Laplace Transform of Periodic functions:

Let $f(t)$ be a given function $(t>0)$.

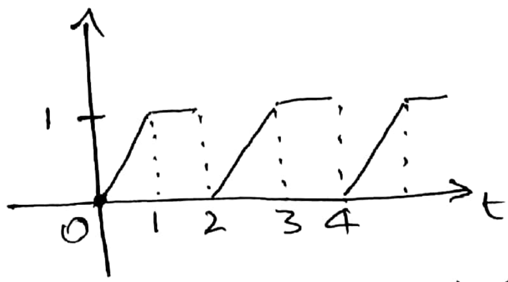
If there is some positive K such that $f(t+K) = f(t)$, then we say that f is a periodic function. If there exists smallest such K, then K is called period of f.

that is the graph of $f(t)$ is repeated in regular interval of K.

Def: If $f(t)$ is a periodic function with period K, then we define

$$L\{f(t)\} = \frac{1}{1-e^{-sK}} \int_0^K e^{-st} f(t)\, dt$$

e.g. The Laplace Transform of the following periodic signal with period 2



i.e., $f(t) = \begin{cases} t & ; 0 < t < 1 \\ 1 & ; 1 < t < 2 \end{cases}$

is $L\{f(t)\} = \frac{1}{1-e^{-2s}} \left\{ \int_0^2 e^{-st} f(t)\, dt \right\}$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st} f(t)\,dt + \int_1^2 e^{-st} f(t)\,dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st}\cdot t\,dt + \int_1^2 e^{-st}\cdot 1\,dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \left[t\cdot\left(\frac{e^{-st}}{-s}\right)\right]_0^1 - \left[1\cdot\left(\frac{e^{-st}}{s^2}\right)\right]_0^1 + \left[\frac{e^{-st}}{-s}\right]_1^2 \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{-1}{s}\left(e^{-s}-0\right) - \frac{1}{s^2}\left(e^{-s}-1\right) - \frac{1}{s}\left(e^{-2s}-e^{-s}\right) \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{1}{s^2}\left(1+e^{-s}\right) - \frac{1}{s}\, e^{-2s} \right\}$$

Exercise (1) Find the Laplace transform of the periodic function $f(t) = \begin{cases} \sin\omega t & \text{if } 0<t<\pi/\omega \\ 0 & \text{if } \pi/\omega < t < 2\pi/\omega \end{cases}$

with period $2\pi/\omega$.

(2) Find the Laplace transform of $f(t) = \begin{cases} t & \text{if } 0<t<1 \\ 0 & \text{if } 1<t<2 \end{cases}$

with period 2.

Some problems on Laplace Transform :

① Find $L\{f(t)\}$, where $f(t) = |t| + |t-1|$

Sol: If $0 < t < 1$, then $f(t) = t - (t-1) = 1$

If $t > 1$, then $f(t) = t + (t-1) = 2t - 1$

Therefore $f(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 2t-1 & \text{if } t > 1 \end{cases}$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} . 1\, dt + \int_1^\infty e^{-st} (2t-1)\, dt$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^1 + \left\{\left[(2t-1).\left(\frac{e^{-st}}{-s}\right)\right]_1^\infty - \left[2\left(\frac{e^{-st}}{s^2}\right)\right]_1^\infty\right\}$$

$$= \frac{1}{s}(1 - e^{-s}) + \left\{-\frac{1}{s}(0 - e^{-s}) - \frac{2}{s^2}(0 - e^{-s})\right\}$$

$$= \frac{1}{s}(1 - e^{-s}) + \frac{1}{s} e^{-s} + \frac{2}{s^2} e^{-s}.$$

$$= \frac{1}{s} + \frac{2}{s^2} e^{-s}.$$

(2) Find $L\left\{ e^{-2t} \int_0^t \left(\frac{1-\cos t}{t}\right) dt \right\}$

Sol. Let $f(t) = \int_0^t \frac{1-\cos t}{t}\, dt$

we know that $L\left\{\frac{1-\cos t}{t}\right\} = \log\left[\frac{\sqrt{s^2+1}}{s}\right]$

Therefore $L\{f(t)\} = L\left\{\int_0^t \frac{1-\cos t}{t}\, dt\right\} = \frac{1}{s}\log\left[\frac{\sqrt{s^2+1}}{s}\right]$

and hence by the first shifting property, we have

$$L\left\{ e^{-2t} f(t) \right\} = L\left\{ e^{-2t} \int_0^t \frac{1-\cos t}{t}\, dt \right\}$$

$$= \frac{1}{s+2} \log\left[\frac{\sqrt{(s+2)^2+1}}{s+2}\right].$$

(3) Find $L\{\cosh 2t . \sin t\}$

$$L\{\cosh 2t . \sin t\} = L\left\{\left(\frac{e^{2t}+e^{-2t}}{2}\right)\sin t\right\}$$

$$= \frac{1}{2}\left\{ L\{e^{2t}\sin t\} + L\{e^{-2t}\sin t\} \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{(s-2)^2+1} + \frac{1}{(s+2)^2+1} \right\}.$$

Exercise :

① Find $L\{f(t)\}$, where $f(t) = |t-2| + |t-3|$

② Find $L\{(\sinh 3t) \cdot t^2\}$

③ Find $L\{e^{-t} \cdot t \cdot \cosh t\}$

④ Find $L\left\{\int_0^t \int_0^t t\cos t \, dt \, dt\right\}$

⑤ Find Laplace transform of the triangular wave of period $2a$ given by

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases}$$

⑥ Find the Laplace transform of the ~~Square~~ full-wave rectifier $f(t) = \begin{cases} E \sin \omega t \; ; & 0 < t < \pi/\omega \end{cases}$

with period π/ω.