Matrices

Let A be a square matrix, then A is

~~Let A matrix~~

(a) Symmetric matrix if

$$\boxed{A^T = A}$$

(b) Skew Symmetric if

$$\boxed{A^T = -A}$$

(c) A is said to be orthogonal if

$$\boxed{AA^T = I}$$

Ex:-(1) $A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ is symmetric.

(2) $B = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$ is skew symmetric.

(3) $C = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$ is orthogonal.

Eigenvalue and Eigenvector :-

Let $A_{n\times n}$ be a square matrix and $X_{n\times 1}$ is any nonzero vector then X is said to be eigenvector of A iff there exist a scalar λ such that

$$\boxed{AX = \lambda X}$$

Ex:- verify that for the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, eigenvalue corresponding to eigenvectors $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are 2 and -1.

$$AX_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2X_1$$

$$AX_1 = 2X_1$$

i.e. $\lambda = 2$ is eigenvalue corresponding to $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now $AX_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -1 \cdot X_2$$

$$AX_2 = (-1)X_2$$

so $\lambda = -1$ is also an eigenvalue.

Ex:- Verify that for $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is eigenvect

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Proceedure for finding eigenvalues and eigenvectors:-

· Let $A_{n\times n}$ be a square matrix, Let $X_{n\times 1}$ be a nonzero eigenvector then we have $AX = \lambda X$ or $AX - \lambda X = 0$ or $(A-\lambda I)X = 0$

i.e $(A-\lambda I)X = 0$, where I is identity matrix

step I:- find characterstic eq^n

$$|A - \lambda I| = 0$$

which gives an equation in λ, solve it to find the values of λ ie. eigenvalues.

characterstic eq^n takes the form

$$\left[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0\right]$$

Q1:- find eigenvalues of $A = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}$

Put $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4 & -5 \\ 1 & -2 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0$

$$\begin{vmatrix} 4-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-2-\lambda) + 5 = 0$$

$$-8 - 4\lambda + 2\lambda + \lambda^2 + 5 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\boxed{\lambda = 3, -1}$$

Q2:- find eigenvalues of $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda^2 - 3\lambda + 2) = 0$

$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2$

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Eigenvectors:- Let $\lambda_1, \lambda_2, ---$ be the eigenvalues of A then eigenvector X, corresponding to any eigenvalue λ can be computed by solving

$$\boxed{(A-\lambda I)X = 0}$$

Q. Find all eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$$

Solⁿ characterstic eqⁿ

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 4-\lambda & 1 \\ -2 & -4 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(4-\lambda)(-1-\lambda)+4] - 2[(-1-\lambda)+2] + 2[-4+2(4-\lambda)] = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

which gives

$$\lambda = 1, 2, 3$$

Now Let $\lambda_1 = 1$ and $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the eigenvector corresponding to λ_1 then

$$\therefore (A-\lambda_1 I)X_1 = 0 \Rightarrow (A-I)X_1 = 0$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x+2y+2z=0$

$x+3y+z=0$

$-2x-4y-2z=0$

$\Rightarrow \quad x+y+z=0$

$x+3y+z=0$

$x+2y+z=0$

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 3 & 1 \end{matrix}$$

solving first two eq^{n}'s

$$\frac{x}{1-3} = \frac{y}{-(0)} = \frac{z}{2}$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{z}{2}$$

so eigenvector $X_1 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$ or

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now Let $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, be the eigenvector corresponding to eigenvalue $\lambda = 2$

then $(A-\lambda_2 I)X_2 = 0 \Rightarrow (A-2I)X_2 = 0$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x+2y+2z=0$

$x+2y+z=0$

$-2x-4y-3z=0$

solving first two

$$\frac{x}{-2} = \frac{y}{-(0-1)} = \frac{z}{0}$$

$$\frac{x}{-2} = \frac{y}{1} = \frac{z}{0}$$

so $X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Now Let X_3 is eigenvector corresponding to $\lambda = 3$ then

$(A - \lambda_3 I) X_3 = 0 \Rightarrow (A - 3I) X_3 = 0$

$$\begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0 \cdot x + 2y + 2z = 0$
$x + y + z = 0$
$-2x - 4y - 4z = 0$

$\Rightarrow$ $0x + 2y + 2z = 0$
$x + y + z = 0$
$x + 2y + 2z = 0$

0 2 2
1 1 1

solving first two

$$\frac{x}{0} = \frac{y}{-(-2)} = \frac{z}{-2}$$

$X_3 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$ or $X_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

eigenvectors corresponding to $\lambda = 1, 2, 3$ are

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$ respectively.

