Recurrence-1

Monday, 6 March 2023 8:02 am

1) $$T(n) = \begin{cases} 1, & n = 0 \\ T(n-1) + 1, & \text{if } n > 0 \end{cases}$$

Recurrence Tree Method

$T(n) \longrightarrow 1$

$1 \quad T(n-1) \longrightarrow 1$

$1 \quad T(n-2) \text{ — } 1$

$1 \quad T(n-3) \text{ — } 1$

$\vdots \qquad \vdots$

$T(2) \text{ — } 1$

$1 \quad T(1) \text{ — } 1$

$1 \quad T(0) \text{ — } 0$

$T(n) = 1 + 1 + 1 + \dots + (n+1) \text{ times}$

$= n$

$= \underline{\underline{O(n)}}$

Substitution Method / Induction

$$T(n) = \begin{cases} 1 & n = 0 \\ T(n-1) + 1, & n > 0 \end{cases}$$

$T(n-1) = T(n-2) + 1$

$T(n-2) = T(n-3) + 1$

$T(n) = T(n-1) + 1$

$= [T(n-2) + 1] + 1 = T(n-2) + 2$

$= [T(n-3) + 1] + 2 = T(n-3) + 3$

$\dots\dots\dots$ k times

$= T(n-k) + k$ ——— 1)

$T(0)$ — Base case

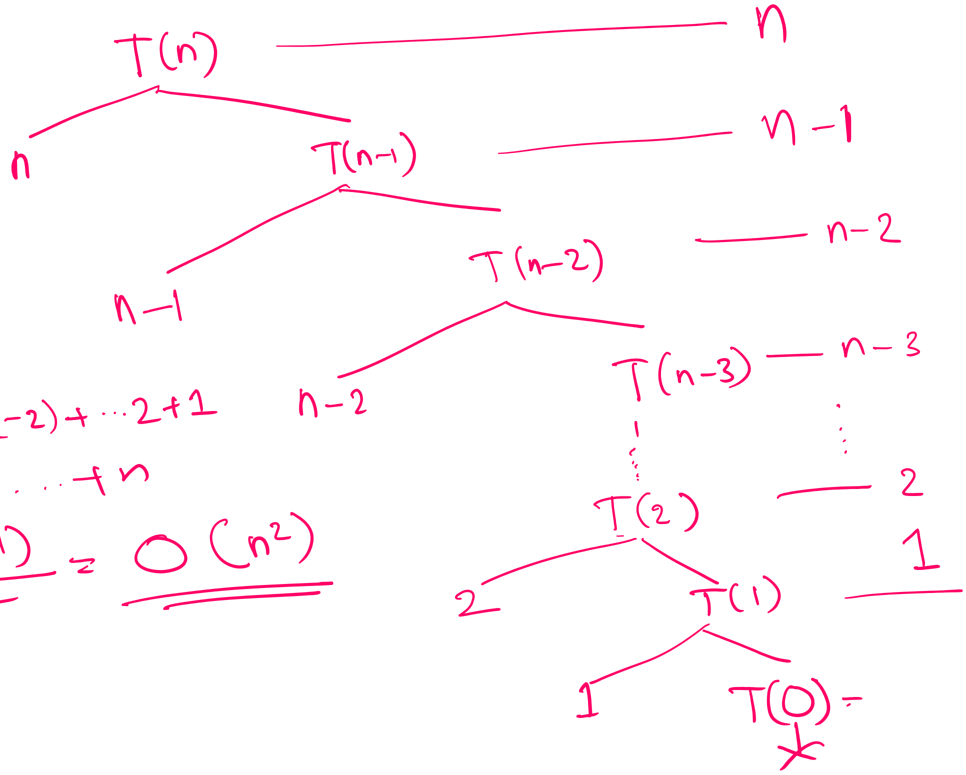
Assume $n - k = 0$

$n = k$

① $\Rightarrow T(n) = T(n-n) + n$

$= \cancel{T(0)}^{1} + n = 1 + n = \underline{\underline{O(n)}}$

②



$T(n) = n + (n-1) + (n-2) + \cdots 2 + 1$

$= 1 + 2 + \cdots\cdots + n$

$= \frac{n(n+1)}{2} = \underline{\underline{O(n^2)}}$

$T(n) = T(n-1) + n$

$= \left[T(n-2) + (n-1)\right] + n = T(n-2) + (n-1) + n$

$= \left[T(n-3) + n-2\right] + (n-1) + n = T(n-3) + (n-2) + (n-1) + n$

$= \cdots\cdots$ k time

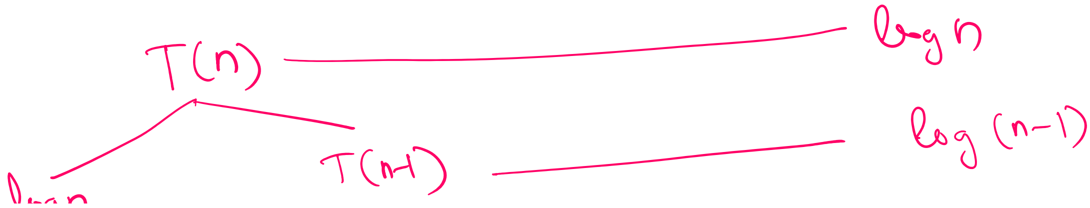
$= T(n-k) + (n-(k-1)) + (n-(k-2)) + \cdots + (n-2) + (n-1) + n$

