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Chain Rule

[For two independent variables and three intermediate variables]

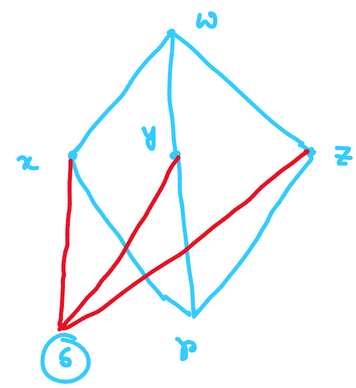
$$w = f(x, y, z) \quad \text{and}$$

$$x = g(r,s), \quad y = h(r,s), \quad z = k(r,s)$$

Hence, $w = f\big(g(r,s), h(r,s), k(r,s)\big)$. and the corresponding partial derivatives are:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}.$$

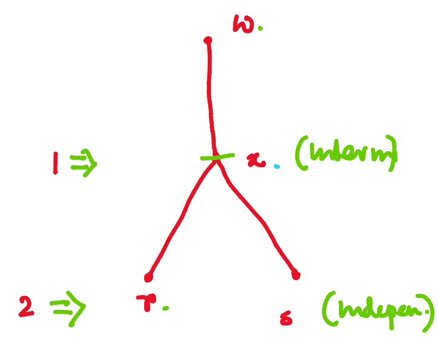
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}.$$



□ If $w = f(x)$ and $x = g(r,s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\cdot\frac{\partial x}{\partial s}$$



□ Prob: Find $\frac{dw}{dt}$ if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$

Find $\frac{dw}{dt}$ at $t = 0$.

Soln: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$ ...

Soln:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$\Rightarrow$ x y z

t

$$= y\cdot(-\sin t) + x\cdot\cos t + 1.1$$

$$= x\cos t - y\sin t + 1$$

$$= \cos^2 t - \sin^2 t + 1$$

Hence,

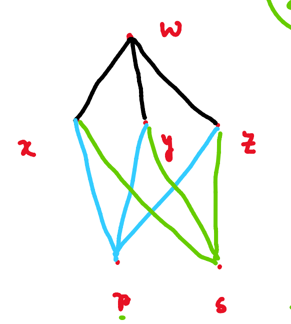
$$\left.\frac{dw}{dt}\right|_{t=0} = 1 - 0 + 1 = 2$$

Prob: Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ if

$$w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r = k(r,s)$$

$\left(\frac{dz}{dr}\right)$

soln:



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= 1.\frac{1}{s} + 2.2r + 2z.2$$

$$= \frac{1}{s} + 4r + 4z \quad (z = 2r)$$

$$= \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= 1.\left(-\frac{r}{s^2}\right) + 2.\frac{1}{s} + 2z.0$$

$$= \frac{2}{s} - \frac{r}{s^2}$$

Problem on the existence of limits

Repeated limits:

$$L = \lim_{(x,y)\to(0,0)} f(x,y)$$ Simultaneous limit

$$R_1 = \lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right)$$

$$R_2 = \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right)$$

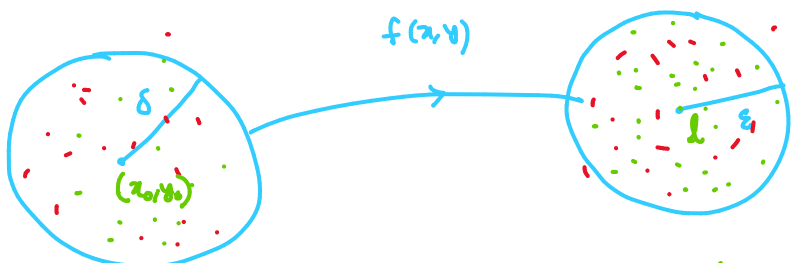
} Repeated limits

Prob: Let $f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$

Prove that: $\lim_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2y^2}{x^2+y^2} = 0$.

$y=0$,	$L=0$
$x=0$,	$L=0$
$y=mx$,	$L=0$

Soln: We are going to use ε-δ definition.



$|f(x,y)-l| < \varepsilon$.

$\Rightarrow l-\varepsilon < f(x,y) < l+\varepsilon$

$\Rightarrow (x-x_0)^2+(y-y_0)^2 < \delta^2$

$x^2+y^2 < \delta^2$

Let $\varepsilon>0$ be given. We have to find $\delta>0$ such that, $|x^2+y^2| < \delta^2$

$|f(x,y)-l| < \varepsilon$

$$|f(x,y)-l| < \varepsilon \quad \left|\frac{x^2y^2}{x^2+y^2} - 0\right|$$

$$= \left|\frac{x^2y^2}{x^2+y^2}\right| < \left|\frac{(x^2+y^2)\,(x^2+y^2)}{(x^2+y^2)}\right|$$

$x^2 < x^2+y^2$

$y^2 < x^2+y^2$

$|x < \sqrt{x^2+y^2}$

$$\left| \frac{\quad}{x^2+y^2} \right| \qquad \left| \frac{\quad}{(x^2+y^2)} \right|$$

$$= |x^2+y^2| < \varepsilon \; ..$$

$x < \sqrt{x^2+y^2}$

$y < \sqrt{x^2+y^2}$

Now, if we consider $0 < x^2+y^2 < \delta^2$, where $\delta^2 = \varepsilon \Rightarrow \delta = \sqrt{\varepsilon}$, which satisfy the requirement. Therefore,

$(x-x_0)^2+(y-y_0)^2<\delta^2$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^2}{x^2+y^2} = 0$$