UNIT II

Discrete Fourier Transform, Properties and its applications

Content

- DFT – Properties - Linear filtering methods - Frequency analysis of signals using DFT - FFT Algorithm - Radix-2 FFT - Sparse FFT - Practical applications

Discrete-Time Fourier Transform

- Definition - The **Discrete-Time Fourier Transform** (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- In general, $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + j\, X_{\text{im}}(e^{j\omega})$$

- $X(e^{j\omega})$ can alternately be expressed as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg\{X(e^{j\omega})\}$$

- $\left|X(e^{j\omega})\right|$ is called the **magnitude function**
- $\theta(\omega)$ is called the **phase function**
- Both quantities are again real functions of ω
- In many applications, the DTFT is called the **Fourier spectrum**
- Likewise, $\left|X(e^{j\omega})\right|$ and $\theta(\omega)$ are called the **magnitude** and **phase spectra**

- The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω
- It is also a periodic function of ω with a period 2π:
- **Inverse Discrete-Time Fourier Transform**:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- **Convergence Condition -** An infinite series of the form

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

may or may not converge

- If $x[n]$ is an absolutely summable sequence, i.e., if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

for all values of ω

- Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT
- Since

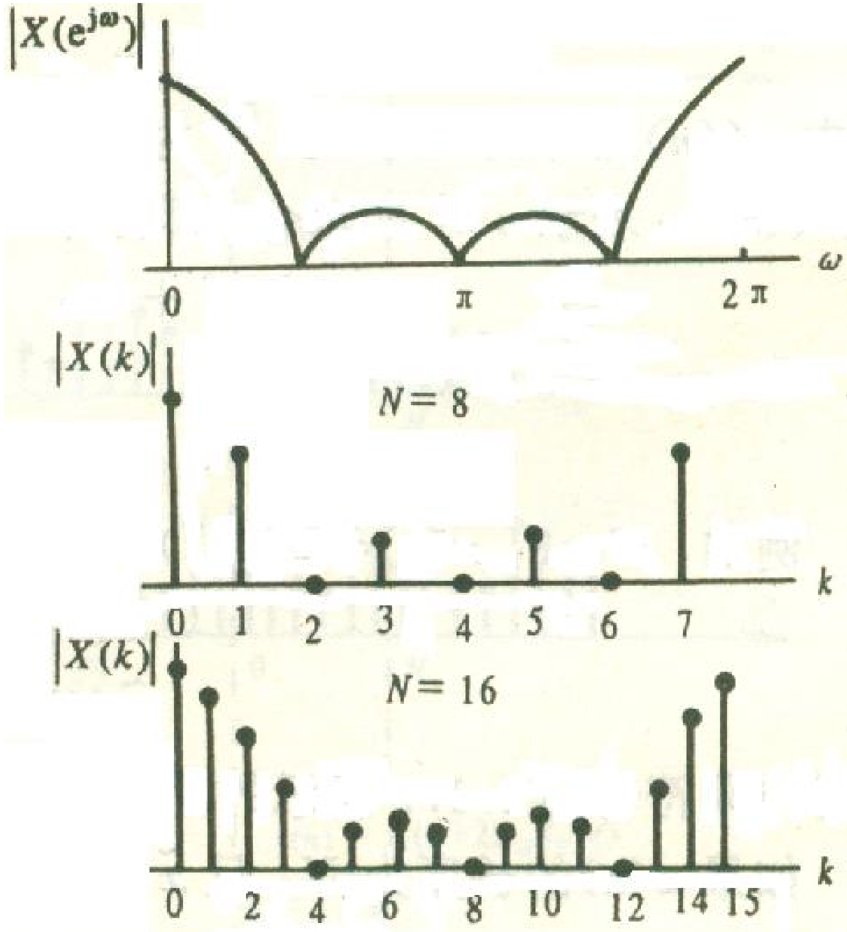
$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left(\sum_{n=-\infty}^{\infty} |x[n]|\right)^2,$$

an absolutely summable sequence has always a finite energy

- However, a finite-energy sequence is not necessarily absolutely summable

DFT, Sequence's Fourier Transform and z-transform

Continuous -time	$x_a(t)$	Fourier Transform	$X_a(j\Omega)$
Sampling			
Discrete-time	$x(n) = x_a(nT)$	Sequence's Fourier Transform	$X(e^{jw})$
	Periodic Copies		
	$x((n))_N$	DFS	$X(k)$
	Extract One period		**Extract One period**
	$x(n)$	DFT	$X(k)$



Relationship between $X(k)$ and $X(e^{j\omega})$

Frequency sampling theorem

$$X(k) \Rightarrow X(e^{j\omega}) \ \text{or} \ X(z)$$

How to realize? Prerequisite for implementation?
What is interpolation formula?