Shortcut method for finding characterstic equation

for 2×2 matrix A

char. eq^n is

$$\lambda^2 - (\text{trace } A)\lambda + (\det A) = 0$$

for 3×3 matrix A

char. eq^n is

$$\lambda^3 - (\text{trace of } A)\lambda^2 + \underbrace{(C_{11} + C_{22} + C_{33})}_{\text{Sum of diagonal cofactors}}\lambda - \det A = 0$$

Q:- find eigenvalues of $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$

characterstic eq^n

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{vmatrix} = 0$$

using shortcut method

trace of $A = 0$

$C_{11} = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -2-3 = -5$ $\qquad C_{22} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$

$C_{33} = \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = -3$ $\qquad$ So $C_{11} + C_{22} + C_{33} = -7$

$\det(A) = |A| = 6$

So characterstic poly. is

$$\lambda^3 - 0\cdot\lambda^2 - 7\lambda - 6 = 0 \Rightarrow \lambda^3 - 7\lambda - 6 = 0$$

$$\lambda^3 - 7\lambda - 6 = 0$$

solving

$$\lambda = -1, -2, 3$$

~~Geometric and Algeb~~

Linear dependence and indepence of vectors :-

The vectors $x_1, x_2, --- x_n$ are linearly dependent if there exist scalars $c_1, c_2, -- c_n$ (at least one of which is non zero), such that

$$c_1 x_1 + c_2 x_2 + --- + c_n x_n = 0$$

The vectors $x_1, x_2, -- x_n$ are linearly independent if $\exists$ $c_1, c_2, --- c_n$ such that

$$c_1 x_1 + c_2 x_2 + --- + c_n x_n = 0.$$

where all c_i's are zero

ie $c_1 = c_2 = c_3 = --- = c_n = 0$

Properties of eigenvalues:-

(1) The eigenvalues of A and A^T are the same.

(2) Sum of eigenvalues is equal to the sum of the principal diagonal elements of A.

(3) The product of eigenvalues is equal to det of A ie $|A|$.

(4) If $\lambda_1, \lambda_2, --- \lambda_n$ are eigenvalues of A then, $\lambda_1^K, \lambda_2^K, --- \lambda_n^K$ are the eigenvalues of A^K, where K is positive integer.

(5) If $\lambda_1, \lambda_2, -- \lambda_n$ are eigenvalues of A then $K\lambda_1, K\lambda_2, --- K\lambda_n$ are eigenvalues of $K \cdot A$, where K is nonzero scalar.

(6) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ---- \frac{1}{\lambda_n}$ are eigenvalues of A^{-1}, provided A is non singular. i.e. $|A| \neq 0$.

(7) If one of the eigenvalue of matrix A is 0 then matrix is ~~sig~~ singular i.e. $|A| = 0$.

(8) The eigenvalues of diagonal matrix are the diagonal elements.

(9) The eigenvalues of upper and lower triangular matrix are the diagonal elements.

(10) The eigenvalues of real symmetric matrix are all real.

Properties of eigenvectors :-

(1) If eigenvalues of matrix $A_{n\times n}$ are distinct then corresponding eigenvectors are linearly independent i.e if $\lambda_1, \lambda_2, --- \lambda_n$ be distinct eigenvalues of A then corresponding eigenvectors $x_1, x_2, -- x_n$ are linearly independent.

2. An eigenvetor x, corresponding to eigenvalue λ, need not to be unique.
i.e. any scalar multiple of eigenvector X ie (KX) is also an eigenvector corresponding to eigenvalue λ.

3. If any two eigenvalues are equal then corresponding eigenvectors are either linearly dependent or independent. i.e. it may or may not be posible to get linearly independent eigenvector corresponding to repeated root.

Algebraic and geometric multiplicity of eigenvalue

The order of eigenvalue λ (i.e. no. of time its repeat) is called algebraic multiplicity. ~~and~~

The number of linearly independent eigenvectors corresponding to eigenvalue λ is called geometric multiplicity.

