Z -TRANSFORM

Standard results

$$(1-x)^{-1} = 1 + x + x^2 + \dots + x^n + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - \dots + (-1)^n x^n + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - \dots + (-1)^n (n+1) x^n + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots + (n+1) x^n + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \dots + (-1)^n \frac{x^n}{n!} + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Definition If $\{f_n\}$ be a sequence $f_0, f_1, f_2, \ldots$

Its Z-transform is defined by

$$Z[f_n] = \sum_{n=0}^{\infty} f_n z^{-n} = F(z)$$

Property 1. Z-transform is linear.*

$$Z[a_n f_n + b_n g_n] = \sum_{n=0}^{\infty} (a_n f_n + b_n g_n) z^{-n}$$

$$= a_n \sum_{n=0}^{\infty} f_n z^{-n} + b_n \sum_{n=0}^{\infty} g_n z^{-n}$$

$$= a_n Z[f_n] + b_n Z[g_n]$$

$$\therefore Z[a_n f_n + b_n g_n] = a_n Z[f_n] + b_n Z[g_n]$$

Property 2 $Z[a^n f_n] = F\left(\frac{z}{a}\right)$ (Shifting Property)

Proof: $Z[a^n f_n] = \sum_{n=0}^{\infty} a^n f_n z^{-n} = \sum_{n=0}^{\infty} f_n \left(\frac{z}{a}\right)^{-n} = F\left(\frac{z}{a}\right)$

Note that $a^n = \frac{1}{a^{-n}}$

Property 3 $Z[nf_n] = -z\dfrac{dF}{dz}$ (Derivative of transform)

Proof: We know that $F(z) = \sum_{n=0}^{\infty} f_n z^{-n}$

$$\frac{dF}{dz} = \sum_{n=0}^{\infty} f_n(-n z^{-n-1}) = -\frac{1}{z}\sum_{n=0}^{\infty} n f_n z^{-n}$$

$$-z\frac{dF}{dz} = -Z[nf_n]$$

$$\boxed{Z[nf_n] = -z\frac{dF}{dz}}$$

Property 4 $Z[n^k] = -z\frac{d}{dz}\{Z[n^{k-1}]\}$

Proof: $$Z[n^k] = \sum_{n=0}^{\infty} n^k z^{-n} \quad \text{——} \quad (*)$$

$$= \sum_{n=0}^{\infty} n^{k-1}\, n\, z^{-(n+1)} z$$

$$Z[n^k] = z\sum_{n=0}^{\infty} n^k z^{-(n+1)} \quad \text{——} \quad (1)$$

Replace k by k-1 in (*)

$$Z[n^{k-1}] = \sum_{n=0}^{\infty} n^{k-1} z^{-n}$$

Differentiate w.r. to z

$$\frac{d}{dz} Z[n^{k-1}] = \sum_{n=0}^{\infty} n^{k-1}(-n) z^{-(n+1)}$$

$$-\frac{d}{dz} Z[n^{k-1}] = \sum_{n=0}^{\infty} n^k z^{-(n+1)} \quad \text{——} \quad (2)$$

From (1) and (2) $$Z[n^k] = -z\frac{d}{dz} Z[n^{k-1}]$$

Problems Find Z-transform of the following:

(a) 1 (b) a^n (c) $(-1)^n$ (d) n (e) $n a^n$

Solution (a) We know that $Z[f_n] = \sum_{n=0}^{\infty} f_n z^{-n}$

$$Z[1] = \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} = \left(\frac{z-1}{z}\right)^{-1} = \frac{z}{z-1}$$

(b) $Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^{-n}$

$$= 1 + \left(\frac{z}{a}\right)^{-1} + \left(\frac{z}{a}\right)^{-2} + \cdots$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^{2} + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1} = \left(\frac{z-a}{z}\right)^{-1}$$

$$= \frac{z}{z-a}$$

(c) $Z[(-1)^n] = \sum_{n=0}^{\infty} (-1)^n z^{-n}$

$$= 1 - z^{-1} + z^{-2} - z^{-3} + \cdots$$

$$= 1 - \frac{1}{z} + \frac{1}{z^2} - \left(\frac{1}{z}\right)^3 + \cdots$$

$$= \left(1 + \frac{1}{z}\right)^{-1} = \left(\frac{z+1}{z}\right)^{-1} = \frac{z}{z+1}$$

