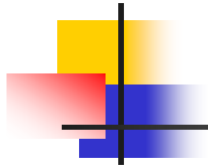




Applications of Linear Differential equations

Spring Mass System

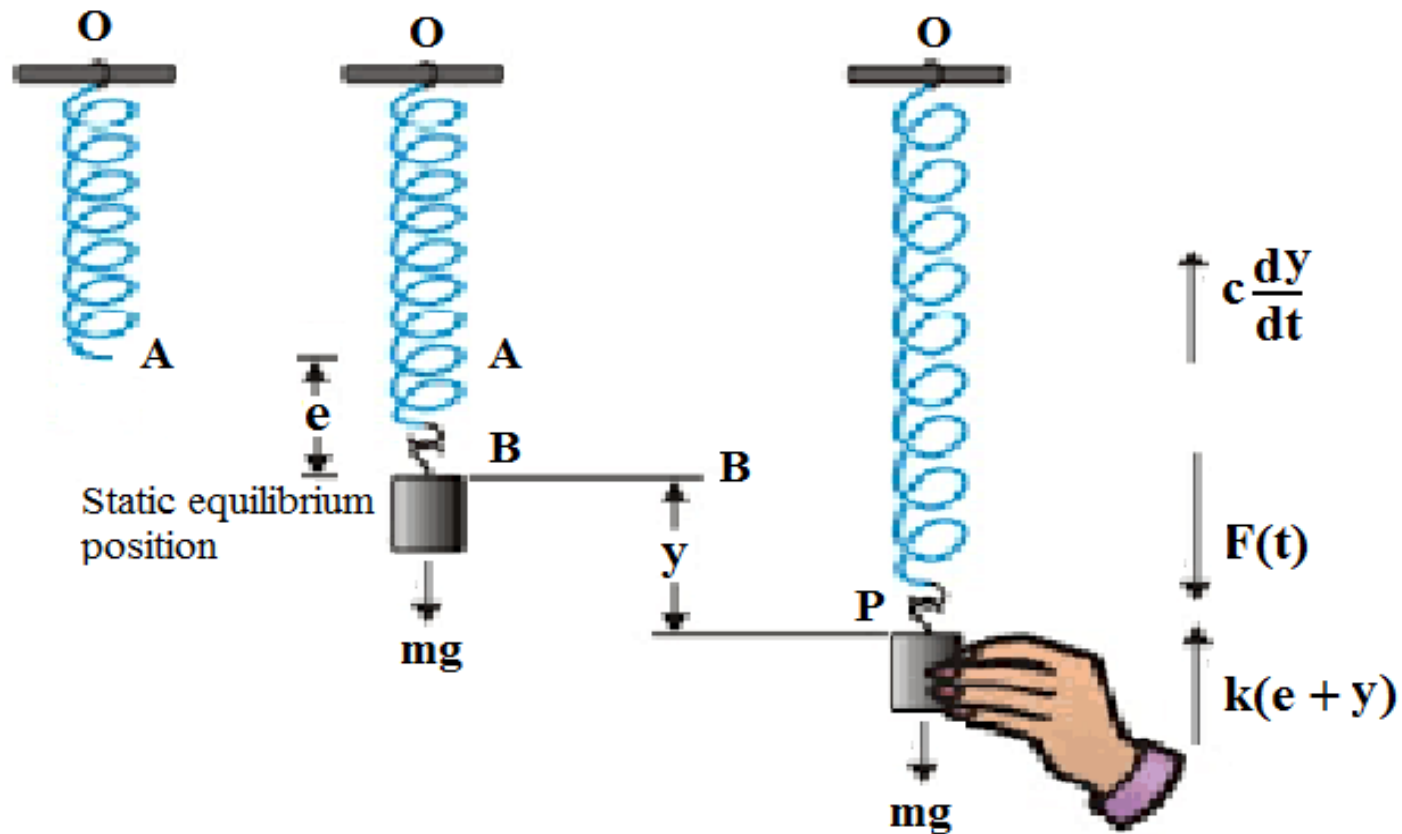


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Mass-spring system



Oscillations of a Spring



Forces acting on the mass

- Gravitational force $mg = ke$
- Restoring force $k(e + y)$
(due to the displacement of the spring)
- Damping (frictional) force $c \frac{dy}{dt}$
- External applied force $F(t)$