Sampling

x(n)'s z-transform:

$$\boldsymbol{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$$

Regular interval sampling on unit circle:

$$X_N(k) = X(z)\big|_{z=W_N^{-k}} = \sum_{n=-\infty}^{\infty} x(n) W_N^{kn}$$

$$z = W_N^{-k} = e^{j\frac{2\pi}{N}k}$$

Discrete Fourier Transform

- Definition - For a length-N sequence $x[n]$, defined for $0 \leq n \leq N-1$ only N samples of its DTFT are required, which are obtained by uniformly sampling $X(e^{j\omega})$ on the ω-axis between $0 \leq \omega \leq 2\pi$ at $\omega_k = 2\pi k/N$, $0 \leq k \leq N-1$

- From the definition of the DTFT we thus have

$$X[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N} = \sum_{k=0}^{N-1} x[n]e^{-j2\pi k/N},$$

$$0 \leq k \leq N-1$$

Discrete Fourier Transform

- Note: $X[k]$ is also a length-N sequence in the frequency domain
- The sequence $X[k]$ is called the **Discrete Fourier Transform** (DFT) of the sequence $x[n]$
- Using the notation $W_N = e^{-j2\pi/N}$ the DFT is usually expressed as:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \; 0 \le k \le N-1$$

Discrete Fourier Transform

- The Inverse Discrete Fourier Transform (IDFT) is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

Matrix Relations

- The DFT samples defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

can be expressed in matrix form as

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}$$

where

$$\mathbf{X} = \begin{bmatrix} X[0] & X[1] & \dots & X[N-1] \end{bmatrix}^T$$
$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & \dots & x[N-1] \end{bmatrix}^T$$

Matrix Relations

$e^{-j\theta} = \cos\theta - j\sin\theta$

$W_N^{x} = e^{-\frac{j2\pi}{N} \times x}$

and $\mathbf{D}_N$ is the $N \times N$ **DFT matrix** given by

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

$N = 4$

$W_N^1 = e^{-\frac{j2\pi}{N} \times 1}$

$W_N^2 = e^{-j2\pi/N \times 2}$

$W_N^3 = e^{-\frac{j2\pi}{N} \times 3}$

$N = 4$

$W_N^1 = e^{-\frac{j2\pi}{4}} = -1$

$W_N^2 = e^{-j2\pi/4 \times 2} = e^{-j\pi} =$

Matrix Relations

IDFT

$D_N^{-1} \Rightarrow \frac{D_N^*}{N}$

- Likewise, the IDFT relation given by

$$x[n] = \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; 0 \le n \le N-1$$

can be expressed in matrix form as

$$\mathbf{x} = \mathbf{D}_N^{-1}\mathbf{X}$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ **IDFT matrix**

Matrix Relations

where

$$\mathbf{D}_N^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

- Note:

$$\mathbf{D}_N^{-1} = \frac{1}{N}\mathbf{D}_N^{*}$$

DFT PROPERTIES

Table 8-2 Basic discrete Fourier transform properties.

Table of DFT Properties		
Property Name	***Time-Domain:*** $x[n]$	***Frequency-Domain:*** $X[k]$
Periodic	$x[n] = x[n+N]$	$X[k] = X[k+N]$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
Conjugate Symmetry	$x[n]$ is real	$X[N-k] = X^*[k]$
Conjugation	$x^*[n]$	$X^*[N-k]$
Time-Reversal	$x[((N-n))_N]$	$X[N-k]$
Delay	$x[((n-n_d))_N]$	$e^{-j(2\pi k/N)n_d}X[k]$
Frequency Shift	$x[n]e^{j(2\pi k_0/N)n}$	$X[k-k_0]$
Modulation	$x[n]\cos((2\pi k_0/N)n)$	$\frac{1}{2}X[k-k_0] + \frac{1}{2}X[k+k_0]$
Convolution	$\sum_{m=0}^{N-1} h[m]x[((n-m))_N]$	$H[k]X[k]$
Parseval's Theorem	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$	

Ex.1

- Derive the DFT for the sequence x[n]=[1,1,2,2,3,3]. Compute the corresponding Amplitude and Phase spectrum.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N},\ k = 0, 1, \ldots, N-1.$$

For $k = 0$

$$X(0) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(0)n/6} = \sum_{n=0}^{5} x(n) = 1 + 1 + 2 + 2 + 3 + 3 = 12$$

