5.4 RMS Value and Parseval's Identity:

Def: The RMS value (root mean square value) of any function f(x) over the range x = a and x = b can be defined as

$$\text{RMS value} = \sqrt{\frac{1}{(b-a)}\int_a^b (f(x))^2\,dx}$$

Example:

1. Find RMS value of $f(x) = x$ over the range $x = 0$ and $x = \pi$.

Sol: We have

$$\text{RMS value} = \sqrt{\frac{1}{\pi - 0}\int_0^\pi (f(x))^2\,dx}$$

$$= \sqrt{\frac{1}{\pi}\int_0^\pi x^2\,dx}$$

$$= \sqrt{\frac{1}{\pi}\left(\frac{\pi^3}{3}\right)}$$

$$= \frac{\pi}{\sqrt{3}}$$

2. A Sinusoidal voltage has a maximum value of 100v. Calculate its RMS value.

Solution: A sinusoidal voltage v having a maximum value of 100V may be written as

$$V(\theta) = 100 \sin\theta$$

Over the range $\theta = 0$ to $\theta = 2\pi$ (a complete cycle),

$$\text{RMS value} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} (100\sin\theta)^2 \, d\theta}$$

$$= \sqrt{\frac{(100)^2}{2\pi}\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta}$$

$$= \sqrt{\frac{(100)^2}{2\pi}\cdot\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{2\pi}}$$

$$= \sqrt{\frac{(100)^2}{2\pi}\,\frac{1}{2}\,(2\pi)} = \frac{100}{\sqrt{2}} = 70.71V.$$

Parseval's Identity:

If the Fourier series for the function $f(x)$ converges uniformly to $f(x)$ in $(-l, l)$, then

$$\frac{1}{l}\int_{-l}^{l}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

Note: In the interval $(0, 2l)$, Parseval's Identity is

$$\frac{1}{l}\int_{0}^{2l}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

Parseval's Identity for

(i) the half range sine series corresponding to $f(x)$ in the interval $(0, l)$ is

$$\frac{2}{l}\int_{0}^{l}(f(x))^2\,dx = \sum_{n=1}^{\infty} b_n^2$$

(ii) the half range cosine series corresponding to $f(x)$ in the interval $(0, l)$ is

$$\frac{2}{l}\int_{0}^{l}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

(iii) the Fourier series corresponding ~~corresponding~~ to the even function $f(x)$ in $(-l, l)$ is

$$\frac{2}{l}\int_{0}^{l}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

(iv) the Fourier series corresponding to the odd function $f(x)$ in $(-l, l)$ is

$$\frac{2}{l}\int_{0}^{l}(f(x))^2\,dx = \sum_{n=1}^{\infty} b_n^2.$$

Problems:

1. Expand the function $f(x) = x^2$ as a Fourier series in $(-\pi, \pi)$ and hence deduce that

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}.$$

Sol: Clearly the given function $f(x) = x^2$ is even in the interval $(-l, l)$. Here $l = \pi$.

Therefore, the Fourier series for $f(x)$ in $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{2\pi^2}{3}$

and $a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) \, dx$

$$= \frac{2}{\pi} \left\{ \left[x^2 \cdot \frac{\sin(nx)}{n} \right]_0^{\pi} - \left[2x \cdot \left(-\frac{\cos(nx)}{n^2} \right) \right]_0^{\pi} + \left[2 \left(-\frac{\sin(nx)}{n^3} \right) \right]_0^{\pi} \right\}$$

$$= \frac{2}{\pi} \left\{ 0 + \frac{2}{n^2} (\pi \cos n\pi - 0) + 0 \right\}$$

$$= \frac{4}{n^2} (-1)^n$$

Hence,

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

By the Parseval's Identity, we have

$$\frac{2}{\pi}\int_0^{\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\Rightarrow \frac{2}{\pi}\int_0^{\pi} (x^2)^2 dx = \frac{\left(\frac{2\pi^2}{3}\right)^2}{2} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^n\right]^2$$

$$\Rightarrow \frac{2}{\pi}\left(\frac{x^5}{5}\right)_0^{\pi} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{2}{\pi}\left(\frac{\pi^5}{5}\right) = \frac{2\pi^4}{9} + 16\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right)$$

Therefore,

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots = \frac{\pi^4}{90}$$

2. ~~Expand $f(x)$~~ Obtain the half range cosine series for $f(x) = x$ in $0 < x < 1$ and hence deduce that

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots = \frac{\pi^4}{96}$$

$a_0 = 1$,

$a_n = \frac{2((-1)^n-1)}{n^2\pi^2}$

3. Find the half range sine series for

$$f(x) = \begin{cases} x & \text{if } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$$

and hence find the sum of the series $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$