Net force acting on the mass

$$\begin{aligned} F_{\text{net}}(t) &= mg - k(e + y) - c \frac{dy}{dt} + F(t) \\ &= ke - ke - ky - c \frac{dy}{dt} + F(t) \\ &= -ky - c \frac{dy}{dt} + F(t) \end{aligned}$$



From Newton's Second Law of Motion

$$F_{net} = ma = my''(t)$$

For our spring-mass system

$$m \frac{d^2y}{dt^2} = -ky - c \frac{dy}{dt} + F(t)$$

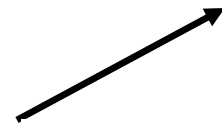
$$m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = F(t)$$



1. Undamped Free Vibrations

$$my''(t) + \cancel{cy'(t)} + ky(t) = \cancel{F(t)}$$


no damping



$$my''(t) + ky(t) = 0$$

$$y''(t) + \omega_0^2 y(t) = 0$$

no external force


$$\omega_0^2 = \frac{k}{m}$$

$\cos \omega_0 t, \quad \sin \omega_0 t$ (particular solutions)

$y(t) = A \cos \omega_0 t + B \sin \omega_0 t$ (general solution)

A and B are arbitrary constants determined from initial conditions $y(0) = y_0$ and $y'(0) = y'_0$



1. Undamped Free Vibrations

$$T = \frac{2\pi}{\omega_0} = 2\pi \left(\frac{m}{k}\right)^{1/2} \quad \text{Period of motion}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{Natural frequency of the vibration}$$

$$R = \sqrt{A^2 + B^2} \quad \text{Amplitude (constant in time)}$$



2. Damped Free Vibrations

$$my''(t) + cy'(t) + ky(t) = \cancel{F(t)}$$

no external force

$$my''(t) + cy'(t) + ky(t) = 0$$

Auxiliary equation is

$$mr^2 + cr + k = 0$$



2. Damped Free Vibrations

$$mr^2 + cr + k = 0$$

Solutions to auxiliary equation are:

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{c}{2m} \left(-1 \pm \sqrt{1 - \frac{4km}{c^2}} \right)$$

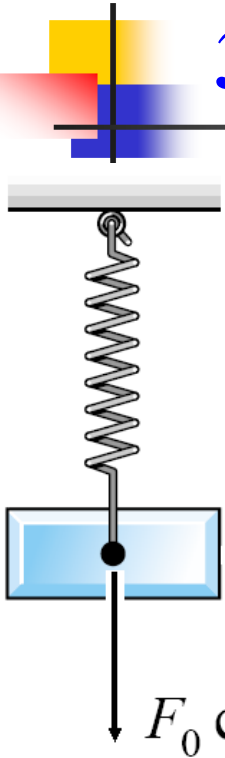
$$c^2 - 4km > 0, \quad y = Ae^{r_1 t} + Be^{r_2 t} \quad \text{over damped}$$

$$c^2 - 4km = 0, \quad y = (A + Bt)e^{-ct/2m} \quad \text{critically damped}$$

$$c^2 - 4km < 0, \quad y = e^{-ct/2m}(A \cos \mu t + B \sin \mu t)$$

Oscillatory or under damped

3. Forced Vibrations (Undamped)



$$my''(t) + cy'(t) + ky(t) = \underbrace{F(t)}_{F_0 \cos \omega t}$$

Periodic external force:

$$my''(t) + \cancel{cy'(t)} + ky(t) = F_0 \cos \omega t$$

no damping

$$my''(t) + ky(t) = F_0 \cos \omega t$$



4. Forced Vibrations (damped)

If the external force $F(t)$ is applied and damping force is present then equation of motion of the spring is

$$m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = F(t)$$



A body weighing 20 Kg is hung from a spring. A pull of 40 Kg weight will stretch the spring to 10 cm. The body is pulled down to 20 cm. below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time 't' seconds and the period of oscillation.

Solution:

The equation of motion of the spring is

$$my''(t) + ky(t) = 0$$

$$\text{Weight} = 20 \text{ Kg} \Rightarrow mg = 20 \Rightarrow m = 20/(9.8)$$

40 Kg weight stretches the spring to 10 cm i.e. 0.1 m

$$40 = k (0.1) \Rightarrow k = 400 \text{ kg/m}$$

$$\text{Hence } k/m = 196$$



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Equation of motion is

$$y'' + 196 y = 0$$

Roots of the auxiliary equation are: $\pm 14i$

Hence $y(t) = A \cos 14t + B \sin 14t$

When $t = 0$, $y = 20\text{cm} = 0.2\text{m}$

and $v = y' = 0$.

From these initial conditions, $A = 0.2$ and $B = 0$.

Therefore $y(t) = 0.2 \cos 14t$

Period of oscillation = $2\pi/\mu = 2\pi/14 = \pi/7$

(where $\mu^2 = k/m$)



**A 3 lb weight on a spring stretches it to 6 inches.
Suppose a damping force 'cv' is present ($c > 0$).**

Show that the motion is

- (i) critically damped if $c = 1.5$**
- (ii) over damped if $c > 1.5$**
- (iii) oscillatory if $c < 1.5$**

Solution:

The equation of motion of the spring is

$$my''(t) + cy'(t) + ky(t) = 0$$

$$\text{Weight} = 3 \text{ lb} \Rightarrow mg = 3 \text{ lb} \Rightarrow m = 3/32$$

3 lb weight stretches the spring to 6 inches i.e. $\frac{1}{2}$ ft

$$3 = k \left(\frac{1}{2}\right) \Rightarrow k = 6 \text{ lb/ft}$$



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Equation of motion is

$$(3/32) y'' + c y' + 6 y = 0$$

$$\text{Discriminant} = 1024 c^2 - 2304$$

(i) The motion is critically damped if the roots are real and equal


$$\text{Discriminant} = 1024 c^2 - 2304 = 0 \Rightarrow c = 1.5$$

(ii) The motion is over damped if the roots are real and distinct.

$$\text{Discriminant} = 1024 c^2 - 2304 > 0 \Rightarrow c > 1.5$$

(iii) The motion is oscillatory if the roots are imaginary

$$\text{Discriminant} = 1024 c^2 - 2304 < 0 \Rightarrow \text{if } c < 1.5$$



Determine whether resonance occurs in a system consisting of a weight 32 lb attached to a spring with stiffness constant $k = 4$ lb/ft and external force $16 \sin 2t$ and no damping force present. Initially $y = 1/2$ and $y' = -4$. Find the displacement of the body from its equilibrium position at any time 't'.

Solution:

Frequency of the external force = $2/2\pi = 1/\pi$

Natural frequency = $\mu/(2\pi)$, where $\mu^2 = k/m = 4$

Hence Natural frequency = $2/2\pi = 1/\pi$

Since Frequency of the external force is equal to Natural frequency, resonance occurs in the system

The equation of motion of the spring is

$$my''(t) + ky(t) = 16 \sin 2t$$

Weight = 32 lb $\Rightarrow mg = 32 \Rightarrow m = 32/32 = 1$



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$k = 4 \text{ lb/ft}$. Hence $k/m = 4$

Equation of motion is $y'' + 4y = 16 \sin 2t$

Roots of the auxiliary equation are: $\pm 2i$

Hence C. F. is $y_c(t) = A \cos 2t + B \sin 2t$


$$\begin{aligned} \text{P.I. is } y_p(t) &= \frac{16}{D^2+4} [\sin 2t] = \frac{16t}{2D} [\sin 2t] \\ &= -4t \cos 2t \end{aligned}$$

Complete solution is $y(t) = A \cos 2t + B \sin 2t - 4t \cos 2t$

when $t = 0$, $y = 1/2 \Rightarrow A = 1/2$

when $t = 0$, $y' = 0 \Rightarrow B = 0$

Hence $y(t) = 1/2 \cos 2t - 4t \cos 2t$



Determine the solutions of a mechanical system with weight 8 lb, stiffness constant 4 lb/ft, damping force is 2 times instantaneous velocity and external force is $8 \sin 4t$. The initial conditions are $y = 1$ and $y' = 0$.

Solution:

The equation of motion of the spring is

$$m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = F(t)$$

Weight = 8 lb $\Rightarrow mg = 8$ lb $\Rightarrow m = 8/32 = 1/4$, $k = 4$ lb/ft

Hence equation of motion is

$$\frac{1}{4} \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + 4y = 8 \sin 4t$$

Roots of the auxiliary equation are: - 4, - 4



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Roots of the auxiliary equation are: $-4, -4$

Hence C. F. is $y_c(t) = (A t + B) e^{-4t}$

P.I. is $y_p(t) = -\cos 4t$

Complete solution is $y(t) = (A t + B) e^{-4t} - \cos 4t$

when $t = 0, y = 1 \Rightarrow B = 2$

when $t = 0, y' = 0 \Rightarrow A = 8$

Hence $y(t) = (8 t + 2) e^{-4t} - \cos 4t$