Module 2: Partial Differential Equations:

2.1 Formation of Partial differential equations by

(i) Elimination of arbitrary constants

(ii) Elimination of arbitrary functions.

(i) Elimination of arbitrary constants:

Problems

1. Form the partial differential equation by eliminating the arbitrary constants a and b from

$$z = ax + by + a^2 + b^2.$$

Sol: Given $z = ax + by + a^2 + b^2 \rightarrow (1)$

Differentiating (1) partially with respect to x as well as y, we get

$$\frac{\partial z}{\partial x} = a + 0 \Rightarrow p = a$$

and $\dfrac{\partial z}{\partial y} = b \Rightarrow q = b$

$$\therefore z = px + qy + p^2 + q^2$$

This is the required pde.

Note: Throughout this chapter Consider $\dfrac{\partial z}{\partial x} = p$ and $\dfrac{\partial z}{\partial y} = q$.

② Form a p.d.e. by eliminating the arbitrary constans a and b from

$$z = a \log\left[\frac{b(y-1)}{1-x}\right]$$

Sol: Given $z = a \log\left[\frac{b(y-1)}{1-x}\right] \longrightarrow ①$

Differentiating ① partially w.r.t. x as well as y, we get

$$\frac{\partial z}{\partial x} = a \frac{1}{\left[\frac{b(y-1)}{1-x}\right]} \cdot \frac{\partial}{\partial x}\left[\frac{b(y-1)}{1-x}\right]$$

$$= \frac{a(1-x)}{b(y-1)} \cdot b(y-1)\left(\frac{+1}{(1-x)^2}\right)$$

$$\Rightarrow p = \frac{a}{1-x} \Rightarrow a = (1-x)p \longrightarrow (i)$$

and $$\frac{\partial z}{\partial y} = a \frac{1}{\left[\frac{b(y-1)}{(1-x)}\right]} \frac{\partial}{\partial y}\left[\frac{b(y-1)}{(1-x)}\right]$$

$$\Rightarrow q = \frac{a(1-x)}{b(y-1)} \frac{b}{(1-x)}$$

$$\Rightarrow q = \frac{a}{y-1} \Rightarrow a = (y-1)q \longrightarrow (ii)$$

∴ From (i) and (ii), we have

$$(1-x)p = (y-1)q \text{ (or) } \boxed{(1-x)p + (1-y)q = 0}$$

This is the required p.d.e.

3) Form p.d.e.s by eliminating arbitrary constants from

(i) $2z = \sqrt{x+a} + \sqrt{y+b}$ Ans: $8pqz = p+q$.

(ii) $z = ax^3 + by^3$ Ans: $3z = px + qy$.

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Ans: $pz = xp^2 + xzr$,

where $r = \frac{\partial^2 z}{\partial x^2}$

[Hint: Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \longrightarrow (1)$

Diffn (1) partially w.r.t. x, we get

$$\frac{x}{a^2} + \frac{z}{c^2} p = 0 \longrightarrow (i)$$

Diffn (1) partially w.r.t. y, we get

$$\frac{y}{b^2} + \frac{z}{c^2} q = 0 \longrightarrow (ii)$$

Diffn (i) partially w.r.t. x, we get

$$\frac{1}{a^2} + \frac{p}{c^2}\frac{\partial z}{\partial x} + \frac{z}{c^2}\frac{\partial p}{\partial x} = 0$$

or, $\frac{1}{a^2} + \frac{p^2}{c^2} + \frac{z}{c^2} r = 0 \longrightarrow (iii)$ $\left(r = \frac{\partial^2 z}{\partial x^2}\right)$

Multiplying (iii) by x, we get

$$\frac{x}{a^2} + \frac{xp^2}{c^2} + \frac{xzr}{c^2} = 0 \longrightarrow (iv)$$

(i) − (iv) $\Rightarrow$ $\frac{1}{c^2}(xzr + xp^2 - pz) = 0$

or, $pz = xp^2 + xzr$]

④ obtain the p.d.e. of all spheres whose centres lie on z-axis with a given radius a.

Sol: Hint (Ans) The eqn of the family of spheres having their centres on z-axis and radius a is $(x-0)^2+(y-0)^2+(z-c)^2=a^2$

or, $x^2+y^2+(z-c)^2=a^2 \longrightarrow ①$, where a and c are arbitrary Constants.