Ex :- Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$\lambda = 1, 1, 5$

linearly independent eigenvector corresponding to $\lambda = 1$ are

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

for $\lambda = 5$
eigen vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Algebraic multiplicity for
$\lambda = 1$ is 2
$\lambda = 5$ is 1

Geometric multiplicity for
$\lambda = 1$ is 2
$\lambda = 5$ is 1

Q:- Find the eigen values and eigen vectors for the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

characterstic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-1)(\lambda-5) = 0$$

$$\lambda = 1, 1, 5$$

eigen vectors are given by

$$(A - \lambda I) X = 0$$

for $\lambda = 1$ Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be eigenvector

$\lambda = 1$

so $(A - I) X_1 = 0$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + x_3 = 0$$

Three variable, only 1 eq^n

so $(3-1) = 2$ independent variable

Let $x_3 = K_1$

$x_2 = K_2$

$x_1 = -(2x_2 + x_3)$

$x_1 = -2K_2 \pm K_1$

i.e. eigenvector for $\lambda = 1$ is

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -K_1 - 2K_2 \\ K_2 \\ K_1 \end{bmatrix}$$

$$X_1 = K_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + K_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 1$, there are two linearly independent eigenvectors,

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

for $\lambda = 5$

$$(A - 5I)X = 0$$

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{2x_2 + x_3 = 0}$

$\cancel{x_1 + x_2 + x_3 = 0}$

$\cancel{x_1 + 2x_2 = 0}$

$-3x_1 + 2x_2 + x_3 = 0$

$x_1 - 2x_2 + x_3 = 0$

$x_1 + 2x_2 - 3x_3 = 0$

Using first two eq^n

$$\cancel{0 \cdot x_1 + 2x_2 + x_3 = 0} \qquad \cancel{\begin{matrix} 0 & 2 & 1 \\ 1 & 1 & 1 \end{matrix}}$$
$$\cancel{x_1 + x_2 + x_3 = 0}$$

$$\cancel{\frac{x_1}{2-1} = \frac{x_2}{-(0-1)} = \frac{x_3}{}}$$

$$\frac{x_1}{4} = \frac{x_2}{-(-4)} = \frac{x_3}{4}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

eigen vector for $\lambda = 5$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Q Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

characteristic eq^n

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(\lambda - 2)^3 = 0$$

$$\lambda = 2, 2, 2$$

Let eigen vetor for $\lambda = 2$ be

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

we have

$$(A - \lambda I) X = 0$$

$$(A - 2I) X = 0$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$

$x_3 = 0$

i.e x_1 is free variable. Let $x_1 = K$.

eigenvector is $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K \\ 0 \\ 0 \end{bmatrix}$.

i.e there is only one eigen vector correspond- to $\lambda = 2$.

Algebraic multiplicity for $\lambda = 2$ is 3

Geometric " " $\lambda = 2$ is 1

Q:- find eigenvalues and eigenvector for the matrix.

$$\begin{bmatrix} 3 & -1 & 3 \\ 9 & -1 & 9 \\ 7 & 1 & 7 \end{bmatrix}$$

Hint:- eigenvalues are $\lambda = 0, 1, 8$

for $\lambda = 0$, eigenvector $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

for $\lambda = 1$, eigenvector $X_2 = \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix}$

for $\lambda = 8$ eigenvector $X_3 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

Q find eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

Hint:- char. eqn

$$\lambda^3 - 17\lambda^2 + 42\lambda = 0$$

eigenvalues are

$$\lambda = 0, 3, 14$$

eigenvectors for

$\lambda = 0$ is $X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$

for $\lambda = 3$ $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

for $\lambda = 14$ $X_3 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

Cayley - Hamilton theorem :- (C-H)

Every square matrix satisfies its own characteristic equation.

This theorem states that if

$$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$$

is characteristic equation for a square matrix $A_{n\times n}$ then
A will satisfy its characteristic equation i.e.

$$a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI = O$$

where I is unit matrix and O is null matrix.

