

Computational Geometry Algorithms

- Computational geometry is the branch of computer science that studies algorithms for solving geometric problems.
- In modern engineering and mathematics, computational geometry has applications in, among other fields, computer graphics, robotics, VLSI design, computer-aided design, and statistics.
- The input to a computational-geometry problem is typically a set of geometric objects, such as a set of points, a set of line segments, or the vertices of a polygon in counterclockwise order.
- The output is often a response to a query about the objects, such as whether any of the lines intersect, or perhaps a new geometric object, such as the convex hull (smallest enclosing convex polygon) of the set of points.
- Each input object is represented as a set of points $\{p_1, p_2, p_3, \dots\}$, where each $p_i = (x_i, y_i)$ and $x_i, y_i \in \mathbb{R}$.
- For example, an n -vertex polygon P is represented by a sequence $p_0, p_1, p_2, \dots, p_{n-1}$ of its vertices in order of their appearance on the boundary of P .
- Computational geometry can also be performed in three dimensions, and even in higher-dimensional spaces, but such problems and their solutions can be very difficult to visualize.

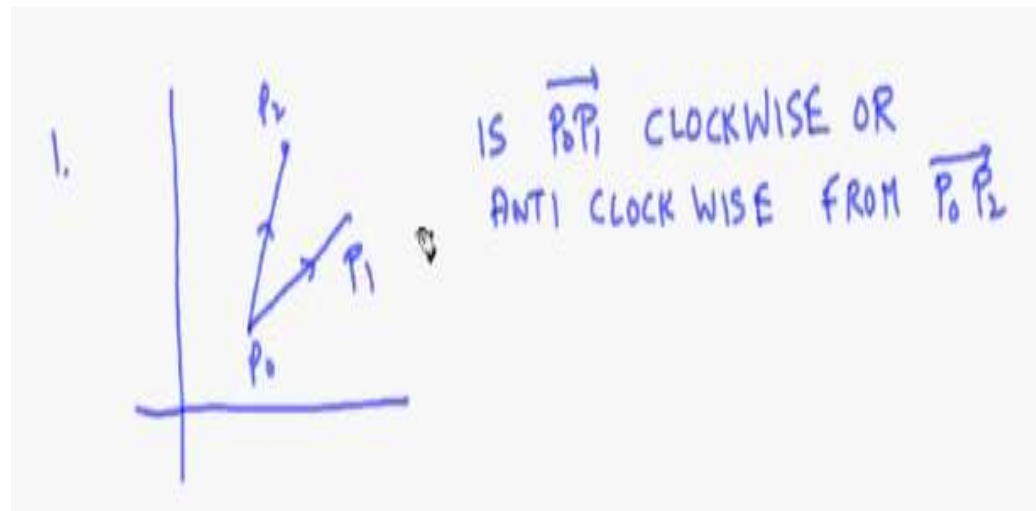
Line-segment properties

- Several of the computational-geometry algorithms will require answers to questions about the properties of line segments.
- We shall explore the following questions:

1. Given two directed segments $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_0p_2}$, is $\overrightarrow{p_0p_1}$ clockwise from $\overrightarrow{p_0p_2}$ with respect to their common endpoint p_0 ?
2. Given two line segments $\overline{p_0p_1}$ and $\overline{p_1p_2}$, if we traverse $\overline{p_0p_1}$ and then $\overline{p_1p_2}$, do we make a left turn at point p_1 ?
3. Do line segments $\overline{p_1p_2}$ and $\overline{p_3p_4}$ intersect?

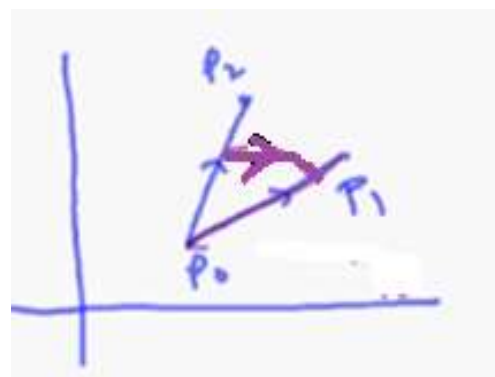
Let us understand the above questions.

Question -1:

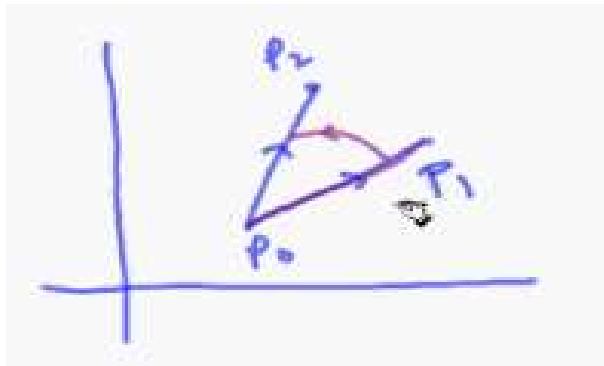


Given two line segments, if we have to reach to p_0p_1 from p_0p_2 , we have to find which is nearer, moving clockwise or anti-clockwise (counter-clockwise) from p_0p_2 .

