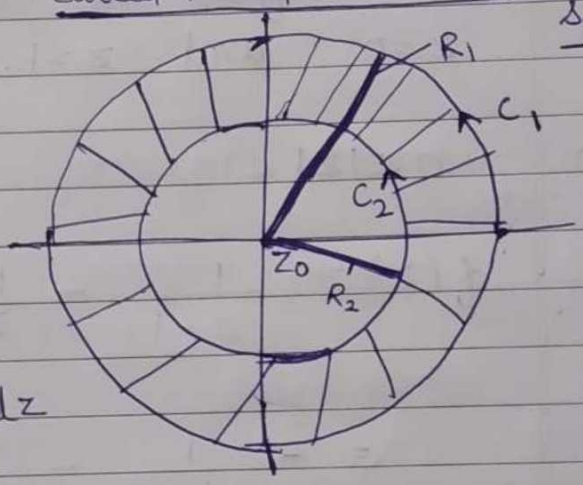
Laurent's series:-

If $f(z)$ is analytic in the closed ring bounded by two concentric circles C_1 and C_2, centered at z_0 and of radius R_1 and R_2. If z is any point in annulus region $(R_1 > R_2)$ then $f(z)$ can be expanded as

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \infty + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots \infty$$

or $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

called Principal part of Laurent series



where

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{(z-z_0)^{-n+1}} dz$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}}$$

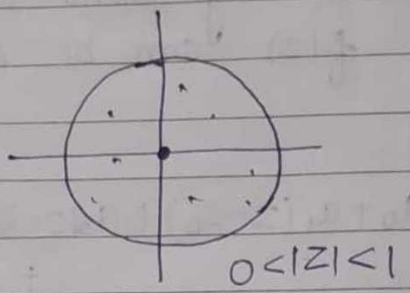
Laurent series

Q1 Find Laurent's series expansion for $f(z) = \frac{1}{z(z-1)}$ for the following regions

(i) $0 < |z| < 1$ (ii) $0 < |z-1| < 1$

Soln:- $f(z) = \frac{1}{z(z-1)}$

$= \frac{z-(z-1)}{z(z-1)}$



$\left[f(z) = \frac{1}{z-1} - \frac{1}{z} \right]$

$f(z)$ is not analytic at $z=0$ and $z=1$.

(i) $0 < |z| < 1$

$f(z) = \frac{1}{z-1} - \frac{1}{z}$

$= -\frac{1}{1-z} - \frac{1}{z}$

$= -(1-z)^{-1} - \frac{1}{z}$

$= -(1+z+z^2+z^3 \cdots\cdots \infty) - \frac{1}{z}$

$= (-1-z-z^2-z^3+\cdots\cdots \infty) - \frac{1}{z}$,

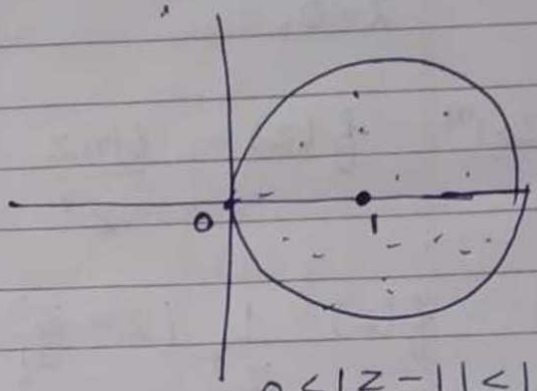
$0 < |z| < 1$

(ii) $0 < |z-1| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z}$$

$z - 1 = t$

$z = t + 1$



$0 < |z-1| < 1$

$$f(z) = \frac{1}{t+1-1} - \frac{1}{t+1}$$

$$f(z) = \frac{1}{t} - \frac{1}{t+1}$$

$$= \frac{1}{t} - (1+t)^{-1}$$

$(1+t)^{-1}$ exp is valid $|t| < 1$, $|z-1| < 1$

$$= \frac{1}{t} - (1 - t + t^2 - t^3 + t^4 - \cdots \infty)$$

$$f(z) = \frac{1}{t} - 1 + t - t^2 + t^3 - t^4 + \cdots \infty$$

$$f(z) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + (z-1)^3 - (z-1)^4 + \cdots \infty$$

$$f(z) = \left(-1 + (z-1) - (z-1)^2 + (z-1)^3 - (z-1)^4 + \cdots \infty\right) + \frac{1}{z-1}$$