Two important applications of C-H Theo :-

(1) If A is non-singular matrix i.e. $|A| \neq 0$ then we can find A^{-1} as :-

If $a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0$ is characteristic equation. then according to C-H theorem

$$a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI = O \quad \text{--- (1)}$$

To find A^{-1}, we multiply (1) with A^{-1}

$$a_0A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \cdots + a_{n-1}I + a_nA^{-1} = O$$

ie $$A^{-1} = -\frac{1}{a_n}\left(a_0A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I\right)$$

2. If we know the lowest degree power of A, then we can compute the highest powers of A (positive integral powers)

Multiplying ① with A, we have

$$a_0 A^{n+1} + a_1 A^n + \cdots + a_{n-1} A^2 + a_n A = 0$$

$$A^{n+1} = -\frac{1}{a_0}\left[a_1 A^n + a_2 A^{n-1} + \cdots + a_n A\right]$$

Q. Verify Cayley Hamiton theorem for

$A = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}$ and hence find A^{-1}.

Solⁿ characteristic eqⁿ is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-2-\lambda) + 5 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0 \quad \text{—— ①}$$

According to C-H theorem, matrix A will satisfy ① so, we have

$$\boxed{A^2 - 2A - 3I = 0} \quad \text{—— ②}$$

$A^2 = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 11 & -10 \\ 2 & -1 \end{bmatrix}$

~~substituting in ②~~

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Now $A^2 - 2A - 3I =$

$$\begin{bmatrix} 11 & -10 \\ 2 & -1 \end{bmatrix} - 2\begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11-8-3 & -10+10-0 \\ 2-2-0 & -1+4-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i.e. $\boxed{A^2 - 2A - 3I = 0}$

i.e. A satisfy its characteristic equation.

Now to find A^{-1}

multiply ② with A^{-1}

$A - 2I - 3A^{-1} = 0$

$A^{-1} = \frac{1}{3}(A - 2I)$

$$= \frac{1}{3}\left(\begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 4-2 & -5 \\ 1-0 & -2-2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -5 \\ 1 & -4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{-5}{3} \\ \frac{1}{3} & \frac{-4}{3} \end{bmatrix}$$

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Q Verify cayley Hamilton theorem for

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \text{ and find } A^{-1}.$$

Sol^n. characteristic eq^n is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

Cayley-Hamilton theorem states that every square matrix satisfies its own charakteristic equation.

i.e. we need to show

$$A^3 - 6A^2 + 9A - 4I = 0 \quad \text{—— (1)}$$

Now $A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$

Now $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9+0 & 21-30+9 \\ -21+30-36+30 & 22-36+18-4 & -21+30-9 \\ 21-30+9+0 & -21+30-9+0 & 22-36+18-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= O$ (null matrix)

i.e Cayley Hamilton theorem verified.

Now for A^{-1},
multiply ① with A^{-1}

$$A^{2} - 6A + 9I - 4A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}\left(0 - A^{2} + 6A - 9I\right)$$

$$A^{-1} = \frac{1}{-4}\left(-A^{2} + 6A - 9I\right)$$

$$= \frac{1}{-4}\begin{bmatrix} -6+12-9 & 5-6+0 & -5+6+0 \\ 5-6+0 & -6+12-9 & 5-6+0 \\ -5+6+0 & 5-6+0 & -6+12-9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ Ans

Q:- Verify Cayley-Hamilton theorem for

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$ and then find A^{-1}.

Hint:- char. eq^n $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

verify cayley Hamilton theorem by showir that

$$A^3 - 5A^2 + 7A - 3I = 0$$

Now find $A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

Q find the inverse of matrix A, using cayley Hamilton theorem

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$$

Hint $A^{-1} = A^2 - 11A + 21I$

$$A^{-1} = \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix}$$

Q:- Verify cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$ and hence compute

A^{-1}. Also find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Sol^n charac. poly is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \quad \text{—— (1)}$$

According to cayley-Hamilton theorem

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$A^2 = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

taking

$A^3 - 5A^2 + 7A - 3I$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - 5\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

i.e $A^3 - 5A^2 + 7A - 3I = 0$ —— ②

To find A^{-1}, multiply ① by A^{-1}

$$A^2 - 5A + 7I - 3A^{-1} = 0$$

$$A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$$

$$\boxed{A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}}$$

To find matrix represented by

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A^4 - 5A^3 + 7A^2 - 3A + A^2 + A + I$$

$$\cancel{+ A^2 + 2A + I}$$

$$= A^6(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + (A^2 + A + I)$$

$$\qquad \| \qquad\qquad\qquad\qquad \|$$

$$\qquad 0 \qquad\qquad\qquad\qquad 0$$

$$= A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 9 \end{bmatrix}$$ Ans

C-H theorem can be used to find the higher power of A, using lower powers of A

Q Using Cayley Hamilton theorem find A^4 If

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

according to Cayley-Hamilton theorem we have

$$A^3 - 6A^2 + 8A - 3I = 0$$

i.e. $A^3 = 6A^2 - 8A + 3I$

On multiplying by A

$$A^4 = 6A^3 - 8A^2 + 3A$$

$$A^4 = 6(6A^2 - 8A + 3I) - 8A^2 + 3A$$

$$A^4 = 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$= 28A^2 - 45A + 18I$$

$A^2 = A \cdot A$

$$= \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

Q If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^n = A^{n-2} + A^2 - I$.

Hence find A^{50}.

Sol^n: char. eq^n

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 1) = 0$$

$$\cancel{\lambda^3 - \lambda^2 + \lambda + \lambda = 0}$$

$$\lambda^2 - 1 - \lambda^3 + \lambda = 0$$

$$\boxed{\lambda^3 - \lambda^2 - \lambda + 1 = 0}$$

According to C-H theorem, matrix A satisfies its characterstic eq^n

i.e $A^3 - A^2 - A + I = 0$ —— (1)

from (1) we have

$$\cancel{A^3} - A^2 = \cancel{A} - I$$

$$\cancel{A^4} - \cancel{A^3} = \cancel{A^2} - \cancel{A}$$

$$\cancel{A^5} - \cancel{A^4} = \cancel{A^3} - \cancel{A^2}$$

$$\vdots$$

$$\cancel{A^{n-1}} - \cancel{A^{n-2}} = \cancel{A^{n-3}} - \cancel{A^{n-4}}$$

$$A^n - \cancel{A^{n-1}} = A^{n-2} - \cancel{A^{n-3}}$$

Adding above, we get

$$A^n - A^2 = A^{n-2} - I$$

i.e $\boxed{A^n = A^{n-2} + A^2 - I}$

To find A^{50},
consider

$$A^n = A^{n-2} + A^2 - I$$

$$A^n = (A^{n-4} + A^2 - I) + A^2 - I \quad \left(\text{using } A^{n-2} = A^{n-4} + A^2 - I \right)$$

i.e $\boxed{A^n = A^{n-4} + 2(A^2 - I)}$

again, putting $A^{n-4} = A^{n-6} + A^2 - I$
we have

$$\boxed{A^n = A^{n-6} + 3(A^2 - I)}$$

$\vdots$

$\cancel{A^n = A^{n(n}}$

Same way we have

$$A^n = A^{n-(n-2)} + \frac{n-2}{2}(A^2 - I)$$

$$A^n = A^2 + \frac{n-2}{2}(A^2 - I)$$

$$A^n = \left(1 + \frac{n-2}{2}\right) A^2 - \left(\frac{n-2}{2}\right) I$$

$$\boxed{A^n = \frac{n}{2} A^2 - \left(\frac{n-2}{2}\right) I}$$

for A^{50} put $n = 50$

$$A^{50} = 25 A^2 - 24 I$$

$$A^{50} = 25 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

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Q:- If a matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, find A^{32}

using cayley-Hamilton theorem.

Q Using cayley-Hamilton theorem find A^4 for the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Soln char. eqn $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-2-\lambda)(3-\lambda) = 0$$

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0 \quad \text{--- (1)}$$

according to C-H theo. A will satisfy its char. eqn

so we have $A^3 - 2A^2 - 5A + 6I = 0$

$$A^3 = 2A^2 + 5A - 6I$$

$$A^4 = 2A^3 + 5A^2 - 6A$$

$$A^4 = 2(2A^2 + 5A - 6I) + 5A^2 - 6A$$

$$= 4A^2 + 10A - 12I + 5A^2 - 6A$$

$$\boxed{A^4 = 9A^2 + 4A - 12I}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A^4 = 9A^2 + 4A - 12I = 9\begin{bmatrix} 1 & -1 & 8 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} + 4\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 12\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & -5 & 80 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \text{ Ans}$$

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