For $k = 1$

$$X(1) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(1)n/6}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-j\pi n/3}$$

$$= 1 + e^{-j\pi/3} + 2\, e^{-j2\pi/3} + 2\, e^{-j\pi} + 3\, e^{-j4\pi/3} + 3\, e^{-j5\pi/3}$$

$$= 1 + 0.5 - j\,0.866 + 2\,(-0.5 - j\,0.866) + 2\,(-1) + 3\,(-0.5 + j\,0.866) + 3(0.5 + j\,0.866)$$

$$= -1.5 + j\,2.598$$

For $k = 2$

$$X(2) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(2)n/6}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-2j\pi n/3}$$

$$= 1 + e^{-j2\pi/3} + 2\, e^{-j4\pi/3} + 2e^{-j2\pi} + 3e^{-j8\pi/3} + 3e^{-j10\pi/3}$$

$$= 1 + (-0.5) - j\,0.866 + 2(-0.5 + j\,0.866) + 2(1) + 3\,(-0.5 - j\,0.866) + 3(-0.5 + j\,0.866)$$

$$= -1.5 + j\,0.866$$

Contd.,

Conjugate Symmetry	$x[n]$ is real	$X[N-k]=X^*[k]$

For $k = 3$

$$X(3) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(3)n/6}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-j\pi n}$$

$$= 1 + e^{-j\pi} + 2e^{-j2\pi} + 2e^{-j3\pi} + 3e^{-j4\pi} + 3e^{-j5\pi}$$

$$= 1 - 1 + 2(1) + 2(-1) + 3(1) + 3(-1) = 0$$

For $k = 4$

$$X(4) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(4)n/6}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-j4\pi n/3}$$

$$= 1 + e^{-j4\pi/3} + 2e^{-j8\pi/3} + 2e^{-j4\pi} + 3e^{-j16\pi/3} + 3e^{-j20\pi/3}$$

$$= 1 + (-0.5 + j\,0.866) + 2(-0.5 - j\,0.866) + 2(1) + 3(-0.5 + j\,0.866) + 3(-0.5 - j\,0.866)$$

$$= -1.5 - j\,0.866$$

For $k = 5$

$$X(5) = \sum_{n=0}^{5} x(n)\, e^{-j2\pi(5)n/6}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-j5\pi n/3}$$

$$= 1 + e^{-j5\pi/3} + 2e^{-j10\pi/3} + 2e^{-j5\pi} + 3e^{-j20\pi/3} + 3e^{-j25\pi/3}$$

$$= 1 + (-0.5 + j\,0.866) + 2(-0.5 + j\,0.866) + 2(-1) + 3(-0.5 - j\,0.866) + 3(0.5 - j\,0.866)$$

$$= -1.5 - j2.598$$

$$X(k) = \{12, -1.5 + j2.598, -1.5 + j0.866, 0, -1.5 - j\,0.866, -1.5 - j\,2.598\}$$

For N=6, X(6-k)=X*(k)

The corresponding amplitude spectrum is given by

$$|X(k)| = \Big\{\sqrt{12\times 12}, \sqrt{(-1.5)^2+(-2.598)^2}, \sqrt{(-1.5)^2+(0.866)^2}, 0,$$

$$\sqrt{(-1.5)^2+(-0.866)^2}, \sqrt{(-1.5)^2+(-2.598)^2}\Big\}$$

$$= \{12, 2.999, 1.732, 0, 1.732, 2.999\}$$

and the corresponding phase spectrum is given by

$$\angle X(k) = \Big\{\tan^{-1}(0), \tan^{-1}\left(\frac{2.598}{-1.5}\right), \tan^{-1}\left(\frac{0.866}{-1.5}\right), \tan^{-1}(0)$$

$$\tan^{-1}\left(\frac{-0.866}{-1.5}\right), \tan^{-1}\left(\frac{-2.598}{-1.5}\right)\Big\}$$

$$= \left\{0, -\frac{\pi}{3}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3}\right\}$$

Ex.2

- Find the Inverse DFT of X(k)={1,2,3,4}

$$x(n)=\left\{\frac{5}{2},-\frac{1}{2}-j\frac{1}{2},-\frac{1}{2},-\frac{1}{2}+j\frac{1}{2}\right\}$$

The inverse DFT is defined as

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1}X(k)\,e^{j2\pi nk/N},\; n=0,1,2,3,\ldots,N-1$$