, $0 < |z-1| < 1$

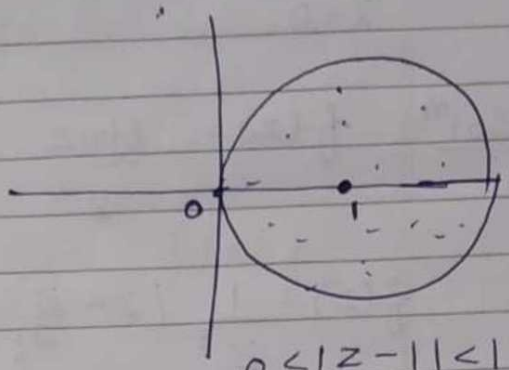
required Laurent's Series

(ii) $0 < |z-1| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z}$$

$z - 1 = t$

$z = t + 1$



$0 < |z-1| < 1$

$$f(z) = \frac{1}{t+1-1} - \frac{1}{t+1}$$

$$f(z) = \frac{1}{t} - \frac{1}{t+1}$$

$$= \frac{1}{t} - (1+t)^{-1}$$

$(1+t)^{-1}$ exp. is valid
$|t| < 1$
$|z-1| < 1$

$$= \frac{1}{t} - (1 - t + t^2 - t^3 + t^4 - \cdots \infty)$$

$$f(z) = \frac{1}{t} - 1 + t - t^2 + t^3 - t^4 + \cdots \infty$$

$$f(z) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + (z-1)^3 - (z-1)^4 + \cdots \infty$$

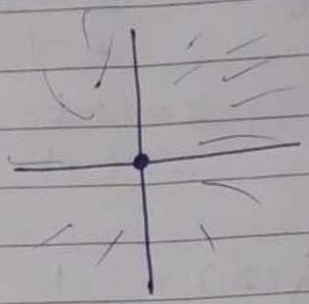
$$f(z) = \left(-1 + (z-1) - (z-1)^2 + (z-1)^3 - (z-1)^4 + \cdots \infty\right) + \frac{1}{z-1}$$

, $0 < |z-1| < 1$

required Laurent's Series

Ex: Find Laurent's series of $\frac{\sin z}{z^7}$ about $z=0$.

Soln $f(z) = \frac{\sin z}{z^7}$.



$$f(z) = \frac{1}{z^7}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} - \cdots \infty\right)$$

region $0<|z|<\infty$

$$f(z) = \left(\frac{1}{z^6} - \frac{1}{z^4}\frac{1}{3!} + \frac{1}{z^2}\frac{1}{5!} - \frac{1}{7!} + \frac{z^2}{9!} - \cdots \infty\right)$$

$$f(z) = \left(\frac{-1}{7!} + \frac{z^2}{9!} - \cdots \infty\right) + \left(\frac{1}{z^2 5!} - \frac{1}{z^4 3!} + \frac{1}{z^6}\right)$$

$0<|z|<\infty$

required Laurent's series.

Ex2 Expand $f(z) = z^3 e^{\frac{1}{2z}}$

Ans:-

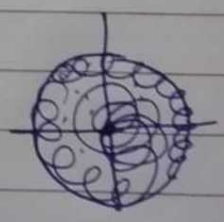
$$\left(\frac{1}{48} + \frac{z}{8} + \frac{z^2}{2} + z^3\right) + \left(\frac{1}{384z} + \cdots \infty\right)$$

region $0<|z|<\infty$

Ex3 Expand $f(z) = \frac{1-e^z}{z}$ in Laurent's series

Ans $f(z)$ is not analytic at $z=0$

$z=0$ is singular point



region of conv. $0<|z|<\infty$

$$f(z) = -1 - \frac{z}{2} - \frac{z^2}{6} - \frac{z^3}{24} - \cdots \infty$$

Q:- Obtain Taylor's and Laurent's series expansion for the function $f(z) = \frac{1}{(z+1)(z+3)}$ in the following regions.

