

Linear combination:-

Let V be a vector space, and $u_1, u_2, u_3, \dots, u_k \in V$.
A vector $v \in V$ is said to be linear combination
of vectors $u_1, u_2, \dots, u_k$, if v can be written
as
$$v = c_1 u_1 + c_2 u_2 + c_3 u_3 + \dots + c_k u_k$$

where $c_1, c_2, \dots, c_k$ are scalars

example:-

Let $v_1 = (1, 3, 1)$, $v_2 = (0, 1, 2)$, $v_3 = (1, 0, -5)$
then v_1 is a linear combination of v_2 &

$$(1, 3, 1) = 3(0, 1, 2) + 1(1, 0, -5)$$

$$v_1 = 3 \cdot v_2 + v_3$$

$$\text{here } c_1 = 3, \quad c_2 = 1$$

ex2. If $V_1 = \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}$

$$V_4 = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$$

here $V_1 = V_2 + 2V_3 - V_4$ (check it)

$$c_1 = 1, c_2 = 2, c_3 = -1$$

i.e V_1 is a linear combination of V_2, V_3, V_4 .

Finding the linear combination:-

- (1) check whether the vector $V = (1, 1, 1)$ can be written as linear combination of vectors $V_1 = (1, 2, 3)$, $V_2 = (0, 1, 2)$ and $V_3 = (-1, 0, 1)$

solⁿ - V can be written as linear combination of V_1, V_2, V_3 if $\exists$ scalars c_1, c_2, c_3 s.t

$$V = c_1 V_1 + c_2 V_2 + c_3 V_3$$

$$(1, 1, 1) = c_1 (1, 2, 3) + c_2 (0, 1, 2) + c_3 (-1, 0, 1)$$

$$(1, 1, 1) = (c_1 + 0 \cdot c_2 - c_3, 2c_1 + c_2 + 0 \cdot c_3, 3c_1 + 2c_2 + c_3)$$

Now,

$$c_1 - c_3 = 1$$

$$2c_1 + c_2 = 1$$

$$3c_1 + 2c_2 + c_3 = 1$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

unique solⁿ $\rightarrow$ 1 set of value

infinite solⁿ $\rightarrow$ many set

$$AX = B.$$

NO solⁿ $\rightarrow$ NO Linear Combination

B]

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{Rank}(A) = 2 \quad \text{Rank}(A|B) = 2$$

$$n = 3$$

$$R(A) = R(A|B) < n$$

ie infinite many solⁿ

we have

$$c_2 + 2c_3 = -1$$

$$c_1 - c_3 = 1$$

$$\text{let } c_3 = t$$

$$c_2 = -1 - 2t$$

$$c_1 = 1 + t$$

solⁿ set

$$\{c_1, c_2, c_3\} = \{1+t, -1-2t, t \mid t \text{ arb } t \in \mathbb{R}\}$$

ie

$(1, 1, 1)^T$ can be written as linear combination of v_1, v_2, v_3 .

for $t=1$

$$V = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$[V = 2v_1 - 3v_2 + v_3]$$

for $t=2$

$$V = 3v_1 - 5v_2 + 2v_3$$

check where $v = (1, -2, 2)$ is a linear combination of $v_1 = (1, 2, 3)$, $v_2 = (0, 1, 2)$, $v_3 = (-1, 0, 1)$

$$(1, -2, 2) = c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)$$

$$c_1 - c_3 = 1$$

$$2c_1 + c_2 = -2$$

$$3c_1 + 2c_2 + c_3 = 2$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 2 & 4 & -1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array} \right]$$

$$c_1 - c_3 = 1$$

$$\text{or } c_2 + 2c_3 = -4$$

$$0 = 7 \text{ (not possible)}$$

$\Rightarrow$ no solⁿ

$$\text{Rank}(A) = 2$$

$$\text{Rank}(A|B) = 3$$

$$\text{Rank}(A) \neq \text{Rank}(A|B)$$

ie no solⁿ.

