Module:1 Application of Single Variable Calculus

Differentiation

Rolle’s Theorem and the Mean Value Theorem

Extrema on an Interval, Increasing and Decreasing functions

First derivative test-Second derivative test for Maxima and Minima

Integration

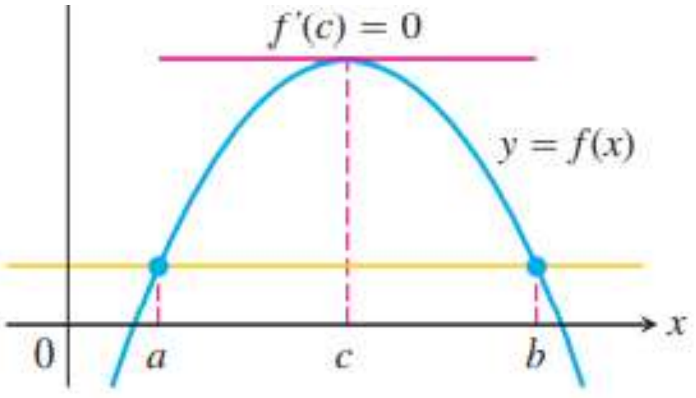
Average function value

Area between curves

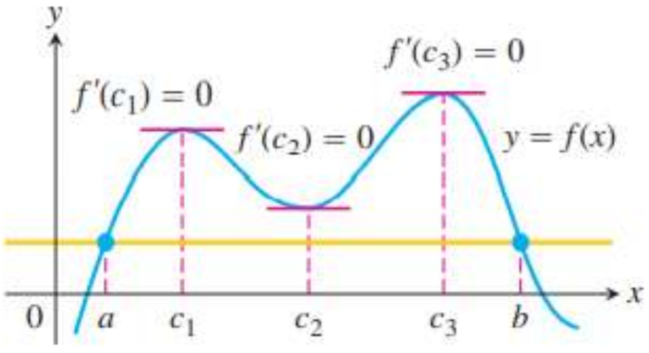
Volumes of solids of revolution

Beta and Gamma functions

Rolle's Theorem Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.



(a)



(b)

Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

From the Figure **it is self-evident that there is at least one point *C* (may be more) of the curve at which the tangent is parallel, to the *x*-axis.**

i.e., **slope of the tangent at *C* ($x = c$) = 0**

1. Define Rolle's theorem, verify Rolle's theorem for $(x-a)^m(x-b)^n$ where m,n are positive integers in $[a,b]$

Sol.: Rolle's Theorem (statement): If (i) f(x) is continuous in the closed interval [a,b],

(ii) f '(x) exists for every value of x in the open interval (a,b) and

(iii) f(a)=f(b) , then there is at least one value 'c' of 'x' in (a,b) such that f'(c) =0

Verification: Given f(x) =$(x-a)^m(x-b)^n$, it is continuous in [a,b].

Also, f(a) =0 ; f(b)=0 ⇒f(a)= f(b)

f '(x)= n$(x-a)^m(x-b)^{n-1}$ + m $(x-a)^{m-1}(x-b)^n$ ⇒derivative exists in (a,b)

According to Rolle's theorem, ∃ c∈ (a, b) ∋ f '(c)=0

⇒n$(c-a)^m(c-b)^{n-1}$+ m $(c-a)^{m-1}(c-b)^n$ =0

⇒ $(c-a)^{m-1}(c-b)^{n-1}[n(c-a)+m(c-b)]$ =0

⇒ $[n(c-a)+m(c-b)]$ =0 ⇒nc-na +mc-mb=0 ⇒c(n+m) =na+mb ⇒ c=$\frac{na+mb}{n+m}$∈ (a,b)

Hence, Rolle's theorem is verified.

2. Verify Rolle's theorem for $f(x) = (x+2)^3(x-3)^4$ in (-2,3)

Sol.: Since$(x+2)^3(x-3)^4$ is polynomial function, it is continuous in [-2,3]

f(-2) = $(-2+2)^3(-2-3)^4$ =0 ; f(3) =$(3+2)^3(3-3)^4$ =0. ⇒f(-2)= f(3)

f '(x)= $4(x+2)^3(x-3)^3 + 3\,(x+2)^2(x-3)^4$

⇒f ' exists , i.e, f(x) is derivable in (-2,3)

Hence the given function satisfies all the conditions ofRolle's theorem.

