f(x) - g(x)] dx = \int_{-1}^2 [(2 - x^2) - (-x)] dx \\
 &= \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^2 \\
 &= \left(4 + \frac{4}{2} - \frac{8}{3} \right) - \left(-2 + \frac{1}{2} + \frac{1}{3} \right) = \frac{9}{2}
 \end{aligned}$$

EXAMPLE

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x -axis and the line $y = x - 2$.



$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x} dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2) dx}_{\text{area of } B}$$

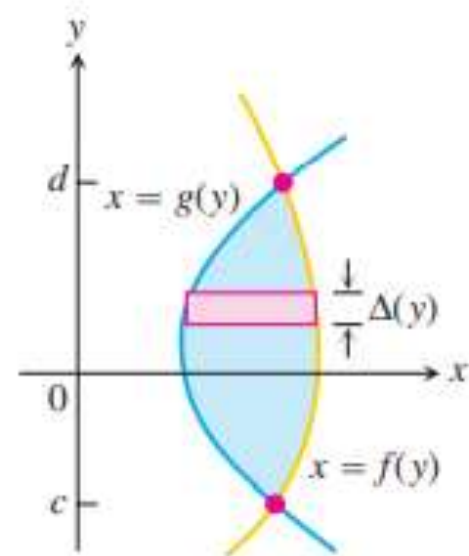
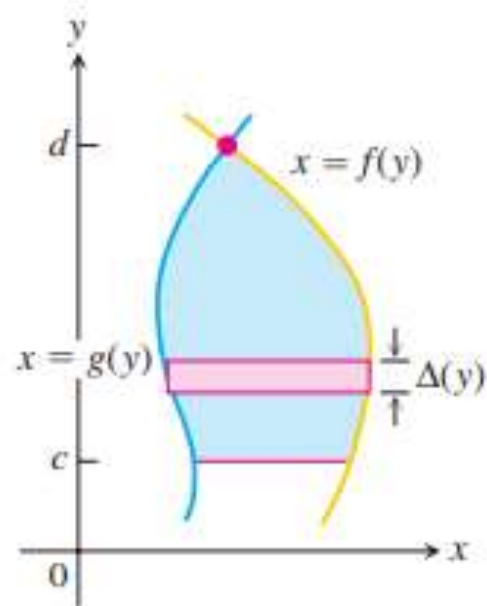
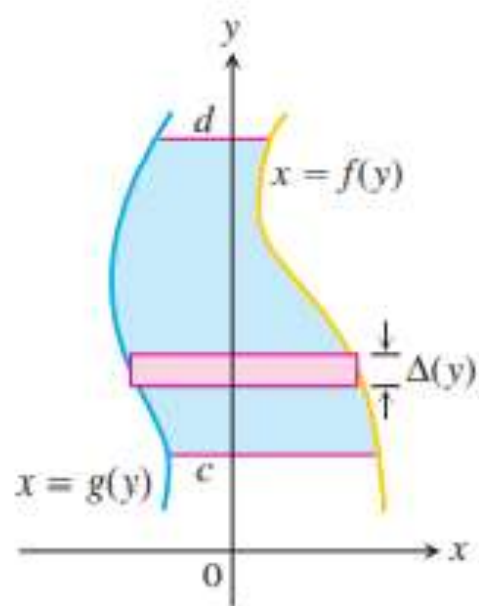
The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x :

$$\begin{aligned}\sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x). \\ x &= (x - 2)^2 = x^2 - 4x + 4 && \text{Square both sides.} \\ x^2 - 5x + 4 &= 0 && \text{Rewrite.} \\ (x - 1)(x - 4) &= 0 && \text{Factor.} \\ x &= 1, \quad x = 4. && \text{Solve.}\end{aligned}$$

We add the areas of subregions A and B to find the total area:

$$\begin{aligned}\text{Total area} &= \underbrace{\int_0^2 \sqrt{x} \, dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2) \, dx}_{\text{area of } B} \\ &= \left[\frac{2}{3} x^{3/2} \right]_0^2 + \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} + 2x \right]_2^4 \\ &= \frac{2}{3} (2)^{3/2} - 0 + \left(\frac{2}{3} (4)^{3/2} - 8 + 8 \right) - \left(\frac{2}{3} (2)^{3/2} - 2 + 4 \right) \\ &= \frac{2}{3} (8) - 2 = \frac{10}{3}.\end{aligned}$$

Integration with Respect to y

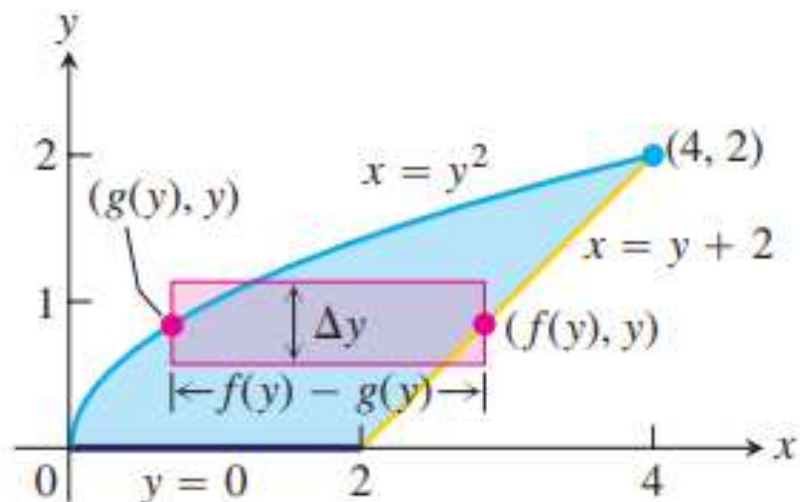


use the formula

$$A = \int_c^d [f(y) - g(y)] dy.$$

EXAMPLE

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x -axis and the line $y = x - 2$. **by integrating with respect to y .**



$$y + 2 = y^2 \quad \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2.$$

$$y^2 - y - 2 = 0 \quad \text{Rewrite.}$$

$$(y + 1)(y - 2) = 0 \quad \text{Factor.}$$

$$y = -1, \quad y = 2 \quad \text{Solve.}$$

The area of the region is

$$A = \int_c^d [f(y) - g(y)] dy = \int_0^2 [y + 2 - y^2] dy$$

$$= \int_0^2 [2 + y - y^2] dy$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3} \right]_0^2$$

$$= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}.$$