MODULE - 1

Fourier Series

1.1

Definition: A function $f(t)$ is said to have period $K(>0)$ if $f(t+K) = f(t)$ for all t.

Definition: Let the function $f(x)$ has period $2l$. Then the Fourier series of $f(x)$ in the interval $[-l, l]$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Here a_0, a_n and b_n are called Fourier coefficients. which are also called as Euler's formulae.

Note: suppose that the interval is $(0, 2l)$, then

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Dirichlet's conditions:

A function $f(x)$ has a valid Fourier Series expansion of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where a_0, a_n and b_n are constants, provided

(i) $f(x)$ is well defined, periodic, single valued and finite.

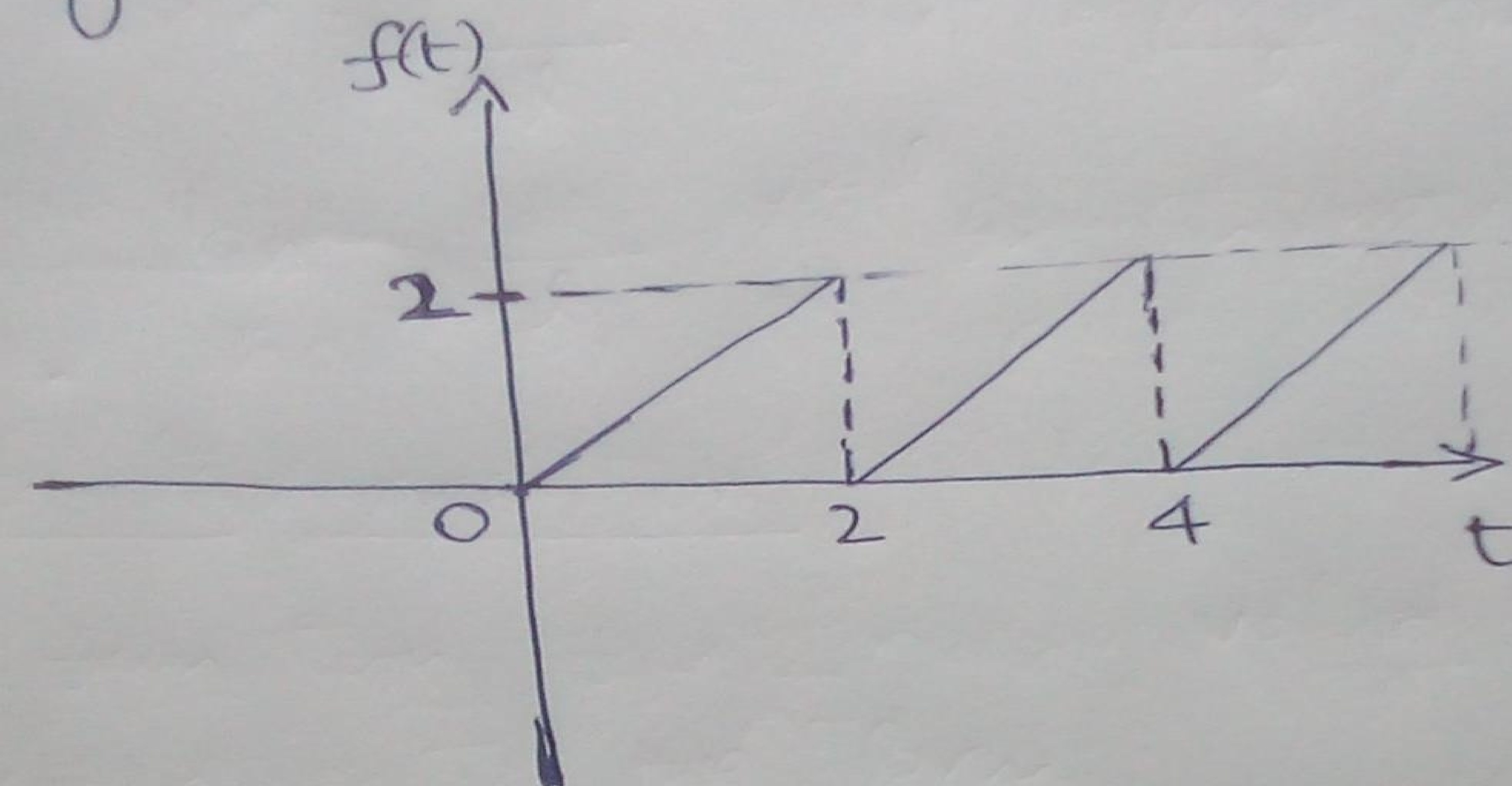
(ii) $f(x)$ has finite number of discontinuities in any one period.

(iii) $f(x)$ has at most a finite number of maxima and minima in the interval of definition.