ie v cannot be written as L.C of v_1, v_2, v_3

Q. Express the matrix $\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

Soln Let $A = \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$

Let $A = K_1 B + K_2 C + K_3 D$, $K_1, K_2, K_3 \in \mathbb{R}$

$$\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix} = K_1 \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + K_2 \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + K_3 \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$K_1 + K_2 + 2K_3 = 5$$

$$-K_1 + K_2 + 2K_3 = 1$$

$$0K_1 + 0K_2 + K_3 = -1$$

$$3K_1 + 2K_2 + K_3 = 9$$

$$\Rightarrow \boxed{K_3 = -1}$$

$$\Rightarrow K_1 + K_2 = 3$$

$$-K_1 + K_2 = -1$$

$$\Rightarrow 3K_1 + 2K_2 = 8$$

$$\left. \begin{array}{l} K_1 + K_2 = 3 \\ -K_1 + K_2 = -1 \end{array} \right\} \Rightarrow K_1 = 2, K_2 = 1$$

ie $\boxed{A = 2B + C + D}$

Spanning set:-

Let V be a vector space and let $S = \{v_1, v_2, \dots, v_k\}$ be a set of vectors in V . Then the set of all possible linear combination of vectors in S , ~~is called span of S and is denoted by~~ is called span of S and is denoted by

$$\text{Span}(S) \text{ or } \text{Span}(v_1, v_2, \dots, v_k)$$

$$\text{span}(S) = \{c_1 v_1 + c_2 v_2 + \dots + c_k v_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}$$

Def:- If every vector in a vector space V can be written as linear combination of vectors in set S , then S is called spanning set of vector space V . we call it as S spans V or $V = \text{span}(S)$ or V is spanned by S or S is a spanning set of V .

in other words we say S spans V if for every $v \in V$, $\exists$ scalars $c_1, c_2, \dots, c_k$ such that

$$v = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$

Ex:- A subset $S = \{(1, 0), (0, 1)\}$ is a spanning set of $V = \mathbb{R}^2$.
as for $(x, y) \in \mathbb{R}^2$

$(x, y) = x(1, 0) + y(0, 1)$ for scalars x & y .
ie (x, y) is a linear combination of elements of S
 $\Rightarrow S$ span $\mathbb{R}^2$.

* A set contains two element, not necessarily spans $\mathbb{R}^2$

Ex $S_1 = \{(1, 1), (2, 2)\}$
does S_1 spans $\mathbb{R}^2$

[these vectors are colinear]



Let $(1, 2) \in \mathbb{R}^2$
if S_1 spans $\mathbb{R}^2$

$$(1, 2) = a(1, 1) + b(2, 2)$$

$$a + 2b = 1$$

$$a + 2b = 2$$

Aug matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & | & 1 \\ 1 & 2 & | & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & | & 1 \\ 0 & 0 & | & 1 \end{bmatrix} \rightarrow \text{no sol}^n$$

ie $\nexists$ a and b s.t.

$$(1, 2) = a(1, 1) + b(2, 2)$$

ie S_1 does not spans $\mathbb{R}^2$.

* $S_2 = \{(1, 1), (1, 0)\}$ spans $\mathbb{R}^2$ (check)

note:- A superset of spanning set S is also a spanning set.

Ex. Say $S = \{(1, 0), (0, 1)\}$

super set of S say $\{(1, 0), (0, 1), (1, 1)\}$ is also a spanning set of $\mathbb{R}^2$

as $(x, y) = x(1, 0) + y(0, 1) + 0 \cdot (1, 1) \quad \forall (x, y) \in \mathbb{R}^2$

Ex:- consider the vector space $V = \mathbb{R}^3$

(a) $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$
forms a spanning set of $\mathbb{R}^3$.

ie for any $v \in \mathbb{R}^3$

$$v = a e_1 + b e_2 + c e_3$$

vector

Note:- Spanning set of a n space need not to be unique.