Given $N=4$, $x(n)=\frac{1}{4}\sum_{k=0}^{3}X(k)\,e^{j2\pi nk/N}, n=0,1,2,3$

When $n=0$

$$x(0)=\frac{1}{4}\sum_{k=0}^{3}X(k)\,e^{j\pi(0)k/2}$$

$$=\frac{1}{4}(1+2+3+4)=\frac{5}{2}$$

When $n=1$

$$x(1)=\frac{1}{4}\sum_{k=0}^{3}X(k)\,e^{j\pi(1)k/2}$$

$$=\frac{1}{4}\left(1+2e^{j\pi/2}+3e^{j\pi}+4e^{j3\pi/2}\right)$$

$$=\frac{1}{4}(1+2(j)+3(-1)+4(-j))$$

$$=\frac{1}{4}(-2-j2)=-\frac{1}{2}-j\frac{1}{2}$$

When $n=2$

$$x(2)=\frac{1}{4}\sum_{k=0}^{3}X(k)\,e^{j\pi k}$$

$$=\frac{1}{4}\left(1+2e^{j\pi}+3e^{j2\pi}+4e^{j3\pi}\right)$$

$$=\frac{1}{4}(1+2(-1)+3(1)+4(-1))$$

$$=\frac{1}{4}(-2)=-1/2$$

When $n=3$

$$x(3)=\frac{1}{4}\sum_{k=0}^{3}X(k)\,e^{j3\pi k/2}$$

$$=\frac{1}{4}\left(1+2e^{j3\pi/2}+3e^{j3\pi}+4e^{j9\pi/2}\right)$$

$$=\frac{1}{4}(1+2(-j)+3(-1)+4j)$$

$$=\frac{1}{4}(-2+2j)=-\frac{1}{2}+j\frac{1}{2}$$

Example

Let $X(\omega)$ denote the Fourier transform of the sequence $x(n) = \left(\frac{1}{2}\right)^n u(n)$. Let $x_1(n)$ denote a sequence of finite duration of length 10; that is, $x_1(n) = 0$, for $n < 0$ and $x_1(n) = 0$ for $n \geqslant 10$. The 10-point DFT of $x_1(n)$ denoted by $X_1(k)$, corresponds to 10 equally spaced samples of $X(\omega)$; that is, $X_1(k) = X(\omega)\big|_{\omega=\frac{2\pi k}{10}}$. Determine $x_1(n)$.

To find DTFT of $x(n) = \left(\frac{1}{2}\right)^n u(n)$:

$$X(\omega) \triangleq \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$\Rightarrow \quad X(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n}$$

Recall the formula: $$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

Hence, $$X(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$W_N^k = e^{-j\frac{2\pi}{N} \times k}$

$N = 10$

To find $X_1(k)$:

$$X_1(k) = X(\omega)\big|_{\omega=\frac{2\pi k}{10}}, \quad 0 \leqslant k \leqslant 9$$

$$\Rightarrow \quad X_1(k) = \frac{1}{1-\frac{1}{2}e^{-j\frac{2\pi k}{10}}}, \quad 0 \leqslant k \leqslant 9$$

$$= \frac{1}{1-\frac{1}{2}W_{10}^k}$$

Recall the DFT pair:

$$a^n \xleftrightarrow[N-\text{point}]{\text{DFT}} \frac{1-a^N}{1-aW_N^k}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n \xleftrightarrow[10-\text{point}]{\text{DFT}} \frac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}W_{10}^k}$$

$$\Rightarrow \quad \frac{\left(\frac{1}{2}\right)^n}{1-\left(\frac{1}{2}\right)^{10}} \xleftrightarrow[10-\text{point}]{\text{DFT}} \frac{1}{1-\frac{1}{2}W_{10}^k}$$

Hence the IDFT of $$X_1(k) = \frac{1}{1 - \frac{1}{2}W_{10}^k}$$

is $$x_1(n) = \frac{\left(\frac{1}{2}\right)^n}{1 - \left(\frac{1}{2}\right)^{10}}, \quad 0 \leqslant n \leqslant 9$$

Ex.2

Given the following sequence:

$$x(n) = \begin{cases} e^{j\omega_0 n}, & 0 \leqslant n \leqslant N-1 \\ 0, & \text{elsewhere} \end{cases}$$

a. Find the DTFT of $x(n)$.

b. Find the N-point DFT of $x(n)$.

c. Find the DFT of $x(n)$ for the case $\omega_0 = \frac{2\pi k_0}{N}$, where k_0 is an integer.

a.

$$\text{DTFT}\{x(n)\} = X(\omega) \triangleq \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$\Rightarrow \quad X(\omega) = \sum_{n=0}^{N-1} e^{j\omega_0 n}e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j(\omega-\omega_0)n}$$

Recall the formula:

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}, \quad a \neq 1$$

Hence,

$$X(\omega) = \frac{1-e^{-j(\omega-\omega_0)N}}{1-e^{-j(\omega-\omega_0)}}$$

Given the following sequence:

$$x(n) = \begin{cases} e^{j\omega_0 n}, & 0 \leqslant n \leqslant N-1 \\ 0, & \text{elsewhere} \end{cases}$$

a. Find the DTFT of $x(n)$.

b. Find the N-point DFT of $x(n)$.

c. Find the DFT of $x(n)$ for the case $\omega_0 = \frac{2\pi k_0}{N}$, where k_0 is an integer.

$$X(\omega) = \frac{1 - e^{-j(\omega-\omega_0)N}}{1 - e^{-j(\omega-\omega_0)}}$$

$$= e^{-j(\omega-\omega_0)(\frac{N-1}{2})} \frac{\sin\left[(\omega-\omega_0)\frac{N}{2}\right]}{\sin\left[(\omega-\omega_0)\frac{1}{2}\right]}$$

b. Recall the relation:

$$X(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}}$$

Hence,

$$X(k) = e^{-j\left(\frac{2\pi k}{N}-\omega_0\right)\left(\frac{N-1}{2}\right)} \frac{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)\frac{N}{2}\right]}{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)\frac{1}{2}\right]}$$

c. Suppose $\omega_0 = \frac{2\pi k_0}{N}$, where, k_0 is an integer.

Then,

$$X(k) = e^{-j\frac{2\pi}{N}(k-k_0)\left(\frac{N-1}{2}\right)} \frac{\sin\left[\pi(k-k_0)\right]}{\sin\left[\frac{\pi}{N}(k-k_0)\right]}$$

DFT pairs

$$\delta(n) \overset{\text{DFT}}{\longleftrightarrow} (1, 1, \cdots 1) \quad (\text{constant})$$

$$(1, 1, \cdots 1)\ (\text{constant}) \overset{\text{DFT}}{\longleftrightarrow} (N, 0, \cdots 0) = N\delta(k)$$

$$a^n(\text{exponential}) \overset{\text{DFT}}{\longleftrightarrow} \frac{a^N - 1}{aW_N^k - 1}$$

$$\cos\left(\frac{2\pi n k_0}{N}\right)\ (\text{sinusoid}) \overset{\text{DFT}}{\longleftrightarrow} \frac{N}{2}[\delta(k - k_0) + \delta(k + k_0)]$$

$$= \frac{N}{2}[\delta(k - k_0) + \delta(k - (N - k_0))]$$

Table: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k] = \text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\}$
$x_{pe}[n]$	$\text{Re}\{X[k]\}$
$x_{po}[n]$	$j\,\text{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$ $\text{Re}\,X[k] = \text{Re}\,X[\langle -k\rangle_N]$ $\text{Im}\,X[k] = -\text{Im}\,X[\langle -k\rangle_N]$ $\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$ $\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x_{pe}[n]$ and $x_{po}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

$x[n]$ is a real sequence

Table: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$\text{Re}\{x[n]\}$	$X_{pcs}[k] = \frac{1}{2}\{X[\langle k\rangle_N] + X^*[\langle -k\rangle_N]\}$
$j\,\text{Im}\{x[n]\}$	$X_{pca}[k] = \frac{1}{2}\{X[\langle k\rangle_N] - X^*[\langle -k\rangle_N]\}$
$x_{pcs}[n]$	$\text{Re}\{X[k]\}$
$x_{pca}[n]$	$j\,\text{Im}\{X[k]\}$

Note: $x_{pcs}[n]$ and $x_{pca}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{pcs}[k]$ and $X_{pca}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

$x[n]$ is a complex sequence

Circular Convolution

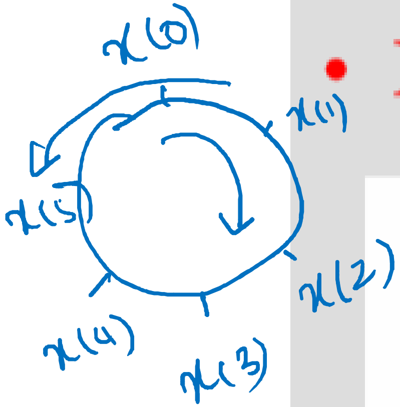
- Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as