Note: The above conditions are sufficient but not necessary.

Examples

1. obtain the Fourier Series for the periodic signal in the time domain.



Sol: From the given figure, we can take $f(t) = t, \quad 0 \leq t \leq 2$.

clearly it is periodic function with period 2.

[$y = mt$
here slop $m = 1$]

Therefore, the Fourier series for $f(t) = t$ in the interval $[0, 2]$ is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{l}\right)$$

Here $l = 1$.

Now,

$$a_0 = \frac{1}{l}\int_0^{2l} f(t)\,dt = \int_0^2 t\,dt = \left[\frac{t^2}{2}\right]_0^2 = 2.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(t)\cos\left(\frac{n\pi t}{l}\right) dt$$

$$= \int_0^2 t\cdot\cos(n\pi t)\,dt$$

$$= \left[t\cdot\left(\frac{\sin(n\pi t)}{n\pi}\right)\right]_0^2 - \left[1\cdot\left(-\frac{\cos(n\pi t)}{n^2\pi^2}\right)\right]_0^2$$

$$+ \left[0\cdot\left(-\frac{\sin(n\pi t)}{n^3\pi^3}\right)\right]_0^2$$

$$= \frac{1}{n\pi}(2\sin 2n\pi - 0) + \frac{1}{n^2\pi^2}(\cos 2n\pi - \cos 0)$$

$$+ 0$$

$$= \frac{1}{n\pi}\, 2(0) + \frac{1}{n^2\pi^2}(1-1)$$

$$= 0$$

Here $\sin 2n\pi = 0$ for $n = 1, 2, 3, \ldots$

$\cos 2n\pi = 1$ for $n = 1, 2, 3, \ldots$

$$b_n = \frac{1}{l}\int_0^{2l} f(t)\, \sin\left(\frac{n\pi t}{l}\right) dt$$

$$= \int_0^2 t \cdot \sin(n\pi t)\, dt$$

$$= \left[t \cdot \left(-\frac{\cos n\pi t}{n\pi}\right)\right]_0^2 - \left[1 \cdot \left(-\frac{\sin(n\pi t)}{n^2\pi^2}\right)\right]_0^2$$

$$+ \left[0 \cdot \left(\frac{\cos(n\pi t)}{n^3\pi^3}\right)\right]_0^2$$

$$= -\frac{1}{n\pi}(2\cos 2n\pi - 0) + \frac{1}{n^2\pi^2}(\sin 2n\pi - 0)$$

$$+ \frac{1}{n^3\pi^3}(\cos 2n\pi - \cos 0)$$

$$= \frac{-1}{n\pi} 2(1) + \frac{1}{n^2\pi^2}(0) + \frac{1}{n^3\pi^3}(1-1)$$

$$= -\frac{2}{n\pi} + 0 + 0 = -\frac{2}{n\pi}$$

Hence the Fourier Series for the given periodic signal is

$$f(t) = \frac{2}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \cdot \sin(n\pi t)$$

i.e., $t = 1 - \frac{2}{\pi}\left(\sin t\pi + \frac{1}{2}\sin 2t\pi + \frac{1}{3}\sin 3t\pi \ldots\right)$

Note: (i) $\sin(n\pi) = 0$ for $n = 1, 2, 3, \ldots$

(ii) $\cos(n\pi) = (-1)^n$ for $n = 1, 2, 3, \ldots$

i.e., $\cos(n\pi) = \begin{cases} -1 & \text{for } n = 1, 3, 5, \ldots \\ 1 & \text{for } n = 2, 4, 6, \ldots \end{cases}$

(2) Obtain the Fourier Series for

$f(x) = x - \pi \, , \quad -\pi < x < \pi$

Answer: Here $l = \pi$

$a_0 = -2\pi \, , \quad a_n = 0$ and $b_n = \frac{2}{n}(-1)^{n+1}$

Therefore,

$$f(x) = -\pi + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - \pi) \cos(nx) \, dx$$

$$= \frac{1}{\pi} \left\{ \underbrace{\int_{-\pi}^{\pi} x \cos nx \, dx}_{\text{odd}} - \pi \int_{-\pi}^{\pi} \underbrace{\cos(nx)}_{\text{even}} \, dx \right\}$$

$$= \frac{1}{\pi} \left\{ 0 - 2\pi \int_{0}^{\pi} \cos(nx) dx \right\}$$

$$= \frac{1}{\pi} \left\{ -2\pi \left(\frac{-\sin(nx)}{n} \right)_0^{\pi} \right\} = \frac{-2}{n}(0 - 0) = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x-\pi) \sin(nx)\, dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{\pi} \underbrace{x \sin(nx)}_{\text{even}} dx - \pi \int_{-\pi}^{\pi} \underbrace{\sin(nx)}_{\text{odd}} dx \right\}$$

$$= \frac{1}{\pi} \left\{ 2 \int_{0}^{\pi} x \sin(nx)\, dx - \pi(0) \right\}$$

$$= \frac{1}{\pi} \left\{ 2 \left\{ \left[x \left(\frac{-\cos nx}{n} \right) \right]_0^{\pi} - \left[1 \cdot \left(\frac{-\sin(nx)}{n^2} \right) \right]_0^{\pi} \right\} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{-2}{n} (\pi \cos(n\pi) - 0) + \frac{2}{n^2} (0 - 0) \right\}$$

$$= \frac{-2}{n} \cos(n\pi) = -\frac{2}{n} (-1)^n = \boxed{\frac{2}{n} (-1)^{n+1}}$$

3. Obtain the Fourier series for $f(x) = x^2$ in the interval $(0, 2\pi)$.

<u>Answer:</u> Here $l = \pi$

$$a_0 = \frac{8}{3}\pi^2, \quad a_n = \frac{4}{n^2} \text{ and } b_n = -\frac{4\pi}{n}$$

Hence, $f(x) = \frac{4}{3}\pi^2 + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) - 4\pi \sum_{n=1}^{\infty} \frac{1}{n} \sin(nx)$

4. Find the Fourier Series for $f(x) = \frac{\pi - x}{2}$ in the interval $(0, 2)$.

Answer: Here $l = 1$

$a_0 = \pi - 1$, $a_n = 0$ and $b_n = \frac{1}{n\pi}$.

5. Find the Fourier series for $f(x) = x - x^2$ in $[-\pi, \pi]$

Answer: Here $l = \pi$

$a_0 = -\frac{2\pi^2}{3}$, $a_n = \frac{4}{n}(-1)^{n+1}$

$b_n = \frac{2}{n}(-1)^{n+1}$

6. Obtain the Fourier Series for $f(x) = x \sin x$ in the interval $(0, 2\pi)$

Sol: The F.S. for $f(x) = x \sin x$ in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Here, $\boxed{l=\pi}$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} x.\sin x\,dx$

$= \frac{1}{\pi}\left\{\left[x(-\cos x)\right]_0^{2\pi} - \left[1.(-\sin x)\right]_0^{2\pi}\right\}$

$= \frac{1}{\pi}(-2\pi + 0) = -2$

$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x \cos nx\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin x \cos nx)\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx$

$= \frac{1}{2\pi}\left\{\left[x\left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right)\right]_0^{2\pi} - \left[1.\left(-\frac{\sin(n+1)}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}\right\}$

(for $n \neq 1$)

$= \frac{1}{2\pi}\left\{\left[2\pi\left(\frac{-1}{n+1} + \frac{1}{n-1}\right) + 0\right]\right.$ (for $n \neq 1$)

$= \frac{2}{n^2-1}$ (for $n \neq 1$)

If $n=1$, we have

$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x \cos x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x\,dx$

$\boxed{a_1 = \frac{1}{\pi}\int^{2\pi} x \sin x \cos x\,dx}$

$= -\frac{1}{2}$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x\sin x.\sin nx\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x(2\sin x.\sin nx)\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x-\cos(n+1)x]\,dx$$

$$=\frac{1}{2\pi}\left\{\left[x\left(\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right)\right]_0^{2\pi} - \left[1.\left(-\frac{\cos(n-1)x}{(n-1)^2}+\frac{\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}\right\}$$

(for $n\neq 1$)

$$=\frac{1}{2\pi}\left[\frac{1}{(n-1)^2}-\frac{1}{(n+1)^2}-\frac{1}{(n-1)^2}+\frac{1}{(n+1)^2}\right]$$

(for $n\neq$ [redacted])

$\therefore b_n = 0$ for $n\neq 1$

If $n=1$, we have

$$b_1=\frac{1}{\pi}\int_0^{2\pi} x.\sin x\sin x\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x(2\sin^2 x)\,dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx=\pi$$

Therefore,

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x$$
$$+ \sum_{n=2}^{\infty} b_n \sin nx.$$

i.e., $$x \sin x = -1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \left(\frac{2}{n^2-1}\right) \cos nx + \pi \sin x.$$

⑦ Find the Fourier Series for $f(x) = 2x - x^2$ in the interval $(0, 2)$ and hence deduce that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

Sol: Here, $l = 1$.

Now, $$a_0 = \frac{1}{l} \int_0^2 (2x - x^2) dx = \frac{4}{3}$$

$$a_n = \int_0^2 (2x - x^2) \cos(n\pi x) dx$$

$$= \frac{-4}{n^2\pi^2}$$

and $$b_n = \int_0^2 (2x - x^2) \sin(n\pi x) dx = 0$$

Therefore, $$2x - x^2 = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

Taking $x = 0$, we get

$$0 = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$