⇒∃c∈(-2,3) ∋ f'(c)=0

Now,f'(c)=$4(c+2)^3(c-3)^3 + 3(c-2)^2(c-3)^4$ =0

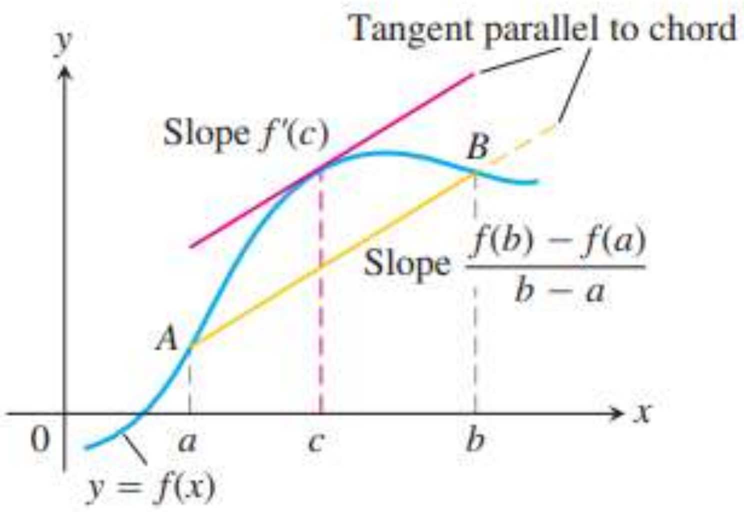
⇒f'(c) =$(c+2)^2(c-3)^3[4(c+2)+3(c-3)]$ =0 ⇒4(c+2)+3(c-3)=0 ⇒ 4c+3c+8-9=0

⇒ 7c=1 ⇒ c=$\frac{1}{7}$∈ (-2,3)

Hence, Rolle's Theorem is verified.

THEOREM —The Mean Value Theorem Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to chord AB.

Verify the mean value theorem for f(x) = sin x in [0,π]

Sol.: f(x) = sin x is continuous over [0,π] and f'(x) = cos x exists in (0,π)

$\Rightarrow \exists c \ni c \in (a,b)$, $f'(c) = \frac{f(b)-f(a)}{b-a}$ a=0, b=π $\Rightarrow f'(c) = \frac{\sin\pi - \sin 0}{\pi - 0} = \frac{0-0}{\pi-0} = 0$

Cos c = 0 $\Rightarrow$ c = $\frac{\pi}{2} \Rightarrow c = \frac{\pi}{2} \in (0,\pi) \ni f'(c) = 0$.

The Mean value theorem is verified.

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$ Ans. 1/2
2. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$ Ans. 1
3. $f(x) = x^3 - x^2, \quad [-1, 2]$

Ans. $\frac{1}{3}(1 + \sqrt{7}) \approx 1.22, \frac{1}{3}(1 - \sqrt{7}) \approx -0.549$

Prove that $\frac{b-a}{1+b^2}<\tan^{-1} b - \tan^{-1} a<\frac{b-a}{1+a^2}$. Hence show that $\frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6}$.

Sol.: Let f(x) $=\tan^{-1}x \Rightarrow$ f'(x)$=\frac{1}{1+x^2}$, 0<a<b<1

By Mean-value theorem, we have f'(c)$=\frac{f(a)-f(b)}{b-a}=\frac{\tan^{-1} b - \tan^{-1} a}{b-a} \Rightarrow \frac{1}{1+c^2}=\frac{\tan^{-1} b - \tan^{-1} a}{b-a}$

Since, a<c<b $\Rightarrow a^2<c^2<b^2 \Rightarrow 1+a^2<1+c^2<1+b^2 \Rightarrow \frac{1}{1+a^2}>\frac{1}{1+c^2}>\frac{1}{1+b^2}$

$\Rightarrow \frac{1}{1+a^2}>\frac{\tan^{-1} b - \tan^{-1} a}{b-a}>\frac{1}{1+b^2} \Rightarrow \frac{b-a}{1+a^2}>\frac{\tan^{-1} b - \tan^{-1} a}{1}>\frac{b-a}{1+b^2}$

$\Rightarrow \frac{b-a}{1+b^2}<\frac{\tan^{-1} b - \tan^{-1} a}{1}<\frac{b-a}{1+a^2}$.

Hence proved.

Now, let a=1, b=$\frac{4}{3}$: $\Rightarrow \frac{\frac{4}{3}-1}{1+\frac{16}{9}}<\tan^{-1}(\frac{4}{3}) - \frac{\pi}{4}<\frac{\frac{4}{3}-1}{1+1} \Rightarrow \frac{\frac{1}{3}}{\frac{25}{9}}<\tan^{-1}(\frac{4}{3}) - \frac{\pi}{4}<\frac{\frac{1}{3}}{2}$

$\Rightarrow \frac{3}{25}<\tan^{-1}(\frac{4}{3})-\frac{\pi}{4}<\frac{1}{6} \Rightarrow \frac{\pi}{4}+\frac{3}{25}<\tan^{-1}\frac{4}{3}<\frac{\pi}{4}+\frac{1}{6}$

Hence shown.