COURSE MATERIAL

MAT3003 Complex Variables and Partial Differential Equations			
Module	**Topics**	**L Hrs**	**SLO**
1	**Analytic Functions:** Complex variable-Analytic functions and Cauchy – Riemann equations - Laplace equation and Harmonic functions - Construction of Harmonic conjugate and analytic functions - Applications of analytic functions to fluid-flow and Field problems	6	1,2,7, 9

TextBooks :
Erwin Kreyszig, *Advanced Engineering Mathematics*, 9th Edition, John Wiley & Sons (Wiley student Edison) (2011).

Reference Books:

1 B. S. Grewal, *Higher Engineering Mathematics*, 42nd Edition (2013), Khanna Publishers, New Delhi.

2 G.Dennis Zill, Patrick D. Shanahan, A first course in complex analysis with applications, 2nd Edition, 2013, Jones and Bartlett Publishers Series in Mathematics: Complex-Michael, D. Greenberg, *Advanced Engineering Mathematics*, 2nd Edition, Pearson Education (2002)

3 Peter V. O' Neil, *Advanced Engineering Mathematics*, 7th Edition, Cengage Learning (2011)

4 JH Mathews, R. W. Howell, *Complex Analysis for Mathematics and Engineers*, Fifth Edition (2013), Narosa Publishers

Complex Numbers

Defi nitions

Def:1 *Complex numbers are defined as ordered pairs* $(x\ ,\ y)$

Points on a complex plane. Real axis, imaginary axis, purely imaginary numbers. Real and imaginary parts of complex number. Equality of two complex numbers.

Def:2 *The sum and product of two complex numbers are de• ned as follows:*

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\end{aligned}$$

In the rest of the chapter use $z, z_1, z_2, \ldots$ for complex numbers and x, y for real numbers. introduce i and $z = x + iy$ notation.

Algebraic Properties

1. Commutativity

$$z_1 + z_2 = z_2 + z_1, \qquad z_1z_2 = z_2z_1$$

2. Associativity

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1z_2)z_3 = z_1(z_2z_3)$$

3. Distributive Law

$$z(z_1 + z_2) = zz_1 + zz_2$$

4. Additive and Multiplicative Indentity

$$z + 0 = z, \qquad z \cdot 1 = z$$

5. Additive and Multiplicative Inverse

$$\begin{aligned}-z &= (-x, -y) \\ z^{-1} &= \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right), \qquad z \neq 0\end{aligned}$$

6. Subtraction and Division
$$z_1 - z_2 = z_1 + (-z_2)\,, \qquad \frac{z_1}{z_2} = z_1 z_2^{-1}$$

7. Modulus or Absolute Value
$$|z| = \sqrt{x^2 + y^2}$$

8. Conjugates and properties
$$\begin{aligned} \overline{z} &= x - iy = (x, -y) \\ \overline{z_1 \pm z_2} &= \overline{z_1} \pm \overline{z_2} \\ \overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2} \\ \overline{\left(\frac{z_1}{z_2}\right)} &= \frac{\overline{z_1}}{\overline{z_2}} \end{aligned}$$

9.
$$\begin{aligned} |z|^2 &= z\overline{z} \\ \operatorname{Re} z &= \frac{z + \overline{z}}{2}, \operatorname{Im} z = \frac{z - \overline{z}}{2i} \end{aligned}$$

10. Triangle Inequality
$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Polar Coordinates and Euler Formula

1. Polar Form: for $z \neq 0$,
$$z = r\,(\cos\theta + i\sin\theta)$$
where $r = |z|$ and $\tan\theta = y/x$. θ is called the argument of z. Since $\theta + 2n\pi$ is also an argument of z,the principle value of argument of z is take such that $-\pi < \theta \leq \pi$. For $z = 0$ the $\arg z$ is undefned.

2. Euler formula: Symbollically,
$$e^{i\theta} = (\cos\theta + i\sin\theta)$$

3.
$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \\ z^n &= r^n e^{n\theta} \end{aligned}$$

4. de Moivre's Formula
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Roots of Complex Numbers

Let $z = re^{i\theta}$ then

$$z^{1/n} = r^{1/n} \exp\left(i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right)$$

There are only n distinct roots which can be given by $k = 0, 1, \ldots, n-1$. If θ is a principle value of $\arg z$ then θ/n is called the principle root.