$$y[n] = g[n] \; \textcircled{N} \; h[n]$$

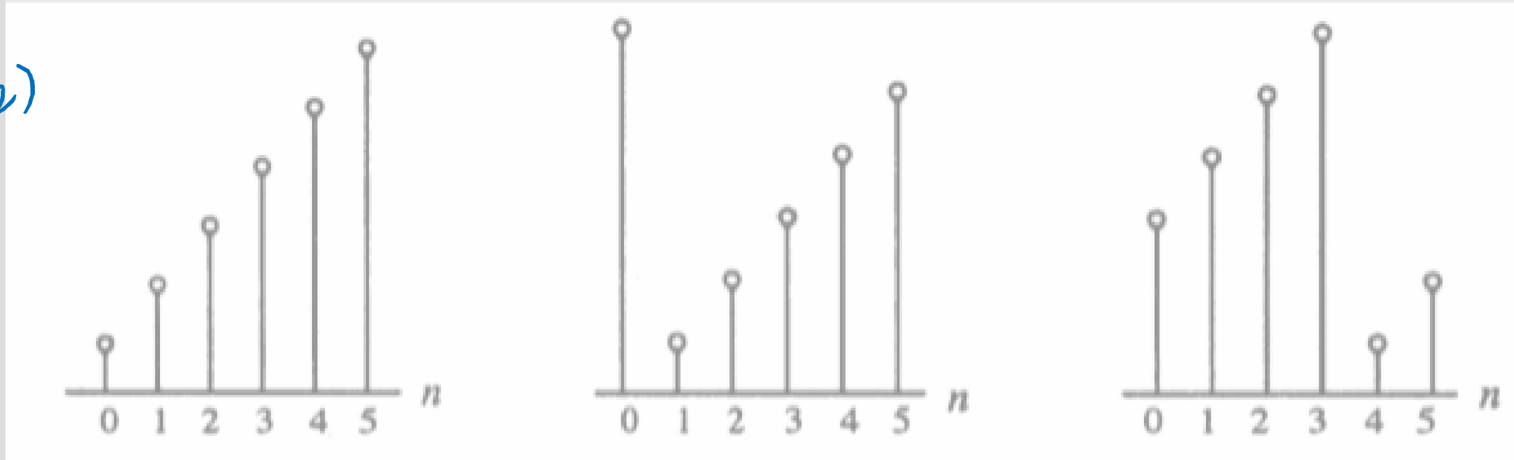
- The circular convolution is commutative, i.e.

$$g[n] \; \textcircled{N} \; h[n] = h[n] \; \textcircled{N} \; g[n]$$

Circular Shift of a Sequence



- Illustration of the concept of a circular shift



$x[n]$ $\qquad$ $x[\langle n-1\rangle_6]$ $\qquad$ $x[\langle n-4\rangle_6]$

$= x[\langle n+5\rangle_6]$ $\qquad$ $= x[\langle n+2\rangle_6]$

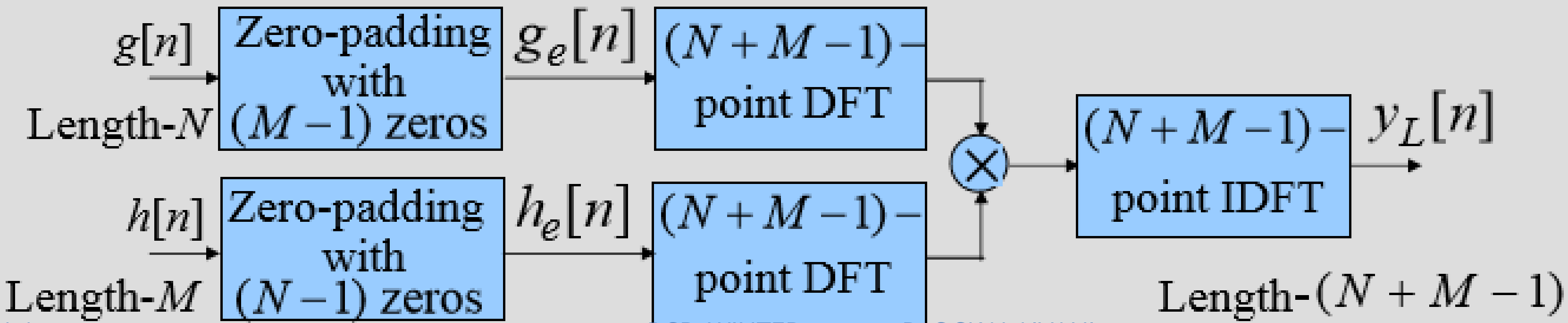
N

Linear Convolution of Two Finite-Length Sequences

- Then

$$y_L[n] = g[n] \circledast h[n] = y_C[n] = g[n] \, \textcircled{L} \, h[n]$$

- The corresponding implementation scheme is illustrated below



MATRIX METHOD

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-2\pi i/4} & e^{-4\pi i/4} & e^{-6\pi i/4} \\ 1 & e^{-4\pi i/4} & e^{-8\pi i/4} & e^{-12\pi i/4} \\ 1 & e^{-6\pi i/4} & e^{-12\pi i/4} & e^{-18\pi i/4} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