(b) The following vectors also form a spanning set of $\mathbb{R}^3$.

$$w_1 = (1, 1, 1), w_2 = (1, 1, 0), w_3 = (1, 0, 0)$$

Let $\{w_1, w_2, w_3\}$ be spanning set of $\mathbb{R}^3$
let $v \in \mathbb{R}^3$

$$\text{then } v = a(1, 1, 1) + b(1, 1, 0) + c(1, 0, 0)$$

$$(x, y, z) = a(1, 1, 1) + b(1, 1, 0) + c(1, 0, 0)$$

~~$$a + b + c = x$$~~

$$a + b + c = x$$

~~$$a + b = y$$~~

$$a + b = y$$

~~$$a = z$$~~

$$a = z$$

$$\Rightarrow b = x - z, a = z, c = x - y$$

$$\Rightarrow a = z, b = y - z, c = x - y$$

$$\text{ie } (x, y, z) = z(1, 1, 1) + (y - z)(1, 1, 0) + (x - y)(1, 0, 0)$$

ie $\{w_1, w_2, w_3\}$ spans $V = \mathbb{R}^3$.

Note Any two vectors in $\mathbb{R}^3$, cannot span $\mathbb{R}^3$.

Note:- This is not always true that any three vectors can span $\mathbb{R}^3$

eg:- $u_1 = (1, 2, 3)$, $u_2 = (1, 3, 5)$, $u_3 = (1, 5, 9)$ do not span $\mathbb{R}^3$.

Ex:- consider the vector space
 $M = M_{2 \times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$

then

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form spanning set of vector space M .

eg $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = 1 \cdot E_{11} + 2 E_{12} + 4 E_{21} + 5 E_{22}$

Eg:- spanning set for P_n , set of all polynomials of degree $\leq n$

$$S = \{1, x, x^2, \dots, x^n\} \text{ spans } P_n.$$

Since each polynomial in P_n can be written as

$$p = a_0 + a_1 x + \dots + a_n x^n.$$

ie

$$\boxed{\text{span}\{1, x, x^2, \dots, x^n\} = P_n}$$

Q:- Determine whether the following polynomials span P_2 .

$$p_1 = 1 - x + 2x^2, \quad p_2 = 5 - x + 4x^2, \quad p_3 = -2 - 2x + 2x^2$$

Note:- ~~but~~ span of an empty set of vector space V is $\{0\}$.

ie If $S = \phi$, $\text{span } S = \{0\}$

(3.4)

Theorem:- Let S be a subset of a vector space V

(i) $\text{span}(S)$ is a subspace of V ; that contains S . ($S \subseteq \text{span}(S)$)

(ii) $\text{span}(S)$ is a smallest subspace of V that contains S .

(ie if W is a subspace of V containing S then $\text{span}(S) \subseteq W$)

Proof (i):- Let $S = \{x_1, x_2, \dots, x_n\}$ be subset of vector space V .

then $\text{span}(S) = \{a_1 x_1 + a_2 x_2 + \dots + a_n x_n : a_i \in \mathbb{R}\}$

to show $\text{span}(S)$ is subspace of V .

we need to show it is closed under vector addition and scalar multiplication.

let $u, v \in \text{span}(S)$

$$u = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$v = b_1 x_1 + b_2 x_2 + \dots + b_n x_n, \quad a_i, b_i \in \mathbb{R}$$

$$u + v = (a_1 + b_1) x_1 + (a_2 + b_2) x_2 + \dots + (a_n + b_n) x_n \\ \in \text{span}(S)$$

and let $k \in \mathbb{R}$

$$ku = k(a_1 x_1 + a_2 x_2 + \dots + a_n x_n) \\ = (ka_1) x_1 + (ka_2) x_2 + \dots + (ka_n) x_n \\ \in \text{span}(S)$$

ie $u + v$ & ku are linear combinations of $x_1, x_2, \dots, x_n$ ie $\text{span}(S)$ is a subspace of V .

[as $1 \cdot v_i = v_i \in \text{span}(S)$ ie $S \subseteq \text{span}(S)$]

Proof 2 $\text{span}(S)$ is the smallest subspace containing S .