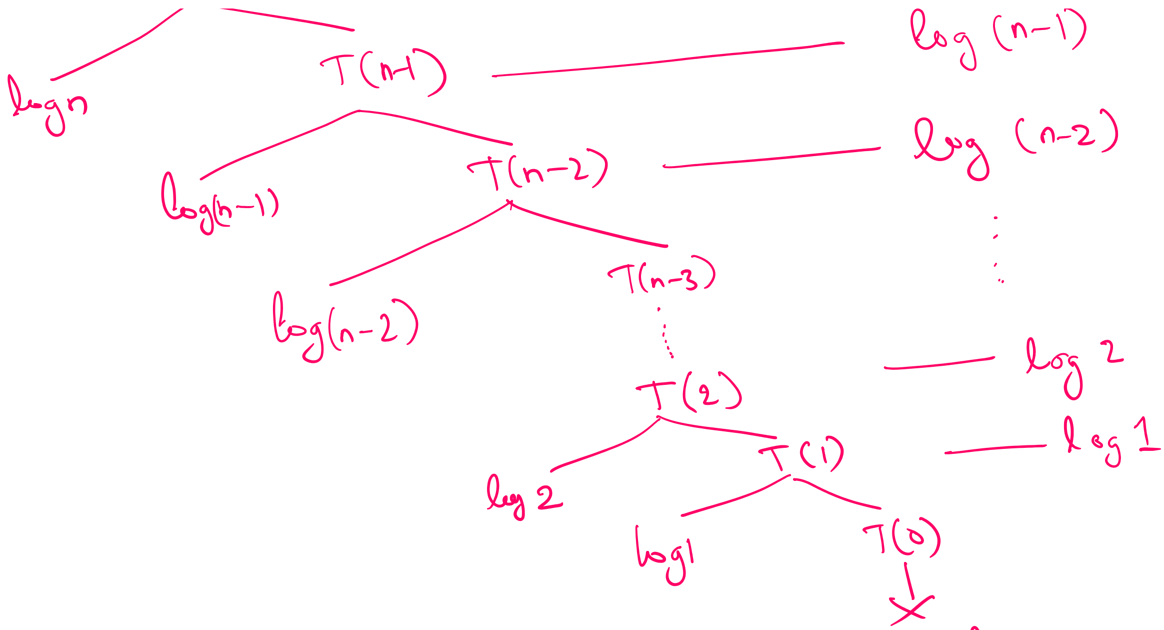
Assume $n - k = 0 \Rightarrow k = n$

$T(n) = T(0) + 1 + 2 + \cdots\cdots + (n-2) + (n-1) + n$

$= 1 + \frac{n(n+1)}{2} = \underline{\underline{O(n^2)}}$

Example 3





$$T(n) = \log 1 + \log 2 + \cdots + \log(n-2) + \log(n-1) + \log n$$

$$= \log(1 \cdot 2 \cdot 3 \cdots n) = \log n!$$

$$= O(n \log n)$$

$$T(n) = T(n-1) + \log n$$

$$= [T(n-2) + \log(n-1)] + \log n$$

$$= T(n-3) + \log(n-2) + \log(n-1) + \log n$$

$$= T(n-k) + \log(n-(k-1)) + \log(n-(k-2)) + \cdots + \log(n-1) + \log n$$

Assume $n = k$

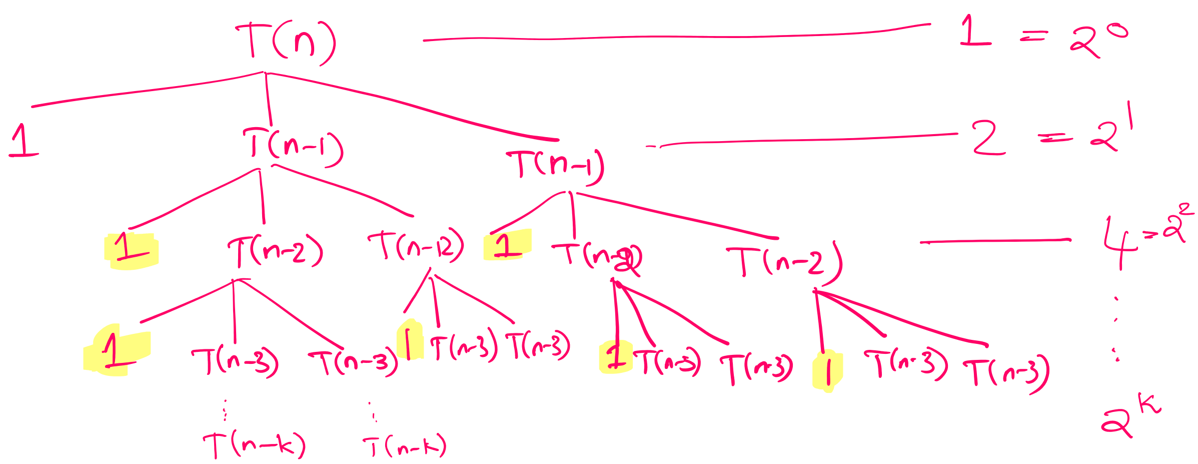
$$\Rightarrow T(n) = T(0) + \log 1 + \log 2 + \cdots + \log(n-1) + \log n$$

$$= 1 + \log(1 \cdot 2 \cdot 3 \cdots n) = 1 + \log n!$$

$$= O(n \log n)$$

Example-4

$$T(n) = \begin{cases} 1, & n = 0 \\ 2T(n-1)+1, & n > 0 \end{cases}$$



$$T(n) = 2^0 + 2^1 + 2^2 + \dots + 2^k$$

$n - k = 0$

$n = k$

$a = 1 \quad r = 2$

$$\frac{a(r^k - 1)}{r - 1} = \frac{1(2^k - 1)}{2 - 1} = 2^k - 1$$

$$T(n) = 2^n - 1 = \underline{\underline{O(2^n)}}$$

Substitution method

$$T(n) = 2T(n-1) + 1$$

$$= 2[2T(n-2) + 1] + 1 = 2^2 T(n-2) + 2 + 1$$

$$= 2^2[2T(n-3) + 1] + 2 + 1 = 2^3 T(n-3) + 2^2 + 2 + 1$$

.... k times

$$= 2^k [T(n-k)] + 2^{k-1} + 2^{k-2} + \dots + 2^2 + 2 + 1$$

$n - k = 0 \Rightarrow n = k$

$$T(n) = 2^n \cdot 1 + 2^{n-1} + 2^{n-2} + \dots + 2^2 + 2 + 1$$

= Sum of gp with $a = 1$ and $r = 2$

$T(n) = O(2^n)$ (as in previous case)

Dividing — Example 1

$$T(n) = \begin{cases} 1 & n = 1 \\ T(n/2) + 1 & \text{if } n > 1 \end{cases}$$

$T(n)$ ——— 1 $\qquad \frac{n/2}{2} = \frac{n}{2^2}$

1 $\qquad T(n/2)$ ——— 1 $\qquad \frac{n/2^2}{2}$

1 $\qquad T(n/2^2)$ — 1 $\qquad n/2^3$

1 $\qquad T(n/2^3)$ — 1

⋮ k times

$T(n/2^k)$ — 1

$T(n) = 1 + 1 + \dots +$ (k times)