Find the DFT of the sequence x(n)={0,1,2,1} for N=4

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$\underset{X[k]}{\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix}} = \underset{W_4}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}} \underset{x(n)}{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

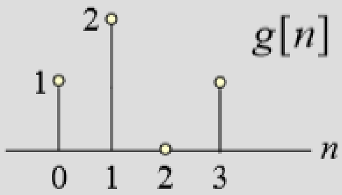
$$X[k] = \{4, -2, 0, -2\}$$

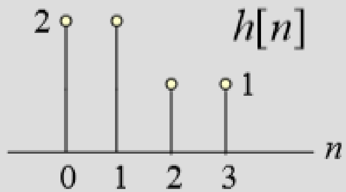
CIRCULAR CONVOLUTION

- Example - Determine the 4-point circular convolution of the two length-4 sequences:

$$\{g[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}, \quad \{h[n]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1$$

as sketched below





Circular Convolution

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$\mathbf{D}_4$ is the 4-point DFT matrix

Circular Convolution

- If $Y_C[k]$ denotes the 4-point DFT of $y_C[n]$ then from Table above we observe

$$Y_C[k] = G[k]H[k], \quad 0 \le k \le 3$$

- Thus

$$\begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix} = \begin{bmatrix} G[0]H[0] \\ G[1]H[1] \\ G[2]H[2] \\ G[3]H[3] \end{bmatrix} = \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix} \begin{matrix} = 4\times 6 \\ = (1-j)\times(1-j) \\ = (0\times -2) \\ = (1+j)(1+j) \end{matrix}$$

Circular Convolution

- A 4-point IDFT of $Y_C[k]$ yields

$$y(n) = \frac{1}{N} D_N^{*} Y(k)$$

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4}\mathbf{D}_4^{*} \begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

Ex.

Example 3.42 Find the 4-point DFTs of two sequences $g(n)$ and $h(n)$ defined below, using a single 4-point DFT.

$$g(n) = (1, 2, 0, 1)$$

and

$$h(n) = (2, 2, 1, 1)$$

$$x(n) = g(n) + jh(n), \quad 0 \leqslant n \leqslant 3$$

$$x(n) = (1 + j2, 2 + j2, 0 + j1, 1 + j1)$$

$$\text{DFT}\{x(n)\} \triangleq X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}, \quad 0 \leqslant k \leqslant 3$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+j2 \\ 2+j2 \\ 0+j1 \\ 1+j1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+j6 \\ 2 \\ -2 \\ j2 \end{bmatrix}$$

CONTd.,

$$X(k) = (4 + j6, 2, -2, j2)$$

$$X^*(k) = (4 - j6, 2, -2, -j2)$$

$$G(k) = \frac{1}{2}[X(k) + X^*(4-k)]$$

$$= \frac{1}{2}[(4 + j6, 2, -2, j2) + (4 - j6, -j2, -2, 2)]$$

$$= (4, 1 - j1, -2, 1 + j)$$

$$H(k) = \frac{1}{2j}[X(k) - X^*(4-k)]$$

$$= \frac{1}{2j}[(4 + j6, 2, -2, j2) - (4, -j6, -j2, -2, 2)]$$

$$= (6, 1 - j, 0, 1 + j)$$

EX

The two 8-point sequences $x_1(n)$ and $x_2(n)$ shown in Fig. RP.3.6 have DFTs $X_1(k)$ and $X_2(k)$ respectively. Determine the relationship between $X_1(k)$ and $X_2(k)$.