Let W be subspace of V , that contains S .
ie $S \subseteq W$
to show $\text{span}(S) \subseteq W$

Let $u \in \text{span}(S)$
to show $u \in W$

Let $u = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$

Now $v_i \in W$, for any scalar $a_i \in \mathbb{R}(F)$

$$a_1 v_1 \in W$$

Similarly $a_2 \in \mathbb{R}$, $v_2 \in W$

$$\Rightarrow a_2 v_2 \in W$$

as W is a subspace of V

$$\text{ie } a_1 v_1 + a_2 v_2 \in W$$

$$\text{ie } a_1 v_1 + a_2 v_2 + \dots + a_n v_n \in W$$

ie for any $u \in \text{span}(S)$, $u \in W$

$$\text{ie } \text{span}(S) \subseteq W$$

ie $\text{span}(S)$ is the smallest subspace containing S .

Linear Dependence and Independence:

→ The set of vectors $\{v_1, v_2, v_3, \dots, v_p\}$ in vector space V are said to be linearly ^{dependent} if there exist scalars $c_1, c_2, \dots, c_p$, not all zero such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_p v_p = \mathbf{0}$$

otherwise, the vectors are called linearly independent. in other words

→ The set of vectors $\{v_1, v_2, \dots, v_p\}$ are said to be linearly independent if only the trivial combination gives zero i.e.

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = \mathbf{0} \text{ happens only when}$$

$$c_1 = c_2 = \dots = c_p = 0.$$

(i.e. all c_i 's are zero)

i.e. if the equation $c_1 v_1 + c_2 v_2 + \dots + c_p v_p = \mathbf{0}$ has only trivial solution ($c_1 = c_2 = \dots = c_p = 0$), vectors are linearly independent.

Note 1 :-

If a set $S = \{v_1, v_2, \dots, v_p\}$ contains a zero vector, say $v_i = \mathbf{0}$, then the set is linearly dependent, as we have following linear combination

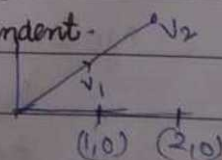
$$1 \cdot v_i + 0 \cdot v_2 + 0 \cdot v_3 + \dots + 0 \cdot v_p = \mathbf{0}$$

i.e. vectors are linearly dependent.

* two vectors v_1 and v_2 are L.D if one is the multiple of other, o.w linearly independent.

Say $v_1 = (1, 1)$ $v_2 = (2, 2)$

i.e. $v_1 - 2v_2 = \mathbf{0}$ or $v_1 = 2v_2$



Note 2:- A set containing only one vector, say v where v is non-zero vector ($v \neq 0$), is always linearly independent.
as $c \cdot v = 0 \Rightarrow c = 0$ (as $v \neq 0$)

Note 3 The singleton set $\{0\}$ is always dependent set i.e. 0 vector is always linearly dependent as $c_1 \cdot 0 = 0$ has nontrivial solution.

Note 4 If a set S of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly independent, then any subset of S say $\{v_1, v_2, \dots, v_m\}$, $m < p$ is also linearly independent.

pf:- set S is L.I. i.e. $\exists c_1, c_2, \dots, c_p$ such that
$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m + \dots + c_p v_p = 0$$

where all c_i , ($i=1$ to p) = 0

$$\Rightarrow c_1 v_1 + c_2 v_2 + \dots + c_m v_m = 0, \text{ where } c_1 = c_2 = \dots = c_m = 0$$

ie subset $\{v_1, v_2, \dots, v_m\}$ is also L.I.

Note 5:- If a set of vectors S is linearly dependent, then any superset of S is also linearly dependent.

in other words, a set containing a linearly dependent set is always linearly dependent.

pf:- let $S = \{v_1, v_2, \dots, v_m\}$ is L.D, i.e. $\exists$ scalars $c_1, c_2, \dots, c_m$ (not all zero) s.t.
$$c_1 v_1 + c_2 v_2 + \dots + c_m v_m = 0$$

Now let $S' = \{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_p\}$ be the superset of S .

we have, $c_1 v_1 + c_2 v_2 + \dots + c_m v_m + c_{m+1} v_{m+1} + \dots + c_p v_p = 0$

as scalars $c_1, c_2, \dots, c_m$, not all are zero. i.e. superset S' is linearly dependent.

Note 6:- A set S with two or more vectors is :-
linearly dependent if at least one of the vector can be written as linear combination of other vectors.
linearly independent if no vector in S is expressible as linear combination of other vectors in S .