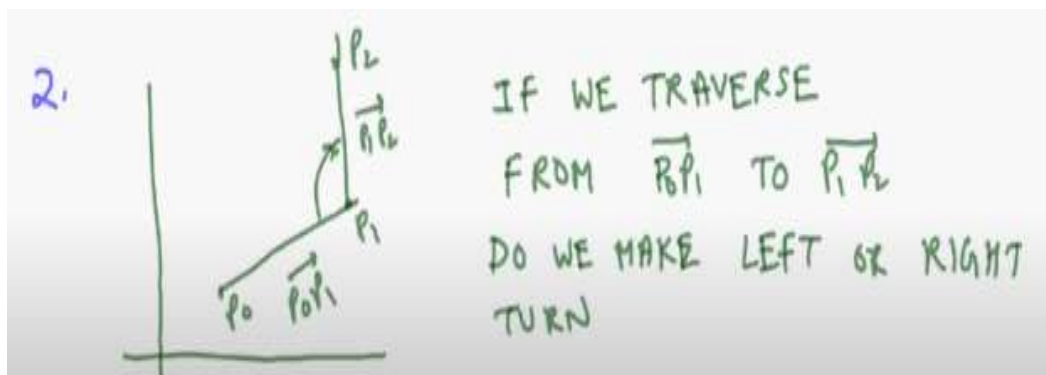
Moving clockwise is nearer.



If we have to reach to POP2 from POP1, then moving anti-clockwise is nearer (shown using purple color line).

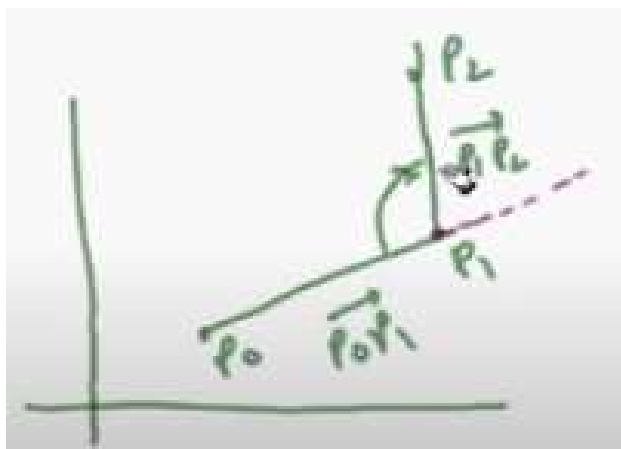


Question-2:

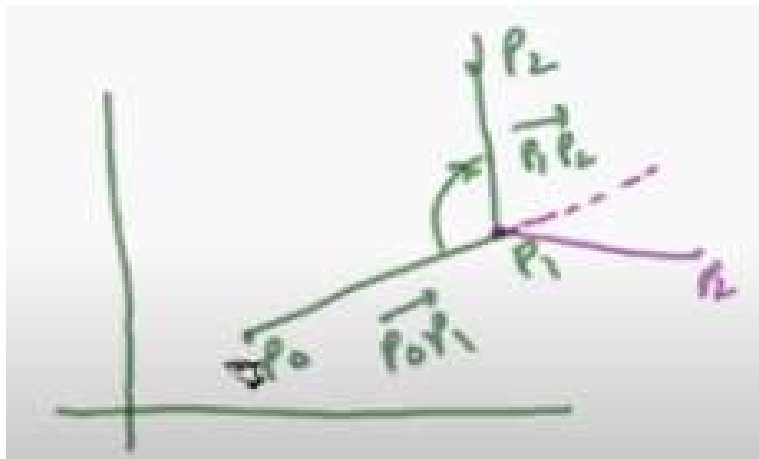


If we move from P0 to P1 and then from P1 to P2, do we make a right turn or left turn?

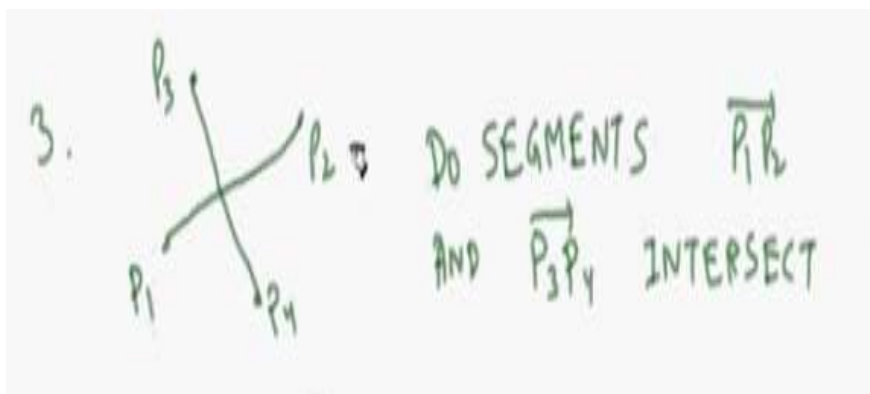
In the below figure, we make a left turn.



If P_2 is as shown below figure (in purple), we make right turn.



Question-3:



P_1P_2 and P_3P_4 are the line segments. Do they intersect each other?

Here, the line segments P_1P_2 and P_3P_4 intersect.

There may be other cases like as shown in below figure, where line segments do not intersect. Here, the line segments P_1P_2 and P_3P_4 do not intersect.



To answer these questions, let us know about vector representation and cross product operation.

Vector Representation:

- A **convex combination** of two distinct points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is any point $p_3 = (x_3, y_3)$.
- Intuitively, p_3 is any point that is on the line passing through p_1 and p_2 and is on or between p_1 and p_2 on the line.
- Given two distinct points p_1 and p_2 , the **line segment** $\overrightarrow{p_1 p_2}$ is the set of convex combinations of p_1 and p_2 .
- If p_1 is the **origin** $(0, 0)$, then we can treat the $\overrightarrow{p_1 p_2}$ as the **vector** $\vec{p_2}$.