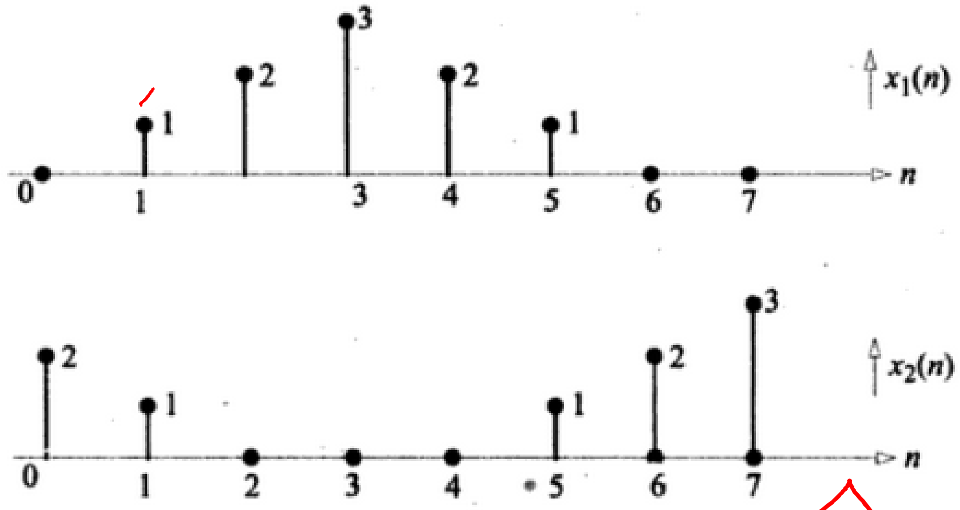
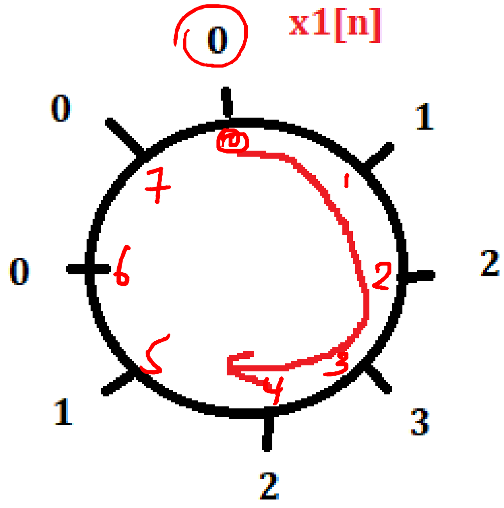
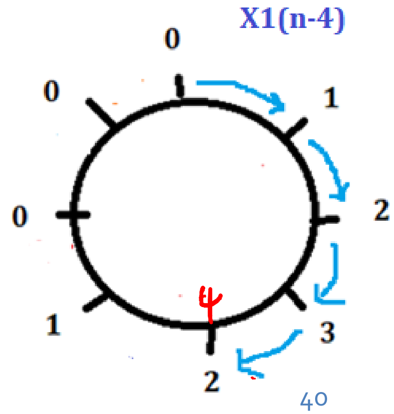
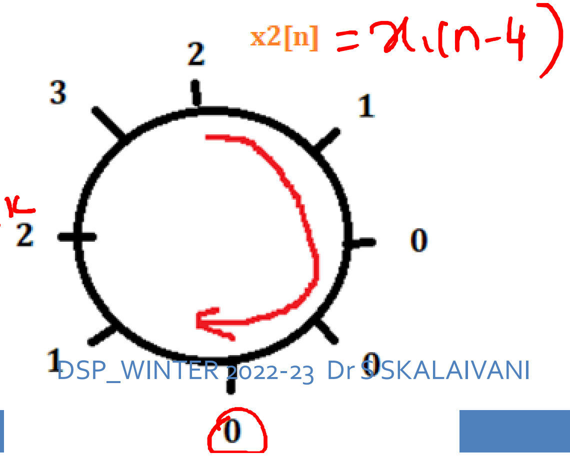


Fig. RP.3.6 Sequences $x_1(n)$ and $x_2(n)$ for RP.3.6.







$$x_2(n) = x_1((n-4))_8$$

$$\text{DFT}\{x((n-m))_N\} = W_N^{km} X(k)$$

$$\begin{aligned} X_2(k) &= \text{DFT}\{x_1((n-4))_8\} \\ &= W_8^{4k} X_1(k) \\ &= e^{-j\pi k} X_1(k) \end{aligned}$$

$$X_2(k) = (-1)^k X_1(k)$$

Ex.

The even samples of the 11-point DFT of a length-11 real sequence are given by

$$X(0) = 2, \quad X(2) = -1 - j3, \quad X(4) = 1 + j4,$$

$$X(6) = 9 + j3, \quad X(8) = 5, \quad X(10) = 2 + j2$$

N =11

Determine the missing odd samples of the DFT.

Since $x(n)$ is a real sequence, the following symmetry condition must be satisfied:

$$X(k) = X^*(N-k)$$

$$\Rightarrow \quad X(k) = X^*(11-k)$$

K =1

$$X(1) = X^*(10) = 2 - j2$$

$$X(3) = X^*(8) = 5$$

$$X(5) = X^*(6) = 9 - j3$$

$$X(7) = X^*(4) = 1 - j4$$

$$X(9) = X^*(2) = -1 + j3$$

$$X(11) = X^*(0) = 2$$