Assume $\frac{n}{2^k} = 1 \Rightarrow n = 2^k \quad k = \log_2 n$

$T(n) = 1 + 1 + \dots +$ ($\log n$) times

$= O(\log n)$

Substitution

$T(n) = T(n/2) + 1$

$= [T(n/2^2) + 1] + 1 = T(n/2^2) + 2$

$= [T(n/2^3) + 1] + 2 = T(n/2^3) + 3$

.... K times

$T(n) = T(n/2^k) + k$

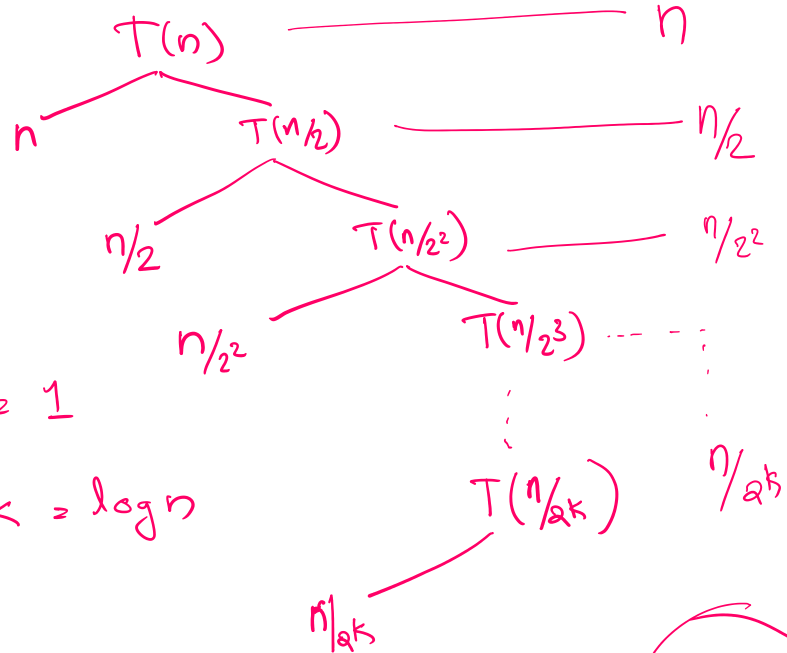
$$T(n) = T(n/2^k) + k$$

Assume $= \frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow k = \log n$

$$T(n) = \cancel{T(1)}^{1} + \log n = O(\log n)$$

Dividing Example - II

$$T(n) = \begin{cases} 1, & n = 1 \\ T(n/2) + n, & n > 1 \end{cases}$$



Assume $\frac{n}{2^k} = 1$

$\Rightarrow n = 2^k ; k = \log n$

$$T(n) = n + \frac{n}{2} + \frac{n}{2^2} + \cdots + \frac{n}{2^k}$$

$$= n\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^k}\right) = n \sum_{i=1}^{k\,(\infty)} \frac{1}{2^i}$$

$$= n(1) = O(n)$$

Substitution

$$T(n) = T(n/2) + n$$

$$= \left[T(n/2^2) + \frac{n}{2}\right] + n = T(n/2^2) + \frac{n}{2} + n$$

$$= \left[T(n/2^3) + \frac{n}{2^2}\right] + \frac{n}{2} + n$$

$\ldots$ k times

$$T(n) = T\left(\frac{n}{2^k}\right) + \frac{n}{2^{k-1}} + \frac{n}{2^{k-2}} + \cdots + \frac{n}{2} + \frac{n}{2^0}$$

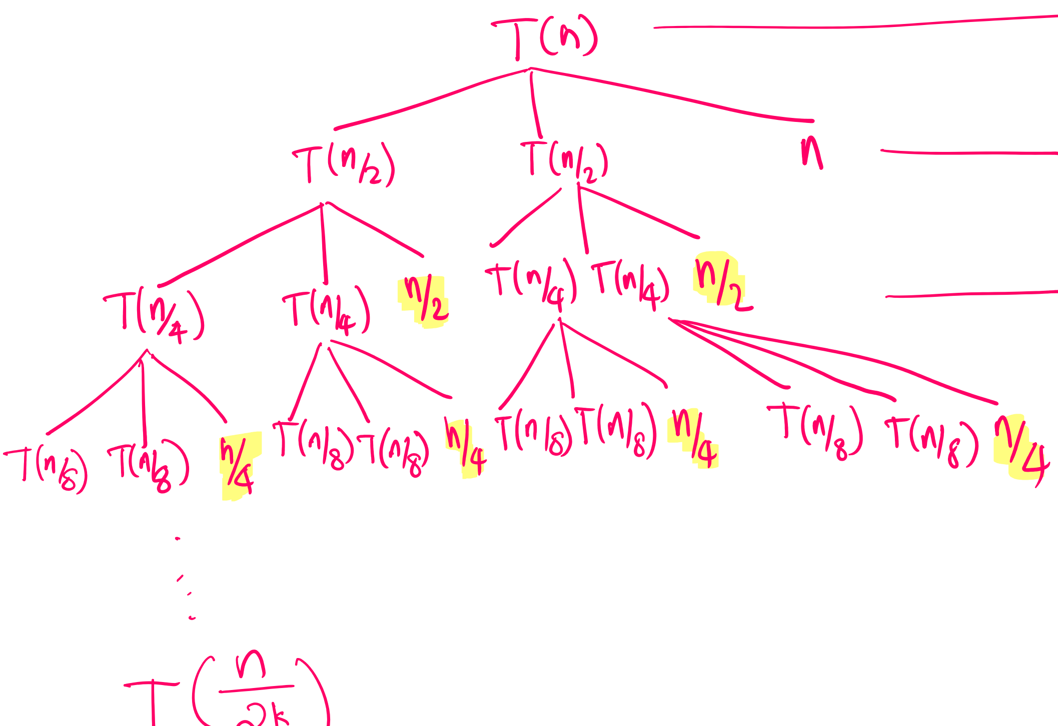
Assume $\frac{n}{2^k} = 1$

$$= T(1) + n\left(\frac{1}{2^{k-1}} + \frac{1}{2^{k-2}} + \cdots + \frac{1}{2} + 1\right)$$

$$= 1 + n(2) = \underline{\underline{O(n)}}$$

Dividing - III

$$T(n) = \begin{cases} 1, & n = 1 \\ 2T(n/2) + n, & \text{if } n > 1 \end{cases}$$



n

$\frac{n}{2} + \frac{n}{2} = n$

$\frac{n}{4} + \frac{n}{4} + \frac{n}{4} + \frac{n}{4} = n$

$$T\left(\frac{n}{2^k}\right)$$

$$k\,\frac{n}{k} = n$$

Assume $\frac{n}{2^k} = 1$ $\quad n = 2^k \quad k = \log n$

$T(n) = n + n + \ldots +$ k times

$= n + n + \ldots \log n$ times

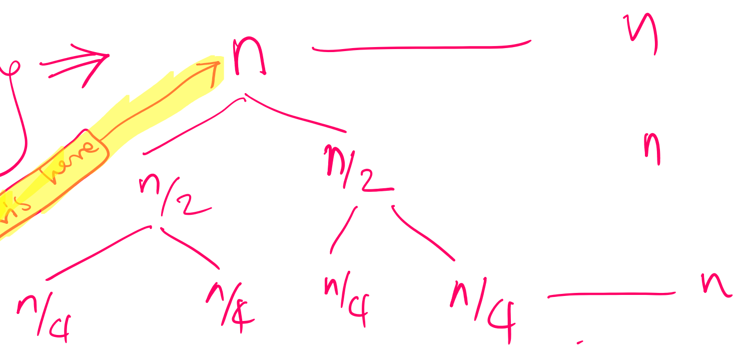
$O(n \log n)$

No need of writing T(n)

Write this way

$$T(n) = \begin{cases} 1, & n = 1 \\ 2T(n/2) + n \end{cases}$$

Use this here



$$T(n) = 2T(n/2) + n$$

$$= 2\left(2T(n/4) + n/2\right) + n = 2^2T(n/4) + 2n$$

$$= 2^2\left[2T\left(n/2^3\right) + \frac{n}{2^2}\right] + 2n$$

$$= 2^3\,T\left(n/2^3\right) + 3n$$

- - - - - k times

$$T(n) = 2^k\,T\left(n/2^k\right) + kn$$

Assume $\frac{n}{2^k} = 1$

$n = 2^k \quad k = \log n$

$$= 2^{\log n}\,T(1) + \log n \cdot n$$

$$= n \cdot 1 + n \log n = \underline{O(n \log n)}$$