(1) $|z| < 1$

(2) $1 < |z| < 3$

(3) $|z| > 3$

(4) $0 < |z+1| < 2$

Soln:- $$f(z) = \frac{1}{(z+1)(z+3)}$$

$f(z)$ has singularity at $z = -1, -3$

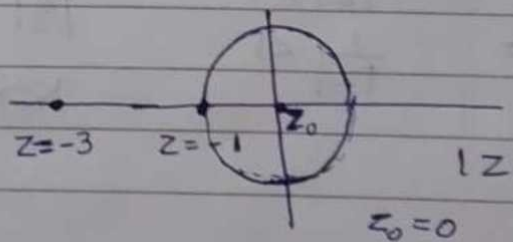
Partial fraction:-

$$f(z) = \frac{1}{(z+1)(z+3)}$$

$$= \frac{1}{2} \frac{(z+3)-(z+1)}{(z+3)(z+1)} = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

(1) $|z| < 1$



$f(z)$ is analytic inside the circle with centre at 0 and radius 1.
i.e $f(z)$ will be expanded through taylor's series in $|z|<1$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2}(1+z)^{-1} - \frac{1}{2\times 3}\left(1+\frac{z}{3}\right)$$

$$= \frac{1}{2}(1-z+z^2-z^3+\cdots\infty) - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$\frac{\cdot 1\cdot}{z+3}$$

$$\frac{1}{z\left(1+\frac{3}{z}\right)} \quad \frac{1}{3\left(1+\frac{z}{3}\right)}$$

$|z|>1$ ✗ $\quad |z|<1$ ✓

as $|z|<1$

$$= \frac{1}{2}\left(1-z+z^2-z^3+\cdots\infty\right) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\infty\right)$$

$$f(z) = \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots\cdots\infty$$

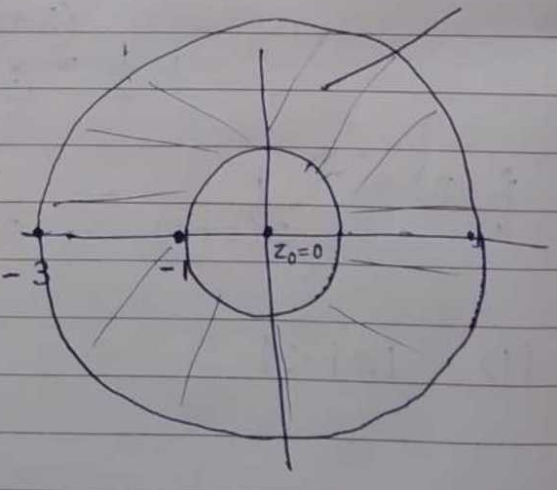
this is taylor's series expansion of $f(z)$ in terms of z in region $|z|<1$.

(ii) $1 \le |z| < 3$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$|z|>1$ $\qquad$ $|\frac{z}{3}|<1$

$$= \frac{1}{|z|}<1$$



$$= \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2\times 3\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}---\infty\right) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}--\infty\right)$$

$$= \left(-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+---\infty\right) + \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}--\infty\right)$$

or

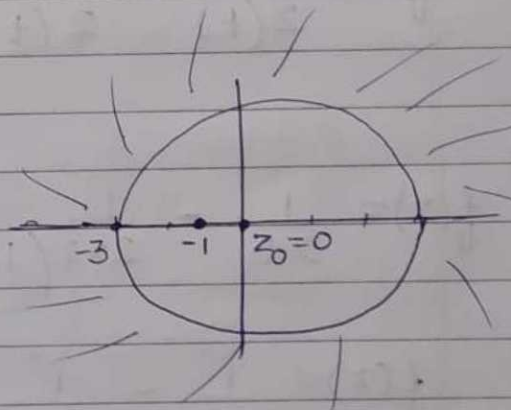
$$f(z) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^n}{3^{n+1}} + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{z^n}$$