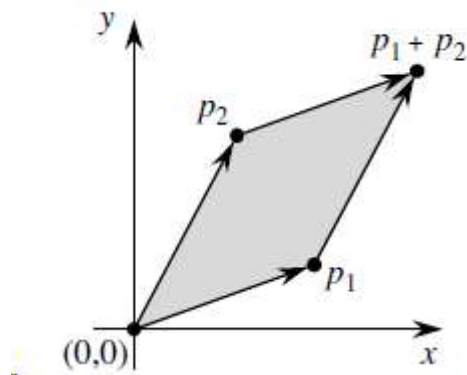
Cross Product Operation:

- Each point is represented by coordinates (x, y) . Example: $P_0 = x_0, y_0$; $P_1 = x_1, y_1$; $P_2 = x_2, y_2$
- Cross product of 2 vectors gives area of parallelogram.

Case-1: P_0 is the origin $(0,0)$

$P_0 = 0, 0$; $P_1 = x_1, y_1$; $P_2 = x_2, y_2$;

The **cross product** $p_1 \times p_2$ can be interpreted as the signed area of the parallelogram **formed by the points $(0, 0)$, p_1 , p_2 , and $p_1 + p_2 = (x_1 + x_2, y_1 + y_2)$** . In the below figure, P_0 is $(0,0)$.



Cross product is the determinant of a matrix.

$$\begin{aligned}\vec{p_0 p_1} \times \vec{p_0 p_2} &= \vec{p_1} \times \vec{p_2} = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \\ &= x_1 y_2 - x_2 y_1\end{aligned}$$

If $\vec{p_1} \times \vec{p_2}$ is positive, then p_1 is clockwise from p_2 with respect to origin $(0,0)$.

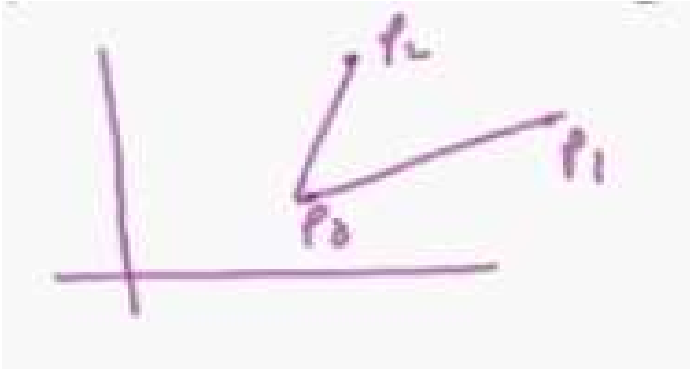
If $\vec{p_1} \times \vec{p_2}$ is negative, then p_1 is counter clockwise from p_2 with respect to origin.

Case-2: P_0 is not $(0,0)$

$P_0 = x_0, y_0$; $P_1 = x_1, y_1$; $P_2 = x_2, y_2$;

If p_0 is not origin, we simply translate to use p_0 as the origin.

In the below figure, P_0 is not $(0,0)$.



$$\begin{aligned}\vec{P_0P_1} \times \vec{P_0P_2} &= (P_1 - P_0) \times (P_2 - P_0) = \begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{vmatrix} \\ &= (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)\end{aligned}$$

If $(P_1 - P_0) \times (P_2 - P_0)$ is positive, then $\vec{P_0P_1}$ is clockwise from $\vec{P_0P_2}$.

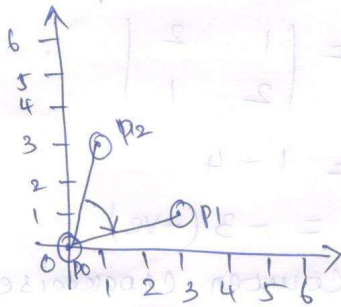
If $(P_1 - P_0) \times (P_2 - P_0)$ is negative, then $\vec{P_0P_1}$ is counter clockwise from $\vec{P_0P_2}$.

Note: If p_0 is not origin, $x_1 - x_0$ becomes new x_1 and $y_1 - y_0$ becomes new y_1 . Similarly, $x_2 - x_0$ becomes new x_2 and $y_2 - y_0$ becomes new y_2 .

Note: In both cases (case 1 and case 2), a boundary condition arises if the cross product is 0; in this case, the vectors are **collinear**, pointing in either the same or opposite directions.

Examples:

1) $P_0(0,0)$ $P_1(3,1)$ $P_2(1,3)$

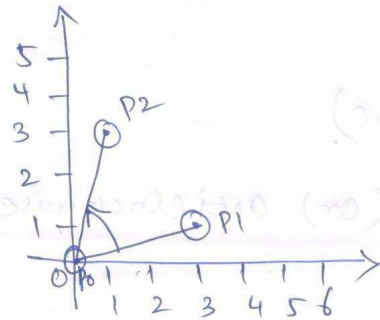


$$\vec{P_0P_1} \times \vec{P_0P_2} = \vec{P_1} \times \vec{P_2} = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 9 - 1 = 8 (+ve)$$

$\therefore \vec{P_1}$ is clockwise from $\vec{P_2}$

2) $P_0(0,0)$ $P_1(3,1)$ $P_2(1,3)$

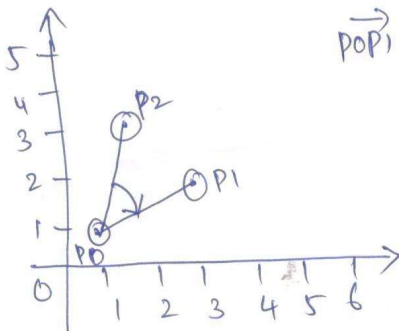


$$\vec{P_0P_2} \times \vec{P_0P_1} = \vec{P_2} \times \vec{P_1} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 1 - 9 = -8 (-ve)$$

$\therefore \vec{P_2}$ is counter clockwise from $\vec{P_1}$

3) $P_0(1,1)$ $P_1(3,2)$ $P_2(2,3)$



$$\vec{P_0P_1} \times \vec{P_0P_2} = P_1 - P_0 \times P_2 - P_0$$

$$P_1 - P_0 = (x_1 - x_0, y_1 - y_0)$$

$$= (3 - 1, 2 - 1) = (2, 1)$$

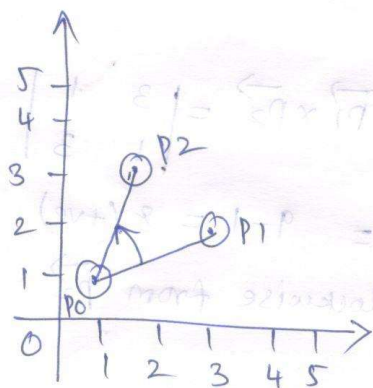
$$P_2 - P_0 = (x_2 - x_0, y_2 - y_0)$$

$$= (2 - 1, 3 - 1) = (1, 2)$$

$$\vec{P_0P_1} \times \vec{P_0P_2} = (P_1 - P_0) \times (P_2 - P_0) = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 (+ve)$$

$\therefore \vec{P_0P_1}$ is clockwise from $\vec{P_0P_2}$

4) $P_0(1,1)$ $P_1(3,2)$ $P_2(2,3)$



$$\vec{P_0P_2} \times \vec{P_0P_1} = (P_2 - P_0) \times (P_1 - P_0)$$

$$= \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 1 - 4$$

$$= -3 \text{ (-ve)}$$

$\therefore \vec{P_0P_2}$ is counter clockwise from $\vec{P_0P_1}$.

Practice problem

1) $a(1,1)$ $b(3,2)$ $c(2,4)$ $d(3,0)$

(i) say whether $\vec{ac}$ is clockwise (or) anticlockwise from $\vec{ab}$

(ii) say whether $\vec{ad}$ is clockwise (or) anticlockwise from $\vec{ab}$

Question-1:

1. Given two directed segments $\vec{p_0p_1}$ and $\vec{p_0p_2}$, is $\vec{p_0p_1}$ clockwise from $\vec{p_0p_2}$ with respect to their common endpoint p_0 ?

Answer:

Apply cross product theorem. From our previous discussion, we can understand that the answer for question 1 can be obtained by calculating the cross product.

Question-2:

2. Given two line segments $\overline{p_0p_1}$ and $\overline{p_1p_2}$, if we traverse $\overline{p_0p_1}$ and then $\overline{p_1p_2}$, do we make a left turn at point p_1 ?

Answer: Apply cross product theorem.

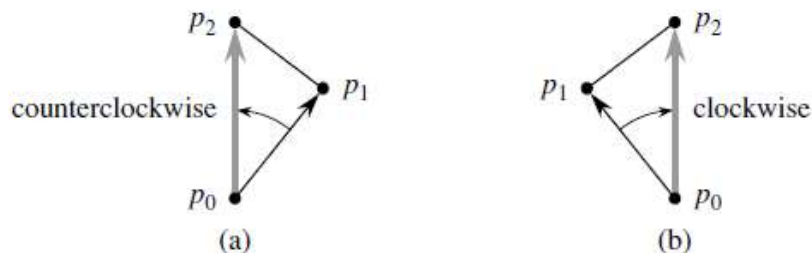
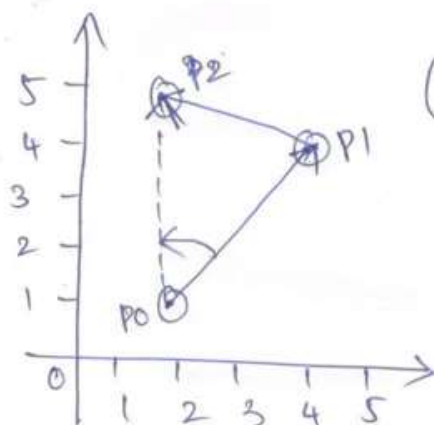


Figure 33.2 Using the cross product to determine how consecutive line segments $\overline{p_0p_1}$ and $\overline{p_1p_2}$ turn at point p_1 . We check whether the directed segment $\overrightarrow{p_0p_2}$ is clockwise or counterclockwise relative to the directed segment $\overrightarrow{p_0p_1}$. (a) If counterclockwise, the points make a left turn. (b) If clockwise, they make a right turn.

Examples :

Consider pts $P_0(2,1)$ $P_1(4,4)$ $P_2(2,5)$

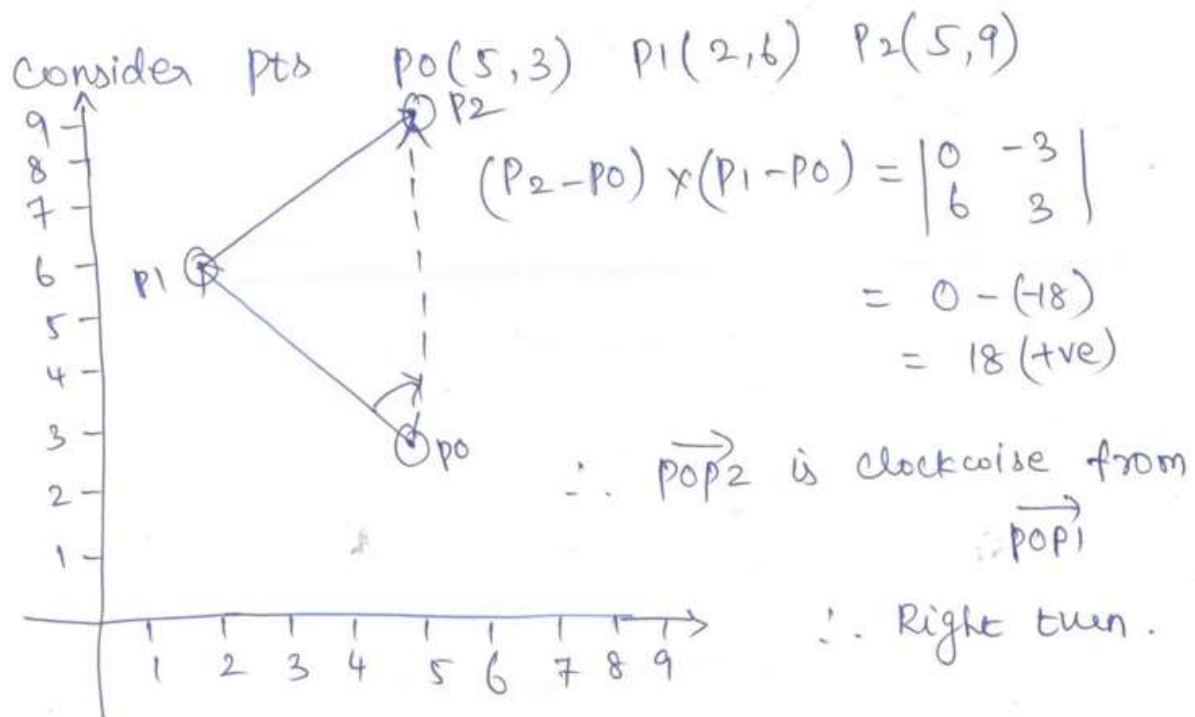


$$(P_2 - P_0) \times (P_1 - P_0) = \begin{vmatrix} 0 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= 0 - 8 = -8 (-ve)$$

$\overrightarrow{P_0P_2}$ is counter clockwise from $\overrightarrow{P_0P_1}$

$\therefore$ Left turn.



So, answer for question 2 can be obtained by calculating the cross product.

Practice Problem:

Consider points $a(2,2)$ $b(5,5)$ and $c(7,3)$. Given 2 line segments ab and bc , do we make left or right turn at common point b .

Question-3:

3. Do line segments $\overline{p_1p_2}$ and $\overline{p_3p_4}$ intersect?

Answer:

To find answer for 3rd questions, we have to check all the necessary conditions.

We will be given with 4 points P_1 , P_2 , P_3 and P_4 .

To determine whether two line segments intersect, we check whether each segment straddles the line containing the other. A segment $\overline{p_1p_2}$ *straddles* a line if point p_1 lies on one side of the line and point p_2 lies on the other side. A boundary case arises if p_1 or p_2 lies directly on the line. Two line segments intersect if and only if either (or both) of the following conditions holds:

1. Each segment straddles the line containing the other.
2. An endpoint of one segment lies on the other segment. (This condition comes from the boundary case.)

The following procedures implement this idea. SEGMENTS-INTERSECT returns TRUE if segments $\overline{p_1p_2}$ and $\overline{p_3p_4}$ intersect and FALSE if they do not. It calls the subroutines DIRECTION, which computes relative orientations using the cross-product method above, and ON-SEGMENT, which determines whether a point known to be collinear with a segment lies on that segment.

SEGMENTS-INTERSECT(p_1, p_2, p_3, p_4)

```

1   $d_1 \leftarrow \text{DIRECTION}(p_3, p_4, p_1)$ 
2   $d_2 \leftarrow \text{DIRECTION}(p_3, p_4, p_2)$ 
3   $d_3 \leftarrow \text{DIRECTION}(p_1, p_2, p_3)$ 
4   $d_4 \leftarrow \text{DIRECTION}(p_1, p_2, p_4)$ 
5  if  $((d_1 > 0 \text{ and } d_2 < 0) \text{ or } (d_1 < 0 \text{ and } d_2 > 0)) \text{ and}$ 
    $((d_3 > 0 \text{ and } d_4 < 0) \text{ or } (d_3 < 0 \text{ and } d_4 > 0))$ 
6    then return TRUE
7  elseif  $d_1 = 0$  and ON-SEGMENT( $p_3, p_4, p_1$ )
8    then return TRUE
9  elseif  $d_2 = 0$  and ON-SEGMENT( $p_3, p_4, p_2$ )
10   then return TRUE
11 elseif  $d_3 = 0$  and ON-SEGMENT( $p_1, p_2, p_3$ )
12   then return TRUE
13 elseif  $d_4 = 0$  and ON-SEGMENT( $p_1, p_2, p_4$ )
14   then return TRUE
15 else return FALSE

```

DIRECTION(p_i, p_j, p_k)

```

1  return  $(p_k - p_i) \times (p_j - p_i)$ 

```

ON-SEGMENT(p_i, p_j, p_k)

```

1  if  $\min(x_i, x_j) \leq x_k \leq \max(x_i, x_j)$  and  $\min(y_i, y_j) \leq y_k \leq \max(y_i, y_j)$ 
2    then return TRUE
3  else return FALSE

```

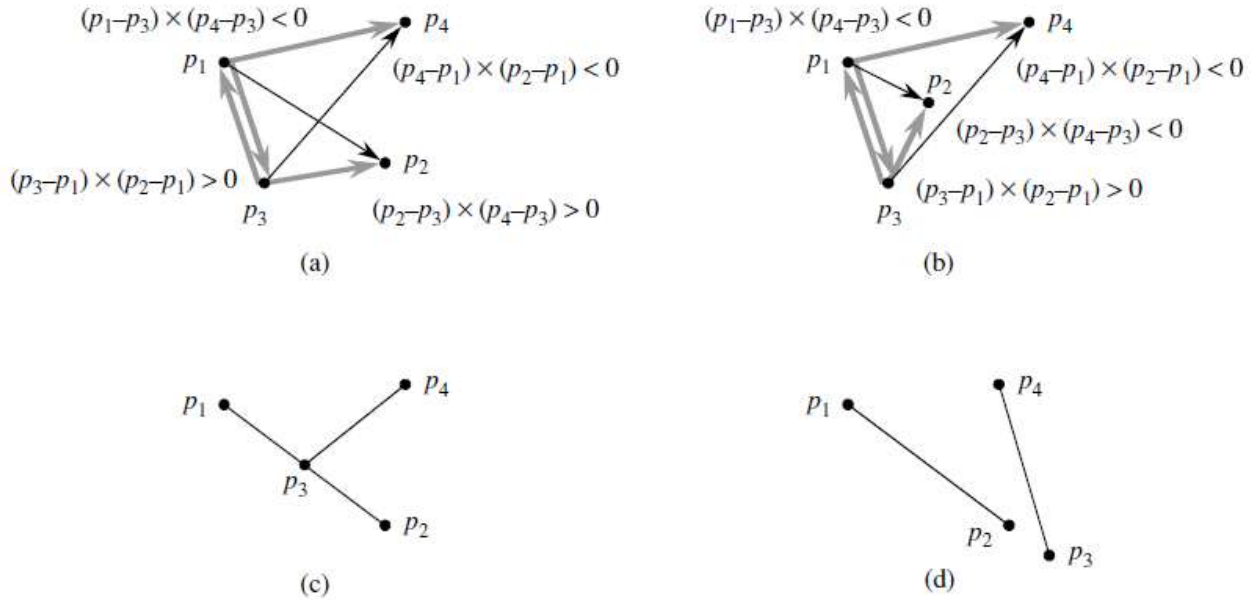
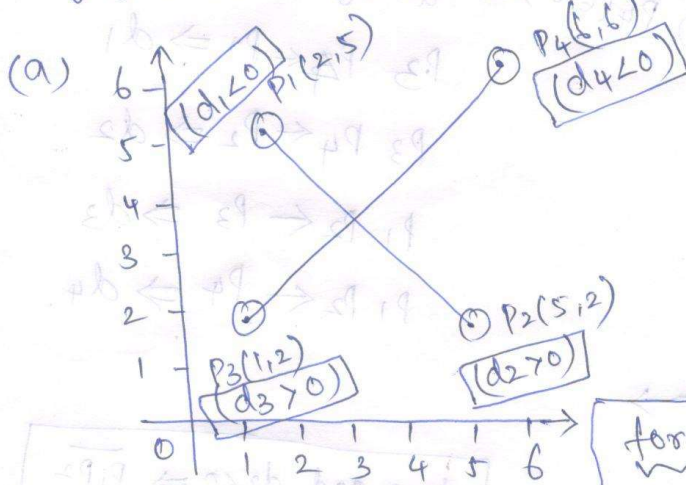


Figure 33.3 Cases in the procedure SEGMENTS-INTERSECT. (a) The segments $\overline{p_1p_2}$ and $\overline{p_3p_4}$ straddle each other's lines. Because $\overline{p_3p_4}$ straddles the line containing $\overline{p_1p_2}$, the signs of the cross products $(p_3 - p_1) \times (p_2 - p_1)$ and $(p_4 - p_1) \times (p_2 - p_1)$ differ. Because $\overline{p_1p_2}$ straddles the line containing $\overline{p_3p_4}$, the signs of the cross products $(p_1 - p_3) \times (p_4 - p_3)$ and $(p_2 - p_3) \times (p_4 - p_3)$ differ. (b) Segment $\overline{p_3p_4}$ straddles the line containing $\overline{p_1p_2}$, but $\overline{p_1p_2}$ does not straddle the line containing $\overline{p_3p_4}$. The signs of the cross products $(p_1 - p_3) \times (p_4 - p_3)$ and $(p_2 - p_3) \times (p_4 - p_3)$ are the same. (c) Point p_3 is collinear with $\overline{p_1p_2}$ and is between p_1 and p_2 . (d) Point p_3 is collinear with $\overline{p_1p_2}$, but it is not between p_1 and p_2 . The segments do not intersect.

Figure 33.3 - examples



$P_3 P_4 \leftarrow P_1 \Rightarrow d_1$
 $P_3 P_4 \leftarrow P_2 \Rightarrow d_2$
 $P_1 P_2 \leftarrow P_3 \Rightarrow d_3$
 $P_1 P_2 \leftarrow P_4 \Rightarrow d_4$

formula:
 $(P_k - P_i) \times (P_j - P_i)$

d_1

$$(P_1 - P_3) \times (P_4 - P_3)$$

$$= \begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix}$$

$$= 4 - 15$$

$$= -11 (-ve)$$

$d_1 < 0$

d_2

$$(P_2 - P_3) \times (P_4 - P_3)$$

$$= \begin{vmatrix} 4 & 5 \\ 0 & 4 \end{vmatrix}$$

$$= 16 - 0$$

$$= 16 (+ve)$$

$d_2 > 0$

d_3

$$(P_3 - P_1) \times (P_2 - P_1)$$

$$= \begin{vmatrix} -1 & 3 \\ -3 & -3 \end{vmatrix}$$

$$= 3 + 9 = 12 (+ve)$$

$d_3 > 0$

d_4

$$(P_4 - P_1) \times (P_2 - P_1)$$

$$= \begin{vmatrix} 4 & 3 \\ -3 & -3 \end{vmatrix}$$

$$= -12 - 3$$

$$= -15 (-ve)$$

$d_4 < 0$

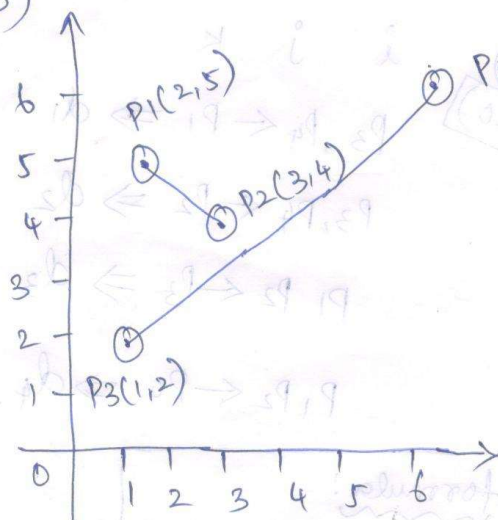
$d_3 < 0$ and $d_2 > 0 \Rightarrow \overline{P_1 P_2}$ straddles the line $\overline{P_3 P_4}$
 $d_3 > 0$ and $d_4 < 0 \Rightarrow \overline{P_3 P_4}$ straddles the line $\overline{P_1 P_2}$

'If' condition in line 5 of algorithm is true.

$\therefore$ returns true.

So, line segments intersect.

(b)



$$\begin{aligned} P_3 P_4 \leftarrow P_1 &\Rightarrow d_1 \\ P_3 P_4 \leftarrow P_2 &\Rightarrow d_2 \\ P_1 P_2 \leftarrow P_3 &\Rightarrow d_3 \\ P_1 P_2 \leftarrow P_4 &\Rightarrow d_4 \end{aligned}$$

$$\begin{aligned} d_1 &= (P_1 - P_3) \times (P_4 - P_3) \\ &= \begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} = 4 - 15 = -11 \text{ (-ve)} \\ &\boxed{d_1 < 0} \end{aligned} \quad \begin{aligned} d_2 &= (P_2 - P_3) \times (P_4 - P_3) \\ &= \begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} = 8 - 10 = -2 \text{ (-ve)} \\ &\boxed{d_2 < 0} \end{aligned}$$

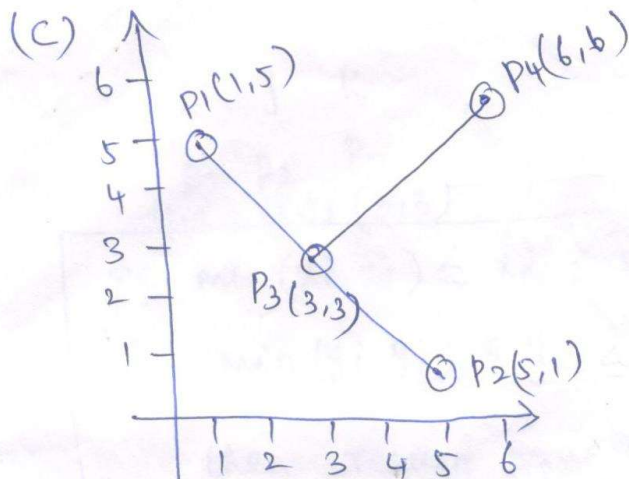
$d_1 < 0$ and $d_2 < 0 \Rightarrow \overline{P_1 P_2}$ does not straddle the line $\overline{P_3 P_4}$
 $d_3 > 0$ and $d_4 < 0 \Rightarrow \overline{P_3 P_4}$ straddles the line $\overline{P_1 P_2}$

$$\begin{aligned} d_3 &= (P_3 - P_1) \times (P_2 - P_1) \\ &= \begin{vmatrix} -1 & 1 \\ -3 & -1 \end{vmatrix} = 1 + 3 = 4 \text{ (+ve)} \\ &\boxed{d_3 > 0} \end{aligned} \quad \begin{aligned} d_4 &= (P_4 - P_1) \times (P_2 - P_1) \\ &= \begin{vmatrix} 4 & 1 \\ 1 & -1 \end{vmatrix} = -4 - 1 = -5 \text{ (-ve)} \\ &\boxed{d_4 < 0} \end{aligned}$$

Here, line 5, 7, 9, 11, 13 'if' conditions false.

$\therefore$ algorithm returns false.

SO, line segments do not intersect.



i	j	k	
P3	P4	P1	$\Rightarrow d_1$
P3	P4	P2	$\Rightarrow d_2$
P1	P2	P3	$\Rightarrow d_3$
P1	P2	P4	$\Rightarrow d_4$

$$\underline{d_1} = (P_1 - P_3) \times (P_4 - P_3)$$

$$= \begin{vmatrix} -2 & 3 \\ 2 & 3 \end{vmatrix}$$

$$= -6 - 6$$

$$= -12 \text{ (-ve)}$$

$$\boxed{d_1 < 0}$$

$$\underline{d_2}$$

$$(P_2 - P_3) \times (P_4 - P_3)$$

$$\begin{vmatrix} 2 & 3 \\ -2 & 3 \end{vmatrix}$$

$$= 6 + 6$$

$$= 12 \text{ (+ve)}$$

$$\boxed{d_2 > 0}$$

$$\boxed{d_1 < 0 \text{ and } d_2 > 0 \Rightarrow \overline{P_1 P_2} \text{ straddles the line } \overline{P_3 P_4}}$$

$$\underline{d_3} = (P_3 - P_1) \times (P_2 - P_1)$$

$$= \begin{vmatrix} 2 & 4 \\ -2 & -4 \end{vmatrix}$$

$$= -8 + 8$$

$$= 0$$

$$\boxed{d_3 = 0} \Rightarrow P_3 \text{ is collinear with } \overline{P_1 P_2}$$

$$\underline{d_4}$$

$$(P_4 - P_1) \times (P_2 - P_1)$$

$$= \begin{vmatrix} 5 & 4 \\ 1 & -4 \end{vmatrix}$$

$$= -20 - 4$$

$$= -24 \text{ (-ve)}$$

$$\boxed{d_4 < 0}$$

As P_3 is collinear with $\overline{P_1P_2}$, we have to check whether P_3 is on the segment $\overline{P_1P_2}$ (or) on the line passing through P_1 and P_2

i	j	k
P_1	P_2	P_3
(1,5)	(5,1)	(3,3)

If $\min(x_i, x_j) \leq x_k \leq \max(x_i, x_j)$ and

$\min(y_i, y_j) \leq y_k \leq \max(y_i, y_j)$

then return true $\Rightarrow$ on the segment

else return false $\Rightarrow$ on the line passing through.

$$\min(x_i, x_j) = (1, 5) = 1$$

$$\max(x_i, x_j) = (1, 5) = 5$$

$$x_k = 3$$

$$\therefore \min(x_i, x_j) \leq x_k \leq \max(x_i, x_j) \Rightarrow \text{True.}$$

$$\min(y_i, y_j) = (5, 1) = 1$$

$$\max(y_i, y_j) = (5, 1) = 5$$

$$y_k = 3$$

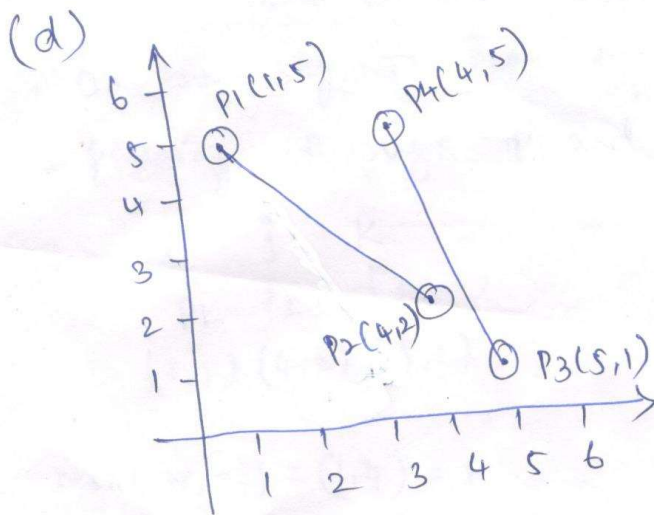
$\therefore \min(y_i, y_j) \leq y_k \leq \max(y_i, y_j) \Rightarrow \text{True}.$

$\boxed{\therefore \text{return value is true}} \Rightarrow P_3 \text{ is on the segment } \underline{P_1 P_2}$

'If' cond. in line 11 is true.

$\therefore$ algorithm returns true.

$\therefore$ line segments intersect.



i	j	k
P3	P4	$\leftarrow P1 \Rightarrow d_1$
P3	P4	$\leftarrow P2 \Rightarrow d_2$
P1	P2	$\leftarrow P3 \Rightarrow d_3$
P1	P2	$\leftarrow P4 \Rightarrow d_4$

$$\underline{d_1} = (P1 - P3) \times (P4 - P3)$$

$$= \begin{vmatrix} -4 & -1 \\ 4 & 4 \end{vmatrix}$$

$$= -16 + 4$$

$$= -12 \text{ (-ve)}$$

$$\underline{d_2} = (P2 - P3) \times (P4 - P3)$$

$$= \begin{vmatrix} -1 & -1 \\ 1 & 4 \end{vmatrix}$$

$$= -4 + 1 = -3 \text{ (-ve)}$$

$$\boxed{d_1 < 0}$$

$$\boxed{d_2 < 0}$$

$d_1 < 0$ and $d_2 < 0 \Rightarrow \overline{P1P2}$ does not straddle the line $\overline{P3P4}$

$$\underline{d_3} = (P3 - P1) \times (P2 - P1)$$

$$= \begin{vmatrix} 4 & 3 \\ -4 & -3 \end{vmatrix}$$

$$= -12 + 12 = 0$$

$$\boxed{d_3 = 0} \Rightarrow P3 \text{ is collinear with } \overline{P1P2}$$

$$\underline{d_4} = (P4 - P1) \times (P2 - P1)$$

$$= \begin{vmatrix} 3 & 3 \\ 0 & -3 \end{vmatrix}$$

$$= -9 - 0$$

$$= -9 \text{ (-ve)}$$

$$\boxed{d_4 < 0}$$

so, we have to check whether P_3 is on the segment $\overline{P_1P_2}$ or on the line passing through P_1 and P_2

i	j	k
P_1	P_2	P_3
(1, 5)	(4, 2)	(5, 1)

$$\min(x_i, x_j) = (1, 4) = 1$$

$$\max(x_i, x_j) = (1, 4) = 4$$

$$\min(y_i, y_j) = (5, 2) = 2$$

$$\max(y_i, y_j) = (5, 2) = 5$$

$$x_k = 5$$

$$y_k = 1$$

$$\min(x_i, x_j) \leq x_k \leq \max(x_i, x_j) \Rightarrow \text{False}$$

$$\min(y_i, y_j) \leq y_k \leq \max(y_i, y_j) \Rightarrow \text{False}$$

$\therefore$ return value is false $\Rightarrow P_3$ is on the line passing through segment $\overline{P_1P_2}$

'If' cond in line 11 is false.
algorithm returns false.

$\therefore$ line segments do not intersect.