(d) $$Z[n] = \sum_{n=0}^{\infty} n z^{-n}$$

$$= z^{-1} + 2z^{-2} + 3z^{-3} + \cdots$$

$$= \frac{1}{z} + \frac{2}{z^2} + 3\left(\frac{1}{z}\right)^3 + \cdots$$

$$= \frac{1}{z}\left(1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{z}{(z-1)^2}$$

Aliter We know that $Z[n f_n] = -z \dfrac{dF}{dz}$ (Property 3)

$$Z[n] = Z[n \cdot 1] = -z \frac{d}{dz} Z[1]$$

$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z \left[\frac{(z-1)\cdot 1 - z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

(e) We know that $Z[n f_n] = -z \dfrac{dF}{dz}$

$$\therefore Z[n a^n] = -z \frac{d}{dz} Z[a^n]$$

$$= -z \frac{d}{dz}\left[\frac{z}{z-a}\right] = -z\left[\frac{(z-a)\cdot 1 - z}{(z-a)^2}\right]$$

$$= \frac{az}{(z-a)^2}$$

<u>Aliter</u>

$$Z[n a^n] = \sum_{n=0}^{\infty} n a^n z^{-n} = \sum_{n=0}^{\infty} n \left(\frac{z}{a}\right)^{-n}$$

$$= \left(\frac{z}{a}\right)^{-1} + 2\left(\frac{z}{a}\right)^{-2} + 3\left(\frac{z}{a}\right)^{-3} + \cdots$$

$$= \frac{a}{z} + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{a}{z}\left(1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + \cdots\right)$$

$$= \frac{a}{z}\left(1 - \frac{a}{z}\right)^{-2} = \frac{a}{z}\cdot\left(\frac{z-a}{z}\right)^{-2} = \frac{az}{(z-a)^2}$$

Problems Find Z-transform of the following:

(a) k (b) $(-k)^n$ (c) $n+1$ (d) n^2 (e) $(n+1)^2$

(f) $\frac{1}{n!}$ (g) $\frac{1}{n}$ (h) $\frac{1}{n+1}$ (i) $\frac{1}{n(n+1)}$ (j) $\frac{(-1)^n}{n!}$

Here k is a constant.

(a) $$Z[k] = \sum_{n=0}^{\infty} k\, z^{-n} = k \sum_{n=0}^{\infty} z^{-n} = k\left(1 + z^{-1} + z^{-2} + \cdots\right)$$

$$= k\left(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots\right)$$

$$= k\left(1 - \frac{1}{z}\right)^{-1} = \frac{kz}{z-1}$$

(b) $$Z[(-k)^n] = \sum_{n=0}^{\infty} (-k)^n z^{-n} = \sum_{n=0}^{\infty} \left(-\frac{z}{k}\right)^{-n}$$

$$= 1 - \frac{k}{z} + \left(\frac{k}{z}\right)^2 - (k/z)^3 + \cdots$$

$$= \left(1 + \frac{k}{z}\right)^{-1} = \frac{z}{z+k}$$

(c) $$Z[n+1] = Z[n] + Z[1] = \frac{z}{(z-1)^2} + \frac{z}{z-1}$$

$$= \frac{z + z(z-1)}{(z-1)^2}$$

$$= \frac{z^2}{(z-1)^2}$$

(d) $$Z[n^2] = Z[n \cdot n]$$

$$= -z\frac{d}{dz}\left[Z(n)\right]$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2 z - z\,2(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{(z-1)z - 2z}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}$$

(e) $$Z\left[(n+1)^2\right] = Z\left[n^2 + 2n + 1\right]$$

$$= Z\left[n^2\right] + 2\,Z\left[n\right] + Z\left[1\right]$$

$$= \frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2(z+1)}{(z-1)^3}$$

(f) $$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots$$

$$= e^{1/z}$$

(g) $$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} = z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \cdots \quad (n \neq 0)$$

$$= \frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots$$

$$= -\log\left(1 - \frac{1}{z}\right)$$

$$= -\log\left(\frac{z-1}{z}\right) = \log\left(\frac{z}{z-1}\right)$$

(h) $$Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{z^{-n}}{n+1} = 1 + \frac{z^{-1}}{2} + \frac{z^{-2}}{3} + \frac{z^{-3}}{4} + \cdots$$ (Nov 2005)

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots$$

$$= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= z\left[-\log\left(1 - \frac{1}{z}\right)\right]$$

$$= -z\log\left(\frac{z-1}{z}\right) = z\log\left(\frac{z}{z-1}\right)$$

(i) Note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$\therefore Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{n+1}\right]$$

$$= \log\left(\frac{z}{z-1}\right) - z\log\left(\frac{z}{z-1}\right)$$

$$= (1-z)\log\left(\frac{z}{z-1}\right)$$

(j) $$Z\left[\frac{(-1)^n}{n!}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-n}}{n!}$$

$$= 1 - \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 \cdots = e^{-1/z}.$$

Problems Find Z-transform of the following:

(a) $\cos n\theta$, (b) $\sin n\theta$ (c) $a^n \cos n\theta$ (d) $a^n \sin n\theta$

(e) $\cos \frac{n\pi}{2}$ (f) $\sin \frac{n\pi}{2}$

Solution We know that $Z[a^n] = \frac{z}{z-a}$

(a) & (b) Put $a = e^{i\theta}$

$$Z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}} = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$Z[(\cos\theta + i\sin\theta)^n] = \frac{z}{(z-\cos\theta) - i\sin\theta} \cdot \frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$Z[\cos n\theta + i\sin n\theta] = \frac{z(z-\cos\theta) + iz\sin\theta}{(z-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1} + \frac{iz\sin\theta}{z^2-2z\cos\theta+1}$$

Equate real and imaginary Parts, we get

$$Z[\cos n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}.$$

(c) We know that $Z[a^n f_n] = F\left(\frac{z}{a}\right) = Z[f_n]_{z \to \frac{z}{a}}$

$$Z[a^n \cos n\theta] = Z[\cos n\theta]_{z \to \frac{z}{a}}$$

$$= \left(\frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}\right)_{z \to \frac{z}{a}}$$

$$= \frac{\frac{z}{a}\left(\frac{z}{a} - \cos\theta\right)}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\theta + 1}$$

$$= \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$$

(d) $Z\left[a^n \sin n\theta\right] = Z\left[\sin n\theta\right]_{z \to z/a}$

$$= \left(\frac{z \sin\theta}{z^2 - 2az\cos\theta + 1}\right)_{z \to \frac{z}{a}}$$

$$= \frac{\frac{z}{a}\sin\theta}{\left(\frac{z}{a}\right)^2 - 2z/a\cos\theta + 1}$$

$$= \frac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$$

(e) We know that

$$Z[\cos n\theta] = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$$

Put $\theta = \frac{\pi}{2}$

$$Z\left[\cos \frac{n\pi}{2}\right] = \frac{z^2}{z^2 + 1}$$

(f) we know that $Z[\sin\theta] = \dfrac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

Put $\theta = \dfrac{\pi}{2}$

$$Z\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$$

Factorial Polynomial

$$n^{(0)} = 1$$

$$n^{(1)} = n$$

$$n^{(2)} = n(n-1)$$

$$n^{(3)} = n(n-1)(n-2)$$

$$\therefore \; Z[n^{(0)}] = Z[1] = \frac{z}{z-1}$$

$$Z[n^{(1)}] = Z[n] = \frac{z}{(z-1)^2}$$

$$Z[n^{(2)}] = Z[n(n-1)] = Z[n^2] - Z[n] = \frac{2z}{(z-1)^3}$$

$$\therefore \; Z[n^{(k)}] = \frac{k!\, z}{(z-1)^{k+1}}$$

Inverse Z - transform

Type - I : Partial fraction Method

Step 1 Given that $F(z)$

Write down $\frac{F(z)}{z}$

Step 2 Perform Partial fractions for $\frac{F(z)}{z}$

Step 3 Multiply z and find inverse Z-tr.

Note $Z[f_n] = F(z) \Rightarrow f_n = Z^{-1}[F(z)]$

Problems Find inverse Z-transforms of

(a) $\dfrac{z}{z^2-3z+2}$ (b) $\dfrac{z^2}{(z-a)(z-b)}$ (c) $\dfrac{2z}{(z-1)(z^2+1)}$

(d) $\dfrac{z^2+z}{(z-1)(z^2+1)}$ (e) $\dfrac{3z^2+z}{(z-0.3)(z+0.4)}$

Sol. (a) $Z^{-1}\left[\dfrac{z}{z^2-3z+2}\right]$

$$F(z) = \frac{z}{z^2-3z+2}$$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

Put $z = 1 \Rightarrow \boxed{A = -1}$

Put $z = 2 \Rightarrow \boxed{1 = B}$

$$\therefore \frac{F(z)}{z} = -\frac{1}{z-1} + \frac{1}{z-2}$$

$$F(z) = -\frac{z}{z-1} + \frac{z}{z-2}$$

$$Z[f_n] = -\frac{z}{z-1} + \frac{z}{z-2}$$

$$f_n = Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{z}{z-1}\right]$$

$$f_n = 2^n - 1$$

(b) $$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$$

$$F(z) = \frac{z^2}{(z-a)(z-b)}$$

$$\frac{F(z)}{z} = \frac{z}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}$$

$$\frac{z}{(z-a)(z-b)} = \frac{A(z-b) + B(z-a)}{(z-a)(z-b)}$$

$$z = A(z-b) + B(z-a)$$

Put $z = a \Rightarrow A = \frac{a}{a-b}$

Put $z = b \Rightarrow B = \frac{-b}{a-b}$

$$\therefore \frac{F(z)}{z} = \frac{a}{a-b} \cdot \frac{1}{z-a} - \frac{b}{a-b} \frac{1}{z-b}$$

$$F(z) = \frac{a}{a-b} \frac{z}{z-a} - \frac{b}{a-b} \frac{z}{z-b}$$

$$Z[f_n] = \frac{a}{a-b} \cdot \frac{z}{z-a} - \frac{b}{a-b} \frac{z}{z-b}$$

$$f_n = \frac{a}{a-b} a^n - \frac{b}{a-b} b^n = \frac{a^{n+1} - b^{n+1}}{a-b}.$$

(C) $F(z) = \dfrac{2z}{(z-1)(z^2+1)}$

$$\frac{F(z)}{z} = \frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$2 = A(z^2+1) + (Bz+C)(z-1)$$

Put $z=1 \Rightarrow 2 = 2A \Rightarrow \boxed{A=1}$

Equate coefficient of z^2 : $0 = A + B$

$\boxed{B=-1}$

Equate " " Const. term $2 = A - C$

$\boxed{C=-1}$

$$\therefore \quad \frac{F(z)}{z} = \frac{1}{z-1} - \frac{(z+1)}{z^2+1}$$

$$F(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1}$$

$$\therefore \quad f_n = 1 - \sin\frac{n\pi}{2} - \cos\frac{n\pi}{2}$$

(d) $F(z) = \dfrac{z(z+1)}{(z-1)(z^2+1)}$

$$\frac{F(z)}{z} = \frac{z+1}{(z-1)(z^2+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+1}$$

$$z+1 = A(z^2+1) + (Bz+C)(z-1)$$

Put $z=1 \Rightarrow \quad 2 = 2A \Rightarrow \boxed{A=1}$

Equate coeff. of z^2 : $\quad 0 = A+B \Rightarrow \boxed{B=-1}$

Equate " " const. term $\quad 1 = A - C \Rightarrow \boxed{C=0}$

$$\therefore \frac{F(z)}{z} = \frac{1}{z-1} - \frac{z}{z^2+1}$$

$$F(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1}$$

$$\therefore f_n = 1 - \cos\frac{n\pi}{2}$$

(e) $$F(z) = \frac{z(3z+1)}{(z-0.2)(z+0.4)}$$

$$\frac{F(z)}{z} = \frac{3z+1}{(z-0.2)(z+0.4)} = \frac{A}{z-0.2} + \frac{B}{z+0.4}$$

Put $z = 0.2 \Rightarrow 0.6A = 1.6 \Rightarrow A = \frac{8}{3}$

Put $z = -0.4 \Rightarrow B = \frac{1}{3}$

$$\therefore \frac{F(z)}{z} = \frac{8}{3}\frac{1}{z-0.2} + \frac{1}{3}\frac{1}{z+0.4}$$

$$F(z) = \frac{8}{3} \frac{z}{z-0.2} + \frac{1}{3} \frac{z}{z+0.4}$$

$$\therefore f_n = \frac{8}{3} (0.2)^n + \frac{1}{3} (-0.4)^n$$

Unit - Impulse function

$$\delta_{n-l} = \begin{cases} 1 & \text{if } n = l \\ 0 & \text{if } n \neq l \end{cases}$$

Note $\quad Z[\delta_n] = 1 \Rightarrow \delta_n = Z^{-1}[1]$

Problems

Find inverse Z-transform of the following:

(a) $$\frac{2z^2+1}{(z-1)(z+2)}$$

(b) $$\frac{z^2+4}{z^2+2z-3}$$

(c) $$\frac{z^2-3z}{z^2-3z-10}$$

(d) $$\frac{z^2}{z^2+7z+10}$$

Solution (a) $F(z) = \dfrac{2z^2+1}{(z-1)(z+2)}$

$$\frac{F(z)}{z} = \frac{2z^2+1}{z(z-1)(z+2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+2}$$

$$2z^2+1 = A(z-1)(z+2) + Bz(z+2) + Cz(z-1)$$

Put $z=0 \Rightarrow \boxed{A = -1/2}$

Put $z=1 \Rightarrow 3 = B(3) \Rightarrow \boxed{B=1}$

Put $z=-2 \Rightarrow 9 = 6C \Rightarrow \boxed{C = 3/2}$

$$\therefore \frac{F(z)}{z} = -\frac{1}{2}\frac{1}{z} + \frac{1}{z-1} + \frac{3}{2}\frac{1}{z+2}$$

$$F(z) = -\frac{1}{2} + \frac{z}{z-1} + \frac{3}{2}\frac{z}{z+2}$$

$$f_n = -\frac{1}{2}\delta_n + 1 + \frac{3}{2}(-2)^n$$

(b) $$F(z) = \frac{z^2+4}{(z+3)(z-1)}$$

$$\frac{F(z)}{z} = \frac{z^2+4}{z(z+3)(z-1)} = \frac{A}{z} + \frac{B}{z+3} + \frac{C}{z-1}$$

$$z^2+4 = A(z+3)(z-1) + Bz(z-1) + Cz(z+3)$$

Put $z=0 \Rightarrow$ $\boxed{A = -4/3}$

Put $z=1 \Rightarrow$ $\boxed{C = 5/4}$

Put $z=-3 \Rightarrow$ $\boxed{B = \frac{13}{12}}$

$$\therefore \quad \frac{F(z)}{z} = -\frac{4}{3}\,\frac{1}{z} + \frac{13}{12}\,\frac{1}{z+3} + \frac{5}{4}\cdot\frac{1}{z-1}$$

$$F(z) = -\frac{4}{3}\,1 + \frac{13}{12}\,\frac{z}{z+3} + \frac{5}{4}\,\frac{z}{z-1}$$

$$f_n = -\frac{4}{3}\,\delta_n + \frac{13}{12}\,(-3)^n + \frac{5}{4}$$

(C) $$F(z) = \frac{z^2 - 3z}{z^2 - 3z - 10}$$

$$\frac{F(z)}{z} = \frac{z-3}{(z-5)(z+2)} = \frac{A}{z-5} + \frac{B}{z+2}$$

$$z - 3 = A(z+2) + B(z-5)$$

Put $z = 5 \Rightarrow \boxed{A = 2/7}$

Put $z = -2 \Rightarrow \boxed{B = 5/7}$

$$\therefore \quad \frac{F(z)}{z} = \frac{2}{7}\,\frac{1}{z-5} + \frac{5}{7}\,\frac{1}{z+2}$$

$$F(z) = \frac{2}{7}\,\frac{z}{z-5} + \frac{5}{7}\,\frac{z}{z+2}$$

$$f_n = \frac{2}{7}\,(5)^n + \frac{5}{7}\,(-2)^n$$

(d) $$F(z) = \frac{z^2}{z^2+7z+10}$$

$$\frac{F(z)}{z} = \frac{z}{(z+5)(z+2)} = \frac{A}{z+5} + \frac{B}{z+2}$$

$$z = A(z+2) + B(z+5)$$

Put $z = -5 \Rightarrow \boxed{A = 5/3}$

Put $z = -2 \Rightarrow \boxed{B = -2/3}$

$$\frac{F(z)}{z} = \frac{5}{3} \cdot \frac{1}{z+5} - \frac{2}{3} \frac{1}{z+2}$$

$$F(z) = \frac{5}{3} \frac{z}{z+5} - \frac{2}{3} \frac{z}{z+2}$$

$$f_n = \frac{5}{3} (-5)^n - \frac{2}{3} (-2)^n$$

<u>Note</u> $Z[n a^n] = \dfrac{az}{(z-a)^2} \Rightarrow Z^{-1}\left[\dfrac{z}{(z-a)^2}\right] = n a^{n-1}$

Problem Find $Z^{-1}\left[\dfrac{z}{(z-2)(z+3)^2}\right]$

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put $z=2 \Rightarrow \quad 1 = 25A \Rightarrow \boxed{A = 1/25}$

Put $z = -3 \Rightarrow \boxed{C = -1/5}$

Equate coeff. of z^2 : $0 = A + B \Rightarrow \boxed{B = -1/25}$

$$\therefore \frac{F(z)}{z} = \frac{1}{25}\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\frac{1}{(z+3)^2}$$

$$F(z) = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$f_n = \frac{1}{25}\left(2^n - (-3)^n\right) - \frac{1}{5}\, n\, (-3)^{n-1}$$

Difference equation : Formation

Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$

$f_n \quad f_{n+1} \quad f_{n+2}$

$$f_{n+2} = f_n + f_{n+1}$$

Problems Form difference equation for the following:

(a) $y_n = A\,2^n + B(-3)^n$ (b) $y_n = (A + Bn)3^n$

Sol. (a) $y_n = A\,2^n + B(-3)^n$ ——— (1)

$y_{n+1} = A\,2^{n+1} + B(-3)^{n+1}$

$y_{n+1} = 2A\,2^n - 3B(-3)^n$ ——— (2)

$y_{n+2} = A\,2^{n+2} + B(-3)^{n+2}$

$y_{n+2} = 4A\,2^n + 9B(-3)^n$ ——— (3)

From (1), (2) and (3)

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0$$

$$y_n(18+12) - y_{n+1}(5) + y_{n+2}(-5) = 0$$

$$y_{n+2} + y_{n+1} - 6y_n = 0$$

(b)

$$y_n = A\,3^n + B\,n\,3^n \quad \text{———} \quad (1)$$

$$y_{n+1} = A\,3^{n+1} + B(n+1)\,3^{n+1}$$

$$y_{n+1} = 3A\,3^n + 3(n+1)\,B\,3^n \quad \text{———} \quad (2)$$

$$y_{n+2} = 9A\,3^n + 9(n+2)\,B\,3^n \quad \text{—} \quad (3)$$

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 3 & 3(n+1) \\ y_{n+2} & 9 & 9(n+2) \end{vmatrix} = 0 \Rightarrow y_{n+2} - 6y_{n+1} + 9y_n = 0$$

Theorem $Z[f_{n+k}] = z^k \left[F(z) - f_0 - \frac{f_1}{z} - \cdots - \frac{f_{k-1}}{z^{k-1}} \right]$

Proof: $Z[f_{n+k}] = \sum_{n=0}^{\infty} f_{n+k} z^{-n}$

$$= \sum_{n=0}^{\infty} f_{n+k} z^{-(n+k)} z^k$$

$$= z^k \sum_{n=0}^{\infty} f_{n+k} z^{-(n+k)}$$ Put $n+k = m$

$$= z^{k} \sum_{m=k}^{\infty} f_m \, z^{-m}$$

$$= z^{k} \left[\sum_{m=0}^{\infty} f_m \, z^{-m} - \sum_{m=0}^{k-1} f_m \, z^{-m} \right]$$

$$= z^{k} \left[F(z) - f_0 - \frac{f_1}{z} - \cdots - \frac{f_{k-1}}{z^{k-1}} \right]$$

Note $Z[f_n] = F(z)$

$$Z[f_{n+1}] = z\left[F(z) - f_0\right]$$

$$Z[f_{n+2}] = z^2\left[F(z) - f_0 - \frac{f_1}{z}\right]$$

Problems Solve the following difference equations using Z-transform.

(a) $u_{n+1} + u_n = 1, \quad u_0 = 0$

(b) $y_{n+1} - 3y_n = 2^n, \quad y_0 = 1$

(c) $y_{n+2} - 3y_{n+1} + 2y_n = 0, \quad y_0 = 0, \; y_1 = 1$

(d) $y_{n+2} = y_{n+1} + y_n, \quad y_0 = 0, \; y_1 = 1$

(or Form difference equation for fibonacci seq. and then solve using Z-transform.