Example 1.1 *The three possible roots of* $\left(\frac{1+i}{\sqrt{2}}\right)^{1/3} = \left(e^{i\pi/4}\right)^{1/3}$ *are* $e^{i\pi/12}, e^{i\pi/12+i2\pi/3}, e^{i\pi/12+i4\pi/3}$.

Regions in Complex Plane

1. $\epsilon - nbd$ of z_0 is defined as a set of all points z which satisfy
$$|z - z_0| < \epsilon$$
2. $Deleted\ nbd$ of z_0 is a nbd of z_0 excluding point z_0.
3. Interior Point, Exterior Point, Boundary Point, Open set and closed set.
4. Domain, Region, Bounded sets, Limit Points.

Functions of Complex Variables

Functions of a Complex Variable

A *function* f de• ned on a set S is a rule that uniquely associates to each point z of S a a complex number w. Set S is called the *domain* of f and w is called the value of f at z and is denoted by $f(z) = w$.

$f(z) = f(x + iy) = u(x, y) + iv(x, y)$

$f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) = F(r, \theta)e^{i\Theta(r,\theta)}$

Example 2.1 *Write $f(z) = 1/z^2$ in $u + iv$ form.*

$u(x, y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$ *and* $v(x, y) = \frac{2xy}{(x^2+y^2)^2}$

$u(r, \theta) = r^{-2} \cos 2\theta$ *and* $v(r, \theta) = -r^{-2} \sin 2\theta$

Domain of f is $C - \{0\}$.

A *multiple-valued function* is a rule that assigns more than one value to each point of domain.

Example 2.2 $f(z) = \sqrt{z}$. *This function assigns two distinct values to each $z\ (\neq 0)$. One can choose the function to be single-valued by specifying*

$$\sqrt{z} = +\sqrt{r}e^{i\theta/2}$$

where θ is the principal value.

Elementary Functions

1. **Polynomials**

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

whrere the coeffcients are real. Rational Functions.

2. **Exponential Function**

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

Converges for all z. For real z the definition coincides with usual exponential function. Easy to see that $e^{i\theta} = \cos\theta + i\sin\theta$. Then

$$e^z = e^x(\cos y + i\sin y)$$

a. $e^{z_1}e^{z_2} = e^{z_1+z_2}$.
b. $e^{z+2\pi i} = e^z$.
c. A line segment from $(x, 0)$ to $(x, 2\pi)$ maps to a circle of radius e^x centered at origin.
d. No Zeros.

3. **Trigonometric Functions**

Define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2}$$
$$\tan z = \frac{\sin z}{\cos z}$$

a. $\sin^2 z + \cos^2 z = 1$
b. $2\sin z_1 \cos z_2 = \sin(z_1 + z_2) + \sin(z_1 - z_2)$
c. $2\cos z_1 \cos z_2 = \cos(z_1 + z_2) + \cos(z_1 - z_2)$
d. $2\sin z_1 \sin z_2 = -\cos(z_1 + z_2) + \cos(z_1 - z_2)$
e. $\sin(z + 2\pi) = \sin z$ and $\cos(z + 2\pi) = \cos z$.
f. $\sin z = 0$ iff $z = n\pi \qquad (n = 0, \pm 1, \ldots)$
g. $\cos z = 0$ iff $z = \frac{\pi}{2} + n\pi \qquad (n = 0, \pm 1, \ldots)$
h. These functions are not bounded.
i. A line segment from $(0, y)$ to $(2\pi, y)$ maps to an ellipse with semimajor axis equal to $\cosh y$ under $\sin$ function.

4. **Hyperbolic Functions**

Define

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$

Analytic Functions

Limits

A function f is defined in a deleted nbd of z_0.

Definition 3.1 *The limit of the function $f(z)$ as $z \to z_0$ is a number w_0 if, for any given $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon.$$

Example 3.1 $f(z) = 5z$. *Show that* $\lim_{z \to z_0} f(z) = 5z_0$.

Example 3.2 $f(z) = z^2$. *Show that* $\lim_{z \to z_0} f(z) = z_0^2$.

Example 3.3 $f(z) = z/\bar{z}$. *Show that the limit of f does not exist as* $z \to 0$.