Ans: $\boxed{xq-yp=0}$ This is the required p.d.e.

(ii) Elimination of arbitrary functions:

Problems:

1. Find a partial differential equation by eliminating the arbitrary function from

$$z=f(x^2-y^2).$$

Sol: we have $z=f(x^2-y^2) \longrightarrow ①$

Taking $u=x^2-y^2$, then $z=f(u)$

Diffn ① partially w.r.t. x, we get

$$\frac{\partial z}{\partial x}=f'(u).\frac{\partial u}{\partial x} \Rightarrow p=f'(u).2x \longrightarrow ②$$

Similarly diff. ① partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = f'(u).\frac{\partial u}{\partial y} \Rightarrow q = f'(u)(-2y) \rightarrow ③$$

Now, ② ÷ ③ $\Rightarrow \frac{p}{q} = \frac{-x}{y}$

or, $\boxed{yp + xq = 0}$

This is the required p.d.e.

② Form the partial differential equation by eliminating the arbitrary function f from $f(x^2+y^2, x^2-z^2)=0$.

Sol: we have $f(x^2+y^2, x^2-z^2)=0$

This can be written as $x^2+y^2 = g(x^2-z^2) \rightarrow ①$

Diff. ① partially w.r.t. x, we get

$$2x = g'(x^2-z^2)\left[2x - 2z\frac{\partial z}{\partial x}\right]$$

or, $x = g'(x^2-z^2)(x - zp) \rightarrow ②$

Similarly Diff. ① partially w.r.t. y, we get

$$2y = g'(x^2-z^2)\left[-2z\frac{\partial z}{\partial y}\right]$$

or, $y = -g'(x^2-z^2)(zq) \rightarrow ③$

Now, ② ÷ ③ $\Rightarrow$ $\dfrac{x}{y} = \dfrac{(x - zp)}{-zq}$

or, $\dfrac{x}{y} = \dfrac{zp - x}{zq}$

$\Rightarrow (zq + y)x - yzp = 0$

This is the required p.d.e.

③ Form a partial differential equation by eliminating the arbitrary functions from

$$z = f(x) + e^y g(x)$$

Sol: we have $z = f(x) + e^y g(x) \longrightarrow ①$

Diff.. ① partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = f'(x) + e^y g'(x) \longrightarrow ②$$

Similarly Diff.. ① partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = e^y g(x) \longrightarrow ③$$

Diff.. ③ partially w.r.t. y, we get

$$\frac{\partial^2 z}{\partial y^2} = e^y g(x)$$

$$\Rightarrow r = \frac{\partial z}{\partial y} \Rightarrow \boxed{r - q = 0}$$

This is the required p.d.e

(4) Form a partial differential equation by eliminating the arbitrary functions from $z = y f(x) + x g(y)$

Sol: Hint: $\frac{\partial z}{\partial x} = y f'(x) + g(y)$

$\Rightarrow p = y f'(x) + g(y) \rightarrow (1)$

and $q = f(x) + x g'(y) \rightarrow (2)$

Also $\frac{\partial^2 z}{\partial x^2} = y f''(x) \Rightarrow r = y f''(x) \rightarrow (3)$

$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y) \Rightarrow s = f'(x) + g'(y) \rightarrow (4)$

$\boxed{s = \frac{\partial^2 z}{\partial x \partial y}}$

and $\frac{\partial^2 z}{\partial y^2} = x g''(y) \Rightarrow t = x g''(y) \rightarrow (5)$

$\boxed{t = \frac{\partial^2 z}{\partial y^2}}$

Now, $s = f'(x) + g'(y)$

$= \frac{1}{y}[p - g(y)] + \frac{1}{x}[q - f(x)]$ (from (1) & (2))

Hence $\boxed{xys = px + qy - z}$

This is the required p.d.e.

5. Form a partial differential equations by eliminating the arbitrary function(s) from

(i) $xyz = f(x^2+y^2+z^2)$

Ans: $(x^2-y^2)z + (px-qy)z^2 = xy(yp-xq)$

(ii) $f(x^2+y^2+z^2, z^2-2xy) = 0$

Ans: $(q-p)z + y - x = 0$

(iii) $z = f(y) + g(x+y)$

Ans: $r - s = 0$

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