Note 7:- If a set contains more vectors than the entries in each vector, then the set is always linearly dependent. i.e. any set of p vectors in $\mathbb{R}^n$ i.e. $S = \{v_1, v_2, \dots, v_p\}$ where $v_i \in \mathbb{R}^n$ is always linearly dependent; if $p > n$.
 [can't say anything about linear independence]

* In particular any three vectors in $\mathbb{R}^2$ will be dependent

Say: $v_1 = (3, 3)$ $v_2 = (4, 1)$ $v_3 = (3, -1.5)$

$$a(3, 3) + b(4, 1) + c(3, -1.5) = 0$$

$$\begin{bmatrix} 3 & 4 & 3 \\ 3 & 1 & -1.5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

System has free variable and hence not all a, b, c are zero i.e. vectors are dependent.

Q:- check whether the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent or not, where $v_1 = (1, -1, 0)$, $v_2 = (0, 1, -1)$, $v_3 = (0, 0, 1)$.

Solⁿ considers $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$
 $c_1 (1, -1, 0) + c_2 (0, 1, -1) + c_3 (0, 0, 1) = 0$
 $(c_1, -c_1 + c_2, -c_2 + c_3) = (0, 0, 0)$

$$c_1 = 0$$

$$-c_1 + c_2 = 0$$

$$-c_2 + c_3 = 0$$

$$\Rightarrow c_1 = 0, c_2 = 0, c_3 = 0$$

ie set of vectors is linearly independent.

Q:- Determine whether the set of vectors $v_1 = (1, 0, 2)$, $v_2 = (-1, 1, 2)$, $v_3 = (-2, 3, 1)$, $v_4 = (2, 1, 1)$ are L.I or L.D.

Solⁿ consider $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$
 $c_1 (1, 0, 2) + c_2 (-1, 1, 2) + c_3 (-2, 3, 1) + c_4 (2, 1, 1) = (0, 0, 0)$

$$\Rightarrow c_1 - c_2 - 2c_3 + 2c_4 = 0$$

$$c_2 + 3c_3 + c_4 = 0$$

$$2c_1 + 2c_2 + c_3 + c_4 = 0$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & 3 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 4 & 5 & -3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 4R_1$$

$$\begin{bmatrix} 1 & -1 & -2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -7 & -7 \end{bmatrix}$$

$$\text{ie } -7c_3 - 7c_4 = 0$$

$$c_3 + 3c_4 = 0$$

$$c_1 - c_2 - 2c_3 + 2c_4 = 0$$

$\rightarrow$ System has infinite many solution

$$\text{Let } c_4 = k$$

$$c_3 = -k$$

$$c_2 = 2k$$

$$c_1 = -2k$$

ie the set of vectors $\{v_1, v_2, v_3, v_4\}$ is ~~linearly~~ linearly dependent.

Q:- Determine whether the vectors

$v_1 = (1, 0, 1, 2)$, $v_2 = (0, 1, 1, 2)$ and $v_3 = (1, 1, 1, 3)$ are linearly independent or ~~independent~~ dependent.

Soln

consider $v_1, v_2, v_3 \in \mathbb{R}^4$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$c_1 (1, 0, 1, 2) + c_2 (0, 1, 1, 2) + c_3 (1, 1, 1, 3) = 0$$

$$c_1 + c_3 = 0$$

$$c_2 + c_3 = 0$$

$$c_1 + c_2 + c_3 = 0$$

$$2c_1 + 2c_2 + 3c_3 = 0$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon form

ie we have

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{c_3 = 0}, \quad \boxed{c_2 + c_3 = 0}, \quad \boxed{c_1 + c_3 = 0}$$

$$\Rightarrow c_1 = 0, c_2 = 0, c_3 = 0$$

ie given set of vectors v_1, v_2, v_3 are L.I.

Q:- For which real values of λ do the following vectors form a linearly dependent set.

$$v_1 = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right), \quad v_2 = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right), \quad v_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$$

Solution:- considers

$$K_1 v_1 + K_2 v_2 + K_3 v_3 = 0$$

$$\text{we have } \lambda K_1 - \frac{K_2}{2} - \frac{1}{2} K_3 = 0$$

$$-\frac{K_1}{2} + \lambda K_2 - \frac{1}{2} K_3 = 0$$

$$-\frac{K_1}{2} - \frac{1}{2} K_2 + \lambda K_3 = 0$$

vectors will be L.D.

if determinant of coeff. matrix is 0.

$$\det(A) = \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = \lambda^3 - \frac{3\lambda}{4} - \frac{1}{4}$$

$$\det(A) = 0$$

$$\lambda = 1, -\frac{1}{2}$$

i.e. for $\lambda = 1, -\frac{1}{2}$, vectors are L.D.

Q:- which of the following set of polynomials in P_2 are L.D.?

(i) $P_1 = 1 - 2x + x^2$ $P_2 = 4 - x + 3x^2$

(ii) $P_1 = 2 - x + 4x^2$ $P_2 = 3 + 6x + 2x^2$ $P_3 = 2 + 10x - 4x^2$

Soln (i) consider $c_1 P_1 + c_2 P_2 = 0$

$$c_1(1 - 2x + x^2) + c_2(4 - x + 3x^2) = 0$$

$$c_1 + 4c_2 = 0$$

$$-2c_1 - c_2 = 0$$

$$c_1 + 3c_2 = 0$$

$$\begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ -2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 \\ 0 & 7 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \sim \begin{bmatrix} 1 & 4 \\ 0 & 7 \\ 0 & 0 \end{bmatrix}$$

as P_1 and P_2 are not multiple of each other i.e. they are L.I. easy to check directly in case of two vectors

$$\Rightarrow c_2 = 0$$

$$c_1 = 0$$

i.e. P_1 & P_2 are L.I.

Q. Determine which of the following set of matrices are linearly dependent or independent.

solⁿ (a) $M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

(b) $M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, $M_4 = \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix}$

solⁿ (a) consider

$$c_1 M_1 + c_2 M_2 + c_3 M_3 = \mathbf{0}$$

$$c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$c_1 + c_2 = 0$$

$$c_1 + c_3 = 0$$

$$c_1 = 0$$

$$c_1 + 2c_2 + 2c_3 = 0$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 1 & 0 & 1 & | & 0 \\ 1 & 0 & 0 & | & 0 \\ 1 & 2 & 3 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 1 & 0 & 1 & | & 0 \\ 1 & 1 & 0 & | & 0 \\ 1 & 2 & 3 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 2 & 3 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 2 & 3 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\text{ie } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

trivial solⁿ

ie M_1, M_2, M_3 are L.I.

$$(b) 4M_1 + 5M_2 + 3M_3 + 4M_4 = 0$$

Aug. matrix

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 3 & 6 & 0 \\ 1 & 0 & 1 & 4 & 0 \\ 2 & 2 & 2 & 6 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 4 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{ie } \left[\begin{array}{cccc} 1 & 1 & 0 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left. \begin{array}{l} -2c_3 - 2c_4 = 0 \\ -c_2 + 3c_3 + 4c_4 = 0 \\ c_1 + c_2 + 2c_4 = 0 \end{array} \right\} \begin{array}{l} \text{ie infinite many} \\ \text{sol}^n \end{array}$$

ie matrices M_1, M_2, M_3, M_4 are L.O.D.

Note:- L.I sets for standard basis vectors
in $\mathbb{R}^3$ and $\mathbb{R}^n$

(a) In $\mathbb{R}^3$ $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$
 $\{e_1, e_2, e_3\}$ is always L.I.

(b) In $\mathbb{R}^n$ $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0)$,
 $e_n = (0, 0, \dots, 1)$ vectors are always L.I.

Note:- L.I set of vectors in P_n :-

$\{1, x, x^2, \dots, x^n\}$ form a L.I set of
vectors in P_n .

ie if $P_0 = 1$, $P_1 = x$, $P_2 = x^2$, $\dots$, $P_n = x^n$.
then

$$a_0 P_0 + a_1 P_1 + \dots + a_n P_n = 0$$

$$\Rightarrow a_0 = a_1 = \dots = a_n = 0$$

ie set is always L.I.

Note:- L.I set for $M_{2 \times 2}$ matrices:-

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is always L.I.