Theorem 3.1 *Let* $f(z) = u(x, y) + iv(x, y)$ *and* $w_0 = u_0 + iv_0$. $\lim\limits_{z \to z_0} f(z) = w_0$ *if and only if* $\lim\limits_{(x,y) \to (x_0,y_0)} u = u_0$ *and* $\lim\limits_{(x,y) \to (x_0,y_0)} v = v_0$.

Example 3.4 $f(z) = \sin z$. *Show that the* $\lim_{z \to z_0} f(z) = \sin z_0$

Example 3.5 $f(z) = 2x + iy^2$. *Show that the* $\lim_{z \to 2i} f(z) = 4i$.

Theorem 3.2 *If* $\lim\limits_{z \to z_0} f(z) = w_0$ *and* $\lim\limits_{z \to z_0} F(z) = W_0$,

$$\begin{aligned} \lim_{z \to z_0} [f(z) + F(z)] &= w_0 + W_0; \\ \lim_{z \to z_0} f(z) F(z) &= w_0 W_0; \\ \lim_{z \to z_0} f(z) / F(z) &= \frac{w_0}{W_0} \qquad W_0 \neq 0. \end{aligned}$$

This theorem immediately makes available the entire machinery and tools used for real analysis to be applied to complex analysis. The rules for finding limits then can be listed as follows:

1. $\lim_{z \to z_0} c = c.$

2. $\lim_{z \to z_0} z^n = z_0^n.$

3. $\lim_{z \to z_0} P(z) = P(z_0)$ if P is a polynomial in z.

4. $\lim_{z \to z_0} \exp(z) = \exp(z_0)\,.$

5. $\lim_{z \to z_0} \sin(z) = \sin z_0.$

Continuity

Def: 3.2 *A function f, defined in some nbd of z_0 is continuous at z_0 if*

$$\lim_{z \to z_0} f(z) = f(z_0)\,.$$

This definition clearly assumes that the function is defined at z_0

If funtions f and g are continuous at z_0 then $f + g$, fg and f/g $(g(z_0) \neq 0)$ are also continuous at z_0.

If a function $f(z) = u(x, y) + iv(x, y)$ is continuous at z_0 then the component functions u and v are also continuous at (x_0, y_0).

Derivative

Def 3.3 *A function f, defined in some nbd of z_0 is differentiable at z_0 if*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit is called the derivative of f at z_0 and is denoted by $f'(z_0)$ or $\frac{df}{dz}(z_0)$.

Example 3.6 $f(z) = z^2$. *Show that $f'(z) = 2z$.*

Example 3.7 $f(z) = |z|^2$. *Show that this function is differentiable only at $z = \dot{0}$. In real analysis $|x|$ is not differentiable but $|x|^2$ is.*

If a function is differentiable at z, then it is continuous at z.

1 Analyticity and the Cauchy-Riemann equations

1.1 Derivation of the Cauchy-Riemann equations

Functions of the complex variable $z = x + iy$

$$w = f(z) \tag{1.1}$$

are expressed in the usual manner except that the independent variable $z = x + iy$ is complex. Thus $f(z)$ has a real part $u(x, y)$ and an imaginary part $v(x, y)$

$$f(z) = u(x, y) + iv(x, y)\,. \tag{1.2}$$

Extra difficulties appear in differentiating and integrating such functions because z varies in a plane and not on a line. For functions of a single real variable the idea of an incremental change δx along the x-axis has to be replaced by an incremental change δz. Because δz is a vector the question of the direction of this limit becomes an issue.

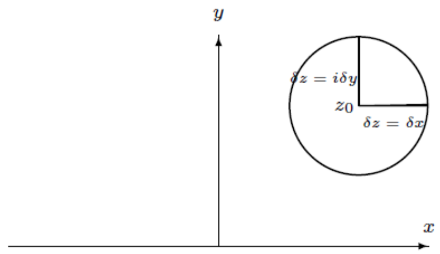
Firstly we look at the concept of differentiation. The definition of a derivative at a point z_0 remains the same as usual ; namely

$$\left.\frac{df(z)}{dz}\right|_{z=z_0} = \lim_{\delta z \to 0}\left(\frac{f(z_0 + \delta z) - f(z_0)}{\delta z}\right)\,. \tag{1.3}$$

The subtlety here lies in the limit $\delta z \to 0$ because δz is itself a vector and therefore the limit $\delta z \to 0$ may be taken in many directions. If the limit in (1.3) is to be unique (to make any sense) *it is required that it be independent of the direction in which the limit $\delta z \to 0$ is taken.* If this is the case then it is said that $f(z)$ is differentiable at the point z.

