Calculus
(BMAT101L)

Question Bank

Dr. T. Phaneendra
M. Sc., Ph. D.

Professor of Mathematics
(Higher Academic Grade)
phaneendra.t@vit.ac.in

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Vellore Institute of Technology
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Contents

Chapter 4

Taylor's Approximations and Constrained Extrema

4.1 Taylor's theorem for Functions of Two Variables

Let $n \geq 1$. Suppose that $f(x, y)$ and its partial derivatives through order $n + 1$ are continuous in a small neighborhood $\mathscr{R}(a, b)$, centered at the point (a, b). Then

$$\begin{aligned} f(a+h, b+k) &= \left| \left\{ e^{h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}} \right\} f \right|_{(a,b)} \\ &= \left\{ 1 + \tfrac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) + \tfrac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 \right. \\ &\quad \left. + \tfrac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 + \cdots + \tfrac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n \right\} f + R_n \\ &= f + \tfrac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \tfrac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) \\ &\quad + \tfrac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) \\ &\qquad + \cdots + \tfrac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f + R_n, \end{aligned} \tag{4.1.1}$$

where f and all its partial derivatives are evaluated at the point (a, b), and

$$R_n = \tfrac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(a + ch + b + ck),$$

for some $0 < c < 1$, and R_n is called the Taylor's remainder after n terms.

In terms of subscript notation, we have

$$\begin{aligned} f(a+h, b+k) &= f + \tfrac{1}{1!}\left(hf_x + kf_y\right) + \tfrac{1}{2!}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right) \\ &\quad + \tfrac{1}{3!}\left(h^3 f_{xxx} + 3h^2 k f_{yxx} + 3hk^2 f_{yyx} + k^3 f_{yyy}\right) + \cdots + R_n \end{aligned} \tag{4.1.2}$$

In view of the continuity of the partial derivatives of all orders, we see that $f_{yxx} = f_{xxy}$, $f_{yyx} = f_{xyy}$ etc. Therefore, (4.1.2) is also written as

$$\begin{aligned} f(a+h, b+k) &= f + \tfrac{1}{1!}\left(hf_x + kf_y\right) + \tfrac{1}{2!}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right) \\ &\quad + \tfrac{1}{3!}\left(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}\right) + \cdots + R_n \end{aligned} \tag{4.1.3}$$

Replacing $(a + h, b + k)$ with (x, y) so that $h = x - a$ $k = y - b$ in (4.1.3), we get

$$\begin{aligned} f(x, y) &= f + \tfrac{1}{1!}\left((x-a)f_x + (y-b)f_y\right) \\ &\quad + \tfrac{1}{2!}\left((x-a)^2 f_{xx} + 2(x-a)(y-b)f_{xy} + (y-b)^2 f_{yy}\right) \\ &\quad + \tfrac{1}{3!}\left((x-a)^3 f_{xxx} + 3(x-a)^2(y-b)f_{xxy} + 3(x-a)(y-b)^2 f_{xyy} \right. \\ &\quad \left. + (y-b)^3 f_{yyy}\right) + \cdots + R_n, \end{aligned} \tag{4.1.4}$$

where f and all its partial derivatives are evaluated at (a, b).

Replacing (a, b) with $(0, 0)$ and then h with x, k with y in (4.1.3), we get

$$\begin{aligned} f(x,y) &= f + \tfrac{1}{1!}\left(xf_x + yf_y\right) + \tfrac{1}{2!}\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right) \\ &\quad + \tfrac{1}{3!}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right) + \cdots + R_n, \end{aligned} \tag{4.1.5}$$

where f and all its partial derivatives are evaluated at $(0, 0)$.

4.2 Approximations using Taylor's theorem

For $n = 1$: the linear approximation of f about the origin $(0, 0)$, is given by

$$f(x,y) \approx f(0,0) + \tfrac{1}{1!}\left(x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right), \tag{4.2.1}$$

and the error in the linear approximation is given by

$$\begin{aligned} E(x,y) &= \tfrac{1}{2!}\left\{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right\} \\ &= \tfrac{1}{2!}\left\{x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right\}, \end{aligned} \tag{4.2.2}$$

where the second order partial derivatives are evaluated at (cx, cy) for some $0 < c < 1$.

For $n = 2$, the quadratic approximation of f is given by

$$\begin{aligned} f(x,y) &\approx f(0,0) + \tfrac{1}{1!}\left(x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right) \\ &\quad + \tfrac{1}{2!}\left(x^2\frac{\partial^2 f(0,0)}{\partial x^2} + 2xy\frac{\partial^2 f(0,0)}{\partial x\partial y} + y^2\frac{\partial^2 f(0,0)}{\partial y^2}\right), \end{aligned} \tag{4.2.3}$$

and the error in the approximation is given by

$$\begin{aligned} E(x,y) &= \tfrac{1}{3!}\left(x^3\frac{\partial^3 f}{\partial x^3} + 3x^2 y\frac{\partial^3 f}{\partial x^2\partial y} + 3xy^2\frac{\partial^3 f}{\partial x\partial y^2} + y^2\frac{\partial^3 f}{\partial y^3}\right) \\ &= \tfrac{1}{3!}\left\{x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{yyx} + y^3 f_{yyy}\right\}, \end{aligned} \tag{4.2.4}$$

where the third order partial derivatives are evaluated at (cx, cy) for some $0 < c < 1$.

For $n = 3$, the cubic approximation of f is given by

$$\begin{aligned} f(x,y) &\approx f + \tfrac{1}{1!}\left(xf_x + yf_y\right) + \tfrac{1}{2!}\left(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}\right) \\ &\quad + \tfrac{1}{3!}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{yyx} + y^3 f_{yyy}\right), \end{aligned} \tag{4.2.5}$$

where f and all its partial derivatives are evaluated at $(0, 0)$.

Example 4.2.1. Find quadratic approximation for $f(x, y) = \cos x \cos y$ at the origin. Also, estimate the error in approximation, if $|x| \le 0.1$ and $|x| \le 0.1$.

Solution. We use (4.2.3) for quadratic approximation, (4.2.4) for the error in the approximation.

$$\begin{aligned} f(x,y) &= \cos x \cos y & &\Rightarrow & f(0,0) &= \cos 0.\cos 0 = 1 \\ f_x(x,y) &= -\sin x \cos y & &\Rightarrow & f_x(0,0) &= -\sin 0.\cos 0 = 0 \\ f_y(x,y) &- \cos x \sin y & &\Rightarrow & f_y(0,0) &= -\cos 0.\sin 0 = 0 \end{aligned}$$

$$f_{xx}(x,y) = -\cos x\cos y \qquad \Rightarrow \qquad f_{xx}(0,0) = -\cos 0.\cos 0 = -1$$
$$f_{yy}(x,y) = -\cos x\cos y \qquad \Rightarrow \qquad f_{yy}(0,0) = -\cos 0.\cos 0 = -1$$
$$f_{xy}(x,y) = \sin x\sin y \qquad \Rightarrow \qquad f_{xy}(0,0) = \sin 0.\sin 0 = 0.$$

Substituting these in (4.2.3), we obtain that

$$f(x,y) \approx 1 + \tfrac{1}{1!}(x\cdot 0 + y\cdot 0) + \tfrac{1}{2!}\left[x^2(-1) + 2xy\cdot 0 + y^2(-1)\right] = 1 - \tfrac{x^2}{2} - \tfrac{y^2}{2},$$

which is the quadratic approximation, we need. Note that the partial derivatives of third order of f, being the products of sines and cosines, have the absolute values less than or equal to 1. Since $|x| \le 0.1$ and $|x| \le 0.1$, the error of approximation is estimated by

$$\begin{aligned} |E(x,y)| &\le \tfrac{1}{3!}\left\{\left|x^3 f_{xxx}\right| + 3\left|x^2 y f_{xxy}\right| + 3\left|xy^2 f_{yyx}\right| + \left|y^3 f_{yyy}\right|\right\} \\ &\le \tfrac{1}{6}\left\{|x|^3 + 3|x|^2|y| + 3|x||y|^2 + |y|^3\right\} \\ &\le \tfrac{1}{6}\left[(0.1)^3 + 3(0.1)^2(0.1) + 3(0.1)(0.1)^2 + (0.1)^3\right] = 0.00134. \end{aligned}$$

Example 4.2.2. Find cubic approximation for $f(x,y) = xe^y$ at the origin, using Taylor's formula.

Solution. We employ (4.2.5) for finding cubic approximation:

$$f(x,y) = xe^y \qquad \Rightarrow \qquad f(0,0) = 0.e^0 = 0;$$
$$f_x(x,y) = \frac{\partial f}{\partial x} = e^y \qquad \Rightarrow \qquad f_x(0,0) = e^0 = 1,$$
$$f_y(x,y) = \frac{\partial f}{\partial y} = xe^y \qquad \Rightarrow \qquad f_y(0,0) = 0.e^0 = 0;$$
$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = 0 \qquad \Rightarrow \qquad f_{xx}(0,0) = 0,$$
$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = xe^y \qquad \Rightarrow \qquad f_{yy}(0,0) = 0.e^0 = 0,$$
$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y\partial x} = e^y \qquad \Rightarrow \qquad f_{xy}(0,0) = e^0 = 1;$$
$$f_{xxx}(x,y) = \frac{\partial^3 f}{\partial x^3} = 0 \qquad \Rightarrow \qquad f_{xxx}(0,0) = 0,$$
$$f_{xxy}(x,y) = \frac{\partial^3 f}{\partial y\partial x^2} = 0 \qquad \Rightarrow \qquad f_{xxy}(0,0) = 0,$$
$$f_{yyx}(x,y) = \frac{\partial^3 f}{\partial x\partial y^2} = e^y \qquad \Rightarrow \qquad f_{yyx}(0,0) = e^0 = 1$$
$$f_{yyy}(x,y) = \frac{\partial^3 f}{\partial y^3} = xe^y \qquad \Rightarrow \qquad f_{yyy}(0,0) = 0.e^0 = 0.$$

Substituting these in (4.2.5), we obtain that

$$\begin{aligned} f(x,y) &\approx 0 + \tfrac{1}{1!}(x\cdot 1 + y\cdot 0) + \tfrac{1}{2!}\left[x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot 0\right] \\ &\quad + \tfrac{1}{3!}\left[x^3\cdot 0 + 3x^2y\cdot 0 + 3xy^2\cdot 1 + y^3.0\right] \\ &= x + xy + \tfrac{1}{2}\cdot xy^2 \end{aligned}$$

which is the required cubic approximation.

Example 4.2.3. Expand $f(x,y) = e^x\log(1+y)$ in ascending powers of x and y about $(0,0)$, up to third degree terms, to obtain a cubic approximation for f.

Solution. We employ (4.2.5) for finding cubic approximation:

$$f(x,y) = e^x\log(1+y) \qquad \Rightarrow \qquad f(0,0) = e^0\log 1 = 0;$$

$$
\begin{aligned}
f_x(x,y) &= \tfrac{\partial f}{\partial x} = e^x \log(1+y) && \Rightarrow\ f_x(0,0) = 0,\\
f_y(x,y) &= \tfrac{\partial f}{\partial y} = \tfrac{e^x}{1+y} && \Rightarrow\ f_y(0,0) = 1;\\
f_{xx}(x,y) &= \tfrac{\partial^2 f}{\partial x^2} = e^x \log(1+y) && \Rightarrow\ f_{xx}(0,0) = 0,\\
f_{yy}(x,y) &= \tfrac{\partial^2 f}{\partial y^2} = -\tfrac{e^x}{(1+y)^2} && \Rightarrow\ f_{yy}(0,0) = -1,\\
f_{xy}(x,y) &= \tfrac{\partial^2 f}{\partial y \partial x} = \tfrac{e^x}{1+y} && \Rightarrow\ f_{xy}(0,0) = 1;\\
f_{xxx}(x,y) &= \tfrac{\partial^3 f}{\partial x^3} = e^x \log(1+y) && \Rightarrow\ f_{xxx}(0,0) = 0,\\
f_{xxy}(x,y) &= \tfrac{\partial^3 f}{\partial y \partial x^2} = \tfrac{e^x}{1+y} && \Rightarrow\ f_{xxy}(0,0) = 1,\\
f_{yyx}(x,y) &= \tfrac{\partial^3 f}{\partial x \partial y^2} = -\tfrac{e^x}{(1+y)^2} && \Rightarrow\ f_{yyx}(0,0) = -1\\
f_{yyy}(x,y) &= \tfrac{\partial^3 f}{\partial y^3} = \tfrac{2e^x}{(1+y)^3} && \Rightarrow\ f_{yyy}(0,0) = 2.
\end{aligned}
$$

Substituting these in (4.2.5), we obtain the cubic approximation

$$
\begin{aligned}
f(x,y) &\approx 0 + \tfrac{1}{1!}\left(x \cdot 0 + y \cdot 1\right) + \tfrac{1}{2!}\left[x^2 \cdot 0 + 2xy \cdot 1 + y^2(-1)\right]\\
&\quad + \tfrac{1}{3!}\left[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2(-1) + y^3 .2\right]\\
&= y + xy + \tfrac{1}{2}(2xy - y^2) + \tfrac{1}{6}(3x^2 y - 3xy^2 + 2y^3),
\end{aligned}
$$

Exercise 4.2.1 (Self-check). Use Taylor's formula to find the quadratic and cubic approximations for each of the following functions $f(x,y)$ at the origin:

(a) $y \sin x$

(b) $\sin x \cos y$

(c) $1/(1-x-y)$

(d) $\sin(x^2+y^2)$

(e) $1/(1-x-y+xy)$

Text and Reference Books

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2. Briggs et al., *Calculus for Scientists and Engineers - Early Transcendentals*, Copyright © (2013) Pearson Education, Inc., Sec. 13.8
3. James Stewart, *Multivariable Calculus*, 8th Edition, Sec. 14.7
4. Smith and Minton, *Calculus - Early Transcendental Functions*, McGraw-Hill (2011), 4th Edition, Sec. 12.7
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Chapter 5

Constrained Maxima and Minima

Lagrange Multiplier Method

The method of Lagrange multiplier depends on the geometric meaning of *gradients*. It is used for maximizing or minimizing functions of several variables subject to one or more side conditions, also called *constraints*. It is an important tool in economics, differential geometry, and advanced theoretical mechanics.

5.1 Two-variable Function with One Constraint

Suppose that x and y are not independent, and connected in fact, by a side condition:

$$g(x, y) = k, \tag{5.1.1}$$

where k is a given constant. We wish to find the extrema for $u = f(x, y)$ subject to the condition (5.1.1). We sketch the graph of $g(x, y) = k$ and various level curves

$$f(x, y) = c \tag{5.1.2}$$

of f, in the direction increasing c. To find the maximum value of f satisfying (5.1.1), we look for the largest c such that the level curve (5.1.2) intersects (5.1.1) in some point, say P_0, where the two curves have the same slopes, and hence the respective normals

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

and

$$\nabla g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j}$$

at P_0 are in the same direction. In other words,

$$\nabla f = \lambda \nabla g$$

at P_0 for some scalar λ, which is known as a *Lagrange multiplier*. We solve the equations

$$\frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x}, \ \frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y} \tag{5.1.3}$$

subject to the constraint (5.1.1) to get the maximum or minimum of f, we need.

Example 5.1.1. Find two non-negative numbers whose sum is 9 and the product of one number and the square of the other number is a maximum.

Solution. Let x and y represent two non-negative numbers. We wish to maximize the product $f(x, y) = xy^2$, subject to the condition that

$$g(x, y) = 0, \tag{5.1.4}$$

where $g(x, y) = x + y - 9$. From the relations (5.1.3), we get $y^2 = \lambda$, $2xy = \lambda$. These imply that $y^2 = 2xy$ or $y(y - 2x) = 0$ so that $y = 2x$, since both x and y are not zero. Substituting $y = 2x$ in (5.1.4), we get $x = 3$, $y = 6$. Thus the critical point is $P(3, 6)$. The maximum value of f is $f(P) = 3(6)^2 = 108$.

Example 5.1.2. A rectangular box with square base, open at the top, is to be made from 48 square feet of material. What dimensions will result in a box with the largest possible volume?

Solution. Let x be the side of the square base, and y be the height of the box. Since the box is open at the top, the surface area of the box equals the area of base plus four times the area of one side plane, that is $x^2 + 4xy$. We wish to maximize the volume $f(x, y) = x^2y$, subject to the condition:

$$\underbrace{x^2 + 4xy - 48}_{g(x,y)} = 0, \tag{5.1.5}$$

The Lagrangian relations (5.1.3) reduce to $2xy = \lambda(2x + 4y)$, $x^2 = 4\lambda x$. That is

$$xy = \lambda(x + 2y) \tag{5.1.6}$$

$$x = 4\lambda. \tag{5.1.7}$$

Dividing (5.1.7) with (5.1.6), we get $\frac{xy}{x+2y} = \frac{x}{4}$ or $x = 2y$. Substituting this in (5.1.5), we get $(2y)^2 + 4(2y)(y) = 48$ or $y = 2$ so that $x = 4$. Thus the critical point is $P(4, 2)$. The maximum volume is $f(P) = (4)^2(2) = 32$ cubic feet.

Example 5.1.3. A right circular cylindrical container with open top has surface area 3π square feet. What height h and base radius r will maximize its volume?

Solution. We maximize the volume $f(r, h) = \pi r^2 h$ of the container, subject to the condition:

$$\underbrace{\pi r^2 + 2\pi rh - 3\pi}_{g(r,h)} = 0, \tag{5.1.8}$$

The Lagrangian relations $\frac{\partial f}{\partial r} = \lambda\frac{\partial g}{\partial r}$, $\frac{\partial f}{\partial h} = \lambda\frac{\partial g}{\partial h}$ reduce to $2\pi rh = \lambda(2\pi r + 2\pi h)$ and $\pi r^2 = \lambda 2\pi r$ or

$$r = \lambda(r + h) \tag{5.1.9}$$

$$r = 2\lambda. \tag{5.1.10}$$

Solving (5.1.9) and (5.1.10), we get $r + h = 2$ or $h = 2 - r$. Substituting this in (5.1.8), we get $\pi r^2 + 2\pi r(2 - r) = 3\pi$ or $r^2 - 4r + 3 = 0$. The two roots are $r = 1, 3$. If $r = 3$, we see that $h = 2 - r = -1$, which is not possible. Therefore, we take $r = 1$, and hence $h = 1$. Thus the largest possible volume of the cylinder is $f(1, 1) = \pi(1)^2(1) = \pi$ cubic feet.

Example 5.1.4. Consider all triangles formed by lines passing through the point $(8/9, 3)$ and both the x and y axes. Find the dimensions of the triangle with the shortest hypotenuse.

Solution. Let a and b be the x and y-intercepts of the hypotenuse respectively. We maximize the square of the length $H(a, b) = a^2 + b^2$, when the hypotenuse passes through the point $(8/9, 3)$, that is when

$$\underbrace{\frac{8}{9a} + \frac{3}{b} - 1}_{g(a,b)} = 0. \tag{5.1.11}$$

Now, the Lagrangian relations $\frac{\partial f}{\partial a} = \lambda \frac{\partial g}{\partial a}$, $\frac{\partial f}{\partial b} = \lambda \frac{\partial g}{\partial b}$ reduce to

$$2a = -\frac{8\lambda}{9a^2} \text{ and } 2b = -\frac{3\lambda}{b^2}\cdot$$

Eliminating λ from these, and simplifying, we get $a = 2b/3$. Substituting this in (5.1.11), we get $\frac{8}{9} \cdot \frac{3}{2b} + \frac{3}{b} = 1$ or $y = 13/3$ and hence $x = 26/9$. Thus the minimum length of the hypotenuse is $f(a, b) = f(13/3, 26/9) = \sqrt{a^2 + b^2} = 13\sqrt{13}/9$ units.

Example 5.1.5. Find a rectangle of largest area, that can be inscribed in the closed region bounded by the x-axis, y-axis, and graph of $y = 8 - x^3$.

Solution. Let R be the largest rectangle, which can be best fit in the region A enclosed by the coordinate axes and the curve C: $y = 8 - x^3$. Then one corner (x, y) of R will lie on the curve C. The problem is to maximize the area function $f(x, y) = xy$ such that

$$\underbrace{x^3 + y - 8}_{g(x,y)} = 0. \tag{5.1.12}$$

The Lagrangian relations $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$ reduce to

$$y = \lambda(3x^2) \text{ and } x = \lambda\cdot$$

Eliminating λ from these, we obtain $y = 3x^3$. Substituting this in (5.1.12), we get $3x^3 = 8 - x^3$ or $x = 2^{1/3}$ and hence $y = 8 - 2 = 6$. Thus the critical point of f is $P(2^{1/3}, 6)$. The largest area of the rectangle is $f(P) = xy = 6 \cdot 2^{2/3}$ square units.

Example 5.1.6. Find the dimensions (radius r and height h) of the cone of maximum volume, which can be inscribed in a sphere of radius 2.

Solution. Let C be a cone of maximum volume, which can be inscribed in a sphere S of radius 2. Then the vertex of the cone lies on the surface of S, and the height of the cone is along the radius of S. If r is the radius of the cross-section of the cone, and h is the length of perpendicular from the centre O of S onto it, then from the geometry, we find that

$$r^2 + h^2 = 2^2 \text{ or } \underbrace{r^2 + h^2 - 4}_{g(r,h)} = 0. \tag{5.1.13}$$

We maximize the volume function $f(r, h) = \pi r^2(h + 2)/3$ subject to the condition (5.1.13). Now, the Lagrangian relations $\frac{\partial f}{\partial r} = \lambda \frac{\partial g}{\partial r}$, $\frac{\partial f}{\partial h} = \lambda \frac{\partial g}{\partial h}$ reduce to

$$\frac{2\pi r(h + 2)}{3} = \lambda(2r) \text{ and } \frac{\pi r^2}{3} = \lambda(2h)\cdot$$

Eliminating λ from these, simplifying, and using (5.1.13), we get $2h^2 + 4h = r^2 = 4 - h^2$ or $3h^2 + 4h - 4 = 0$. The two roots are $h = -2, 2/3$. Discarding the negative value, $h = 2/3$ so that $r^2 = 4 - 4/9 = 32/9$.

Thus the critical point of f is $P(4\sqrt{2}/3, 2/3)$. The maximum volume of the cone is $f(P) = \pi r^2(h + 2)/3 = \pi(32/9)(2 + 2/3)/3 = 256\pi/81$ cubic units.

5.2 Three-variable Function with One Constraint

Suppose that x, y and z are not independent, but connected by a side condition:

$$g(x, y, z) = 0. \tag{5.2.1}$$

To find the extrema for $u = f(x, y, z)$ subject to the condition (5.2.1), we solve the Langrangean equations

$$\frac{\partial f}{\partial x} = \lambda\frac{\partial g}{\partial x},\ \frac{\partial f}{\partial y} = \lambda\frac{\partial g}{\partial y},\ \frac{\partial f}{\partial z} = \lambda\frac{\partial g}{\partial z} \tag{5.2.2}$$

through the condition (5.2.1).

Example 5.2.1. Find the critical points of $f(x, y, z) = z^2$ on the surface of the paraboloid $z = x^2 + y^2$.

Solution. Write $g(x, y, z) = x^2 + y^2 - z$. Then Lagrangian relations (5.2.2) become

$$0 = 2\lambda x,\ 0 = 2\lambda y,\ 2z = -\lambda \text{ or } \lambda x = 0,\ \lambda y = 0,\ z = -\lambda/2. \tag{5.2.3}$$

If $\lambda = 0$, from (5.2.3) and the relation that $z = x^2 + y^2$, we find that $z = 0 = x = y$. On the other hand, if $\lambda \neq 0$, then from the first two relations of (5.2.3) and the relation $z = x^2 + y^2$ imply that $z = 0$, while the third relation of (5.2.3) gives $z \neq 0$. This is a contradiction. Hence $\lambda \neq 0$ is not possible. In other words, $(0, 0, 0)$ is the critical point of f, we need.

Example 5.2.2. Find the maximum and minimum values of $f(x, y, z) = x + y + 2z$ on the surface $x^2 + y^2 + z^2 = 3$.

Solution. Let $g(x, y, z) = x^2 + y^2 + z^2 - 3$. We maximize $f(x, y, z) = x + y + 2z$ subject to the condition

$$g(x, y, z) = x^2 + y^2 + z^2 - 3 = 0 \tag{5.2.4}$$

Then the Lagrangian relations (5.2.2) become

$$1 = 2\lambda x,\ 1 = 2\lambda y,\ 1 = 2\lambda z \text{ or } x = 1/2\lambda,\ y = 1/2\lambda,\ z = 1/\lambda, \tag{5.2.5}$$

since $\lambda \neq 0$. Substituting (5.2.5) in (5.2.4), we see that

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 3 \text{ or } \lambda = \pm\frac{1}{\sqrt{2}}.$$

Now, inserting $\lambda = \frac{1}{\sqrt{2}}$ in (5.2.5), we get one critical point $P(1/\sqrt{2}, 1/\sqrt{2}, \sqrt{2})$. While, using $\lambda = -\frac{1}{\sqrt{2}}$ in (5.2.5), we get other critical point $Q(-1/\sqrt{2}, -1/\sqrt{2}, -\sqrt{2})$.

Finally, $f(P) = 3\sqrt{2}$ and $f(Q) = -3\sqrt{2}$. Therefore, P is a point of maximum and Q is a point of minimum for f.

Example 5.2.3. Find the dimensions of the rectangular box of maximum volume that can be inscribed inside the sphere of radius $a > 0$, centred at the origin O.

Solution. Equation of the sphere S with radius $a > 0$ and centre at the origin O(0,0) is $x^2 + y^2 + z^2 = a^2$. Consider the rectangular box B with dimensions $2x$, $2y$ and $2z$ with the volume

$V = f(x, y, z) = 2x.2y.2z = 8xyz$. The box B can be inscribed in the sphere S, if the centre O of the sphere S coincides with the centre of gravity of the box B, and each corner $P(x, y, z)$ of the box B lies on the sphere S.

Now, we maximize $f = 8xyz$ subject to the condition

$$g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0, \; x > 0, y > 0, z > 0. \tag{5.2.6}$$

Then the Lagrangian relations (5.2.2) reduce to

$$8yz = 2\lambda x, \; 8zx = 2\lambda y, \; 8xy = 2\lambda z \text{ or } 4yz = \lambda x, \; 4zx = \lambda y, \; 4xy = \lambda z \tag{5.2.7}$$

Multiplying the relations in (5.2.7) with x, y and z respectively, then adding, and using (5.2.6), we find that $12xyz = \lambda(x^2 + y^2 + z^2) = a^2\lambda$ or $\lambda = 12xyz/a^2$. Inserting this in (5.2.7), and then simplifying, we get $x^2 = a^2/3$, $y^2 = a^2/3$, $z^2 = a^2/3$ or $x = y = z = a/\sqrt{3}$. Thus the critical point of f is $P\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Now, for fixed x and y, f is an increasing function of z, and for $x = \frac{a}{\sqrt{3}}$, $y = \frac{a}{\sqrt{3}}$, the maximum value of z satisfying the condition (5.2.6) is $\frac{a}{\sqrt{3}}\cdot$ Thus f is maximum at P, and $\max f = f(P) = 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{a}{\sqrt{3}} \cdot \frac{a}{\sqrt{3}} = \frac{8a^3}{3\sqrt{3}}\cdot$

Example 5.2.4. Find the points on the cone $z^2 = x^2 + y^2$, which are nearest the point $(4, 2, 0)$.

Solution. We explore for the extrema of The squared distance function $f(x, y, z) = (x - 4)^2 + (y - 2)^2 + z^2$ subject to the condition that

$$\underbrace{x^2 + y^2 - z^2}_{g(x,y,z)} = 0. \tag{5.2.8}$$

The Lagrangian relations (5.2.2) reduce to $2(x - 4) = 2\lambda x, \; 2(y - 2) = 2\lambda y, \; 2z = -2\lambda z$ or

$$x - 4 = \lambda x, \; y - 2 = \lambda y, \; \lambda = -1. \tag{5.2.9}$$

Solving these, we find that $x = 2$, $y = 1$, and hence $z^2 = 4 + 1 = 15$ so that $z = \pm\sqrt{5}$. The two critical points $(2, 1, \pm\sqrt{5})$ are closest to the point $(4, 2, 0)$.

Example 5.2.5. Find the plane that passes through the point (2,1,2) and cuts off the least volume from the first octant.

Solution. The volume of the tetrahedron formed by the three coordinate planes and the plane $\pi : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ cut off in the first octant is $f(a, b, c) = abc/6$. Since the point $(2, 1, 2)$ lies on π, we see that

$$\underbrace{\frac{2}{a} + \frac{1}{b} + \frac{2}{c} - 1}_{g(a,b,c)} = 0. \tag{5.2.10}$$

Our aim is to maximize f subject to the condition (5.2.10). The Lagrangian relations (5.2.2) reduce to

$$\frac{bc}{6} = -\frac{2\lambda}{a^2}, \; \frac{ca}{6} = -\frac{\lambda}{b^2}, \; \frac{ab}{6} = -\frac{2\lambda}{c^2}. \tag{5.2.11}$$

Multiplying these relations by a, b and c respectively, and then adding, we get

$$\frac{abc}{2} = -\lambda \underbrace{\left(\frac{2}{a} + \frac{1}{b} + \frac{2}{c}\right)}_{=1}$$

so that $\lambda = -abc/2$. Substituting this in each of the relations (5.2.11), we then find that

$$\frac{bc}{6} = -\frac{-abc}{a^2}, \quad \frac{ca}{6} = -\frac{-abc\lambda}{2b^2}, \quad \frac{ab}{6} = -\frac{-abc}{c^2}$$

or that $a = 6$, $b = 3$, $c = 6$. Thus the critical point of f is $(6, 3, 6)$, and the maximum volume is $f(6, 3, 6) = (6)(3)(6)/6 = 18$ cubic units.

Text and Reference Books

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