↓ Principal Part

Q:- $|z|>3$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$\frac{1}{|z|}<1$ $\quad$ $\frac{3}{|z|}<1$



$3<|z|<\infty$

annulus region

$$= \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2z\left(1+\frac{3}{z}\right)}$$

$$f(z) = \frac{1}{2}\left[\frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots \infty\right) - \frac{1}{z}\left(1 - \frac{3}{z} + \frac{3^2}{z^2} - \cdots \infty\right)\right]$$

$$f(z) = \frac{1}{2}\left[\frac{2}{z^2} - \frac{8}{z^3} + \frac{26}{z^4} - \cdots \infty\right]$$

$$\left[f(z) = \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \cdots \infty\right]$$

Laurent's series having only principal Part.

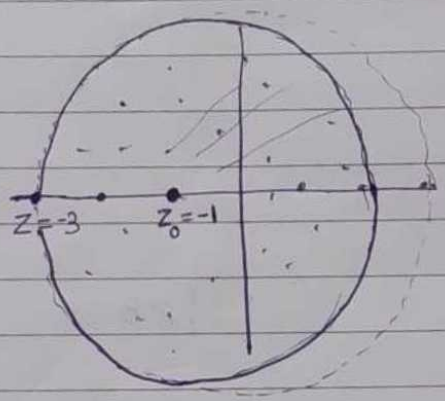
(4) $0 < |z+1| < 2$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

Let $z+1 = t$

$z = t-1$

$$f(z) = \frac{1}{2(t)} - \frac{1}{2(t+2)}$$



$$f(z) = \frac{1}{2t} - \frac{1}{2\times 2\left(1+\frac{t}{2}\right)} \qquad 0 < |z+1| < 3$$

$$f(z) = \frac{1}{2t} - \frac{1}{4\left(1+\frac{t}{2}\right)} \qquad \text{as } \left|\frac{t}{2}\right| < 1$$

$$f(z) = \frac{1}{2t} - \frac{1}{4}\left(1+\frac{t}{2}\right)^{-1}$$

$$= \frac{1}{2t} - \frac{1}{4}\left(1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} - \cdots \infty\right)$$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{4}\left(1 - \frac{(z+1)}{2} + \frac{(z+1)^2}{4} - \frac{(z+1)^3}{8} + \cdots \infty\right)$$

$$f(z) = \left(-\frac{1}{4} + \frac{(z+1)}{2} - \frac{(z+1)^2}{2} + \cdots \infty\right) + \underbrace{\frac{1}{2(z+1)}}$$

Principal part

Q:- Find series expansion of $f(z) = \frac{z^2-1}{z^2+5z+6}$ about $z=0$, in the region

(1) $|z|<2$ (2) $2<|z|<3$

Soln:- $f(z) = \frac{z^2-1}{z^2+5z+6}$

$$f(z) = 1 - \frac{5z+7}{z^2+5z+6} = 1 - g(z)$$

consider

$$\frac{5z+7}{z^2+5z+6} = \frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

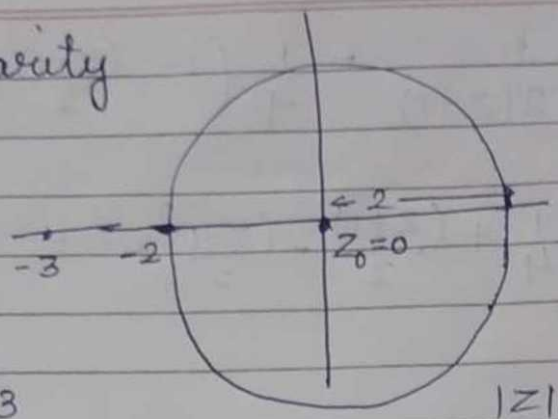
$$B = 8, \quad A = -3$$

$$g(z) = \frac{5z+7}{z^2+5z+6} = \frac{-3}{z+2} + \frac{8}{z+3}$$

(1) $|z|<2$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$f(z)$ has singularity at $z=-2, -3$



$$f(z) = 1 + \frac{3}{z+2} + \frac{8}{z+3}$$

$|z|<2$ region of conv.

$$= 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} + \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} + \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

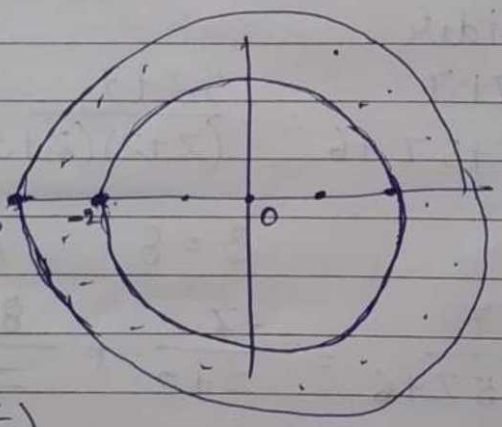
$$f(z) = 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n + \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

Ans

(2) $2<|z|<3$

~~$f(z)$~~

$$f(z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)}$$



$$f(z) = 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$2<|z|<3$

$\downarrow$

$\left|\frac{2}{z}\right|<1, \quad \left|\frac{z}{3}\right|<1$

$$f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$f(z) = 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots \infty\right)$$

$$- \frac{8}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{4}\right)^3 + \cdots \infty\right)$$

$$f(z) = 1 - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n + 3\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{n-1}}{z}.$$

Q:- Find Laurent series expansion of $f(z) = \dfrac{7z-2}{(z+1)(z)(z-2)}$ in the region $1 < |z+1| < 3$.

Q:- Expand $f(z) = \dfrac{1}{z^2(z-i)}$ about $z = i$.

Q:- If $0 < |z-1| < 2$, express $f(z) = \dfrac{z}{(z-1)(z-3)}$ in series of positive and negative powers.

Q:- Find expansion of $f(z) = \frac{z-1}{z^2}$ in taylor's series in powers of $(z-1)$ and Laurent's series for the domain $|z-1|>1$.

Q:- expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in Laurent series valid for

(i) $0<|z-1|<2$ (ii) $0<|z-3|<2$

Q Find Laurent series expansion of $f(z)$

$f(z) = \frac{7z-2}{z(z+1)(z-2)}$ in region

$1<|z+1|<3$.

Q. expand $f(z) = \frac{1}{(z+1)(z-2)}$ in the region

① $|z|<1$ ② $1<|z|<2$ ③ $|z|>2$

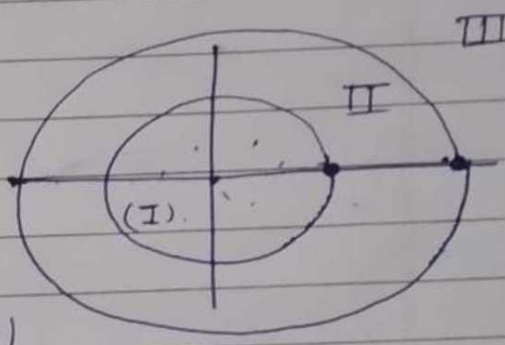
④ $0<|z-1|<2$ ⑤ ~~XXXXXX~~

Find all Taylor Series ~~or~~ and Laurent's series expansion of $f(z) = \frac{-2z+3}{z^2-3z+2}$ with center 0.

Soln Hint: $f(z)$ is not analytic at $z=1, 2$

$$f(z) = -\frac{1}{z-1} - \frac{1}{z-2}$$

possible regions are



(I) $|z| < 1$ (taylor)

(II) $1 < |z| < 2$ (Laurent)

(III) $|z| > 2$ (Laurent)