There is a general test on functions to determine whether (1.3) is independent of the direction of the limit. The simplest way is to firstly take the limit in the horizontal direction : that is $\delta z = \delta x$, in which case

$$\frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \equiv u_x + iv_x\,. \tag{1.4}$$



The z-plane with a point at z_0 and a circle of radius $|\delta z|$ around it. The horizontal radius is drawn for the case when $\delta z = \delta x$ and the vertical for the case when $\delta z = i\delta y$.

Next we take the limit in the vertical direction: that is $\delta z = i\delta y$

$$\frac{df(z)}{dz} = \frac{\partial u}{\partial(iy)} + i\frac{\partial v}{\partial(iy)} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \equiv -iu_y + iv_y\,. \tag{1.5}$$

If the limits in both directions are to be equal df/dz in (1.4) and (1.5) must be equal, which makes

$$\boxed{u_x = v_y\,, \qquad\qquad u_y = -v_x\,.} \tag{1.6}$$

The boxed pair of equations above are known as **the Cauchy-Riemann equations**. If these hold at a point z then $f(z)$ is said to be differentiable at z. There is no such requirement in single variable calculus. Moreover the CR equations bring us to a further idea regarding differentiation in the complex plane:

Definition: *If $f(z)$ is differentiable at all points in a neighbourhood of a point z_0 then $f(z)$ is said to be analytic (regular) at z_0.*

Some functions are analytic everywhere in the complex plane except at certain points: these points are called *singularities*. Three examples illustrate this.

Example 1: $f(z) = z^2$. Writing $z^2 = x^2 - y^2 + 2ixy$ we have

$$u(x,y) = x^2 - y^2\,, \qquad\qquad v(x,y) = 2xy\,. \tag{1.7}$$

Clearly four trivial partial derivatives show that $u_x = 2x$, $u_y = -2y$, $v_x = 2y$ and $v_y = 2x$ thus demonstrating that the CR equations hold for all values of x and y. It follows that $f(z) = z^2$ is differentiable at all points in the z-plane and every point in this plane has an (infinite) neighbourhood in which $f(z) = z^2$ is differentiable. Clearly $f(z) = z^2$ is analytic everywhere.

Example 2: $f(z) = z^{-1}$. Writing $z^{-1} = (x - iy)/(x^2 + y^2)$ we have

$$u(x,y) = \frac{x}{x^2 + y^2} \qquad\qquad v(x,y) = -\frac{y}{x^2 + y^2}\,. \tag{1.8}$$

Without giving the working it is not difficult to show that the CR equations hold everywhere except at the origin $z = 0$ where the limit is indeterminate: $z = 0$ is the point where it fails to be differentiable. Hence $w = z^{-1}$ is analytic everywhere except at $z = 0$.

If we write $z = re^{i\theta}$ then we can write Cauchy-Riemann Conditions in polar coordinates:

$$\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ u_\theta &= -r v_r. \end{aligned}$$

Analytic Functions

Definition 3.4 *A function is analytic in an open set if it has a derivative at each point in that set.*

Example 3.13 $f(z) = 1/z$ *is analytic at all nonzero points.*

Example 3.14 $f(z) = |z|^2$ *is not analytic anywhere.*

A function is not analytic at a point z_0, but is analytic at some point in each nbd of z_0 then z_0 is called the singular point of the function f.

Harmonic Functions

Definition 3.7 *A real valued function $H(x, y)$ is said to be* harmonic *in a domain of xy plane if it has continuous partial derivatives of the first and second order and satisfies* Laplace equation

$$H_{xx}(x,y)+H_{yy}(x,y)=0$$

Theorem 3.5 *If a function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then the functions u and v are harmonic in D.*

Definition 3.8 *If two given functions $u(x, y)$ and $v(x, y)$ are harmonic in domain D and their first order partial derivatives satisfy Cauchy-Riemann Conditions*

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

then v is said to be harmonic conjugate *of u.*

Example 3.15 *Let $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Show that v is hc of u and not vice versa.*

Example 3.16 $u(x, y) = y^3 - 3xy$. *Find harmonic conjugate of u.*

Example The function $f(z) = \bar{z}$ has $f(x+iy) = x - iy$ so that

$$u(x,y) = x \text{ and } v(x,y) = -y$$

The first order partial derivatives of u and v are

$$\begin{aligned} u_x(x,y) &= 1 \quad & v_x(x,y) &= 0 \\ u_y(x,y) &= 0 \quad & v_y(x,y) &= -1 \end{aligned}$$

As the Cauchy–Riemann equation $u_x(x,y) = v_y(x,y)$ is satisfied nowhere, the function $f(z) = \bar{z}$ is differentiable nowhere. We have already seen this in Example 1.

Example The function $f(z) = e^{z}$ has

$$f(x+iy) = e^{x+iy} = e^x\big\{\cos y + i\sin y\big\} = u(x,y) + iv(x,y)$$

with

$$u(x,y) = e^x \cos y \text{ and } v(x,y) = e^x \sin y$$

The first order partial derivatives of u and v are

$$\begin{aligned} u_x(x,y) &= e^x \cos y \qquad & v_x(x,y) &= e^x \sin y \\ u_y(x,y) &= -e^x \sin y \quad & v_y(x,y) &= e^x \cos y \end{aligned}$$

As the Cauchy–Riemann equations $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$ are satisfied for all (x,y), the function $f(z) = e^z$ is entire and its derivative is

$$f'(z) = f'(x+iy) = u_x(x,y) + iv_x(x,y) = e^x \cos y + ie^x \sin y = e^z$$

Example The function $f(x+iy) = x^2 + y + i(y^2 - x)$ has

$$u(x,y) = x^2 + y \text{ and } v(x,y) = y^2 - x$$

The first order partial derivatives of u and v are

$$\begin{aligned} u_x(x,y) &= 2x \quad & v_x(x,y) &= -1 \\ u_y(x,y) &= 1 \quad & v_y(x,y) &= 2y \end{aligned}$$

As the Cauchy–Riemann equations $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$ are satisfied only on the line $y = x$, the function f is differentiable on the line $y = x$ and nowhere else. So it is nowhere analytic.

Example The function $f(x+iy) = x^2 - y^2 + 2ixy$ has

$$u(x,y) = x^2 - y^2 \text{ and } v(x,y) = 2xy$$

The first order partial derivatives of u and v are

$$\begin{aligned} u_x(x,y) &= 2x & v_x(x,y) &= 2y \\ u_y(x,y) &= -2y & v_y(x,y) &= 2x \end{aligned}$$

As the Cauchy–Riemann equations $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$ are satisfied for all (x,y), this function is entire. There is another way to see this. It suffices to observe that $f(z) = z^2$, since $(x+iy)^2 = x^2 - y^2 + 2ixy$. So f is a polynomial in z and we already know that all polynomials are differentiable everywhere.

Example The function $f(x+iy) = x^2 + y^2$ has

$$u(x,y) = x^2 + y^2 \text{ and } v(x,y) = 0$$

The first order partial derivatives of u and v are

$$\begin{aligned} u_x(x,y) &= 2x & v_x(x,y) &= 0 \\ u_y(x,y) &= 2y & v_y(x,y) &= 0 \end{aligned}$$

As the Cauchy–Riemann equations $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$ are satisfied only at $x = y = 0$, the function f is differentiable only at the point $z = 0$. So it is nowhere analytic. There is another way to see that $f(z)$ cannot be differentiable at any $z \neq 0$. Just observe that $f(z) = z\bar{z}$. If $f(z)$ were differentiable at some $z_0 \neq 0$, then $\bar{z} = \frac{f(z)}{z}$ would also be differentiable at z_0 and we already know that this is not case.

Theorem 17.2. *Let $z = re^{i\theta}$. If $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$ is differentiable at $z_0 = r_0 e^{i\theta_0}$, then the Cauchy-Riemann equations in polar coordinates are satisfied at z_0; that is,*

$$\frac{\partial U}{\partial r}(r_0,\theta_0) = \frac{1}{r_0}\frac{\partial V}{\partial \theta}(r_0,\theta_0) \quad \textit{and} \quad \frac{1}{r_0}\frac{\partial U}{\partial \theta}(r_0,\theta_0) = -\frac{\partial V}{\partial r}(r_0,\theta_0).$$

Summary. The Cauchy-Riemann equations in polar coordinates can be remembered as

$$\boxed{U_r = \frac{1}{r}V_\theta \quad \text{and} \quad \frac{1}{r}U_\theta = -V_r.}$$

Example : Suppose that $U(r,\theta) = r^n \cos(n\theta)$ and $V(r,\theta) = r^n \sin(n\theta)$. We find

$$U_r = nr^{n-1}\cos(n\theta)$$
$$V_\theta = nr^n \cos(n\theta)$$

and

$$U_\theta = -nr^n \sin(n\theta)$$
$$V_r = nr^{n-1}\sin(n\theta)$$

so that $U_r = r^{-1}V_\theta$ and $r^{-1}U_\theta = -V_r$. Hence, U and V satisfy the Cauchy-Riemann equations in polar coordinates.

We can now use the Cauchy-Riemann equations to derive Laplace's equation in polar coordinates. (Assume that all second partials exist and are sufficiently smooth so that the mixed partials are equal.) That is, we know

$$u_x = v_y \quad \text{implies} \quad rU_r = V_\theta \quad \text{and} \quad u_y = -v_x \quad \text{implies} \quad U_\theta = -rV_r$$

and so taking derivatives with respect to x of the first equation and derivatives with respect to y of the second equation implies

$$0 = (u_x - v_y)_x + (u_y + v_x)_y = (rU_r - V_\theta)_x + (U_\theta + rV_r)_y.$$

Now, using the chain rule, we find

$$(rU_r - V_\theta)_x = r_x U_r + r(U_{rr}r_x + U_{\theta r}\theta_x) - (V_{\theta\theta}\theta_x + V_{r\theta}r_x)$$

and

$$(U_\theta + rV_r)_y = (U_{\theta\theta}\theta_y + U_{r\theta}r_y) + r_yV_r + r(V_{rr}r_y + V_{\theta r}\theta_y).$$

Adding the previous two terms, using the equality of the mixed partials, and simplifying implies

$$r_xU_r + rr_xU_{rr} + (r\theta_x + r_y)U_{\theta r} + \theta_yU_{\theta\theta} = -r_yV_r - rr_yV_{rr} - (r\theta_y - r_x)V_{r\theta} + \theta_xV_{\theta\theta}. \qquad (*)$$

The next step is to note that

$$r\theta_x + r_y = r \cdot -\frac{\sin\theta}{r} + \sin\theta = 0 \quad \text{and} \quad r\theta_y - r_x = r \cdot \frac{\cos\theta}{r} - \cos\theta = 0.$$

so that $(*)$ becomes

$$r_xU_r + rr_xU_{rr} + \theta_yU_{\theta\theta} = -r_yV_r - rr_yV_{rr} + \theta_xV_{\theta\theta}.$$

Substituting in r_x, θ_x, r_y, θ_y, we conclude

$$\cos\theta\left[U_r + rU_{rr} + \frac{1}{r}U_{\theta\theta}\right] = -\sin\theta\left[V_r + rV_{rr} + \frac{1}{r}V_{\theta\theta}\right]. \qquad (\dagger)$$

If, instead, at the beginning of the derivation we had taken derivatives with respect to y of the first equation and derivatives with respect to x of the second equation, then we would have found

$$\cos\theta\left[V_r + rV_{rr} + \frac{1}{r}V_{\theta\theta}\right] = -\sin\theta\left[U_r + rU_{rr} + \frac{1}{r}U_{\theta\theta}\right]. \qquad (\ddagger)$$

We now multiple $(\dagger)$ by $\cos\theta$, multiply $(\ddagger)$ by $\sin\theta$, and add, then we conclude

$$(\cos^2\theta + \sin^2\theta)\left[U_r + rU_{rr} + \frac{1}{r}U_{\theta\theta}\right] = 0$$

and so we finally arrive at Laplace's equation in polar coordinates

$$\boxed{U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0.}$$

Note that we can also conclude immediately that V satisfies Laplace's equation in polar coordinates as well,

$$V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\theta\theta} = 0.$$

Example: Suppose that $U(r,\theta) = r^n\cos(n\theta)$. We can now show directly that U is harmonic. That is,

$$U_r = nr^{n-1}\cos(n\theta), \quad U_{rr} = n(n-1)r^{n-2}\cos(n\theta), \quad U_\theta = -nr^n\sin(n\theta), \quad U_{\theta\theta} = -n^2r^n\cos(n\theta)$$

so that

$$\begin{aligned} U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} &= n(n-1)r^{n-2}\cos(n\theta) + \frac{1}{r}\cdot nr^{n-1}\cos(n\theta) + \frac{1}{r^2}\cdot -n^2r^n\cos(n\theta) \\ &= r^{n-2}\cos(n\theta)[n(n-1) + n - n^2] \\ &= 0 \end{aligned}$$

Properties of analytic functions

Let us consider the CR equations $u_x = v_y$ and $u_y = -v_x$ as a condition for the analyticity of a function $w = u(x, y) + iv(x, y)$. Cross differentiation and elimination of first u and then v gives

$$u_{xx} + u_{yy} = 0 \qquad v_{xx} + v_{yy} = 0\,,$$

thus showing that u and v must always be a solution of Laplace's equation (without boundary conditions): these are called **harmonic functions**. It also said that $u(x, y)$ and $v(x, y)$ are **conjugate** to one another. In the following set of examples it will be shown how, given a harmonic function $u(x, y)$, its conjugate $v(x, y)$ can be constructed. The pair can then put together as $u + iv = f(z)$ to ultimately find $f(z)$.

Example : Given that $u = x^2 - y^2$ show (i) that it is harmonic; (ii) find $v(x, y)$ and then (iii) construct the corresponding complex function $f(z)$.

With $u = x^2 - y^2$ we have $u_x = 2x$, $u_{xx} = 2$, $u_y = -2y$ and $u_{yy} = -2$. Therefore $u_{xx} + u_{yy} = 0$ so it satisfies Laplace's equation. This is a sufficient condition for v to exist and for us to write $v_y = u_x = 2x$ and $v_x = -u_y = 2y$. While there are two PDEs here there can only be one solution compatible with both. Integrating them both in turn gives

$$v = 2xy + A_(x)\,, \qquad v = 2xy + B(y)\,.$$

It is clear that they are compatible if $A(x) = B(y) = \text{const} = c$ making the result

$$v = 2xy + c\,,$$

with

$$f(z) = x^2 - y^2 = 2ixy + ic = z^2 + ic\,.$$

The ic simply moves $f(z)$ an arbitrary distance along the imaginary axis.

Example : Given that $u = x^3 - 3xy^2$ find its conjugate function $v(x,y)$ and the corresponding complex function $f(z)$.

We first check that $u = x^3 - 3xy^2$ satisfies Laplace's equation: $u_x = 3x^2 - 3y^2$; $u_{xx} = 6x$; $u_y = -6xy$ and $u_{yy} = -6x$. Thus $u_{xx} + u_{yy} = 0$ and so v exists and is found from the CR equations:

$$v_y = 3x^2 - 3y^2 \qquad v_x = 6xy .$$

Partially integrating these gives

$$v = 3x^2y - y^3 + A(x) \qquad v = 3x^2y + B(y) .$$

The way to make these compatible is to choose $B(y) = -y^3 + c$ and $A(x) = c$ finally giving

$$v = 3x^2y - y^3 + c$$

with

$$\begin{aligned} f(z) &= x^3 - 3xy^2 + i(3x^2y - y^3 + c) \\ &= z^3 + ic . \end{aligned}$$

Example 3: Given that $u = e^x\big(x\cos y - y\sin y\big)$ show that it satisfies Laplace's equation. Also find its conjugate v and then $f(z)$.

We find that

$$u_{xx} = e^x\big[(x+2)\cos y - y\sin y\big] ; \qquad u_{yy} = -e^x\big[(x+2)\cos y - y\sin y\big] ,$$

and so Laplace's equation is satisfied. Then

$$v_y = u_x = e^x\big[(x+1)\cos y - y\sin y\big] ; \qquad v_x = -u_y = e^x\big[(x+1)\sin y + y\sin y\big] .$$

Using the indefinite integrals $\int y\sin y\,dy = \sin y - y\cos y$ and $\int x\,e^x dx = e^x(x-1)$ we find

$$v = e^x\big(x\sin y + y\cos y\big) + A(x) ; \qquad v = e^x\big(y\cos y + x\sin y\big) + B(y) .$$

For compatibility we take $A(x) = B(y) = \text{const} = c$. Then

$$\begin{aligned} w &= e^x\big[(x+iy)\cos y - (y-ix)\sin y\big] + ic \\ &= e^x\big[z\cos y + iz\sin y\big] + ic \\ &= ze^{x+iy} + ic \\ &= ze^z + ic . \end{aligned}$$

Orthogonality

Let us finally consider the family of curves on which $u =$ const. From the chain rule

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \tag{1.25}$$

and therefore on curves of constant u we have $du = 0$, giving the gradient on this family as

$$\left.\frac{dy}{dx}\right|_{u=const} = -\frac{u_x}{u_y}\,. \tag{1.26}$$

Likewise, on the family of curves of constant v

$$\left.\frac{dy}{dx}\right|_{v=const} = -\frac{v_x}{v_y} \tag{1.27}$$

giving

$$\left.\frac{dy}{dx}\right|_{u=const} \times \left.\frac{dy}{dx}\right|_{v=const} = \frac{v_x u_x}{v_y u_y}\,. \tag{1.28}$$

Now if $f(z)$ is analytic in a region R then the CR equations hold there, $u_x = v_y$ and $u_y = -v_x$, and (1.28) becomes

$$\left.\frac{dy}{dx}\right|_{u=const} \times \left.\frac{dy}{dx}\right|_{v=const} = -1\,. \tag{1.29}$$

The final result is that in regions of analyticity curves of constant u and curves of constant v are always orthogonal.

Construction of An Analytic Function When real or Imaginary part is Given

(Putting in Exact differential M dx + N dy = 0)

The Cauchy-Riemann equations provide a method of constructing an analytic function $f(z) = u+iv$ when u or v or $u \pm v$ is given. Suppose u is given, we determine the differential dv , since $v = v(x,y)$.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$ using C-R equations ,this becomes

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = M.dx + N\,dy$$

And it is clear that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Because u is harmonic. This shows that M dx + N dy is an exact differential. Consequently , v can be obtained by integrating M w.r.t. x by treating y as a constant and integrating w.r.t. y only those terms in N that do not contain x, and adding the results.

Similarly, if v is given then by using

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy.$$

Following the procedure explained above we find u, and hence f(z) u + iv can be obtained. Analogous procedure is adopted to find u+iv when $u \pm v$ is given.

Milne-Thomson Method:

An alternative method of finding $u \pm iv$ when u or v or $u \pm v$ is given.

Suppose we are required to find an analytic function $f(z) = u + iv$ when u is given. We recall that

$$f'(z) = \left[\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right] \quad \text{-----------------------(I)}$$

Let us we set $\frac{\partial u}{\partial x} = \phi_1(x, y)$ and $\frac{\partial u}{\partial y} = \phi_2(x, y)$ -------(II)

Then $f'(z) = \phi_1(x, y) - i\phi_2(x, y)$ --------(III)

Replacing x by z and y by 0, this becomes

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0) \quad \text{-------(IV)}$$

From which the required analytic function f(z) can be got.
Similarly , if v is given we can find the analytic function $f(z) = u + iv$ by starting with

$f'(z) = \left[\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}\right]$ Analogous procedure is used when $u \pm v$ is given.

Applications to flow problems:

As the real and imaginary parts of an analytic function are the solutions of the Laplace's equation in two variable. The conjugate functions provide solutions to a number of field and flow problems.

Let v be the velocity of a two dimensional incompressible fluid with irrigational motion, $V = \frac{\partial v}{\partial x} i + \frac{\partial v}{\partial y} j$ -----------------------------(1)

Since the motion is irrotational curl V= 0.
Hence V can be written as

$$\nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j \quad \text{----------------------------(II)}$$

Therefore , ϕ is the velocity component which is called the velocity potential. From (I) and (II) we have

$$\frac{\partial v}{\partial x}=\frac{\partial \phi}{\partial x}\ ,\ \frac{\partial v}{\partial y}=\frac{\partial \phi}{\partial y} \qquad \text{-------------------------(III)}$$

Since the fluid is incompressible div $V = 0$.

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)=0 \qquad \text{------------------------(IV)}$$

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0$$ This indicates that ϕ is harmonic.

The function $\phi(x, y)$ is called the velocity potential , and the curves $\phi(x, y) = c$ are known as equi -potential lines.

Note : The existence of conjugate harmonic function $\psi(x, y)$ so that

$w(z) = \phi(x, y) + i\,\psi(x, y)$ is Analytic.

The slope is Given by $\dfrac{dy}{dx}=-\dfrac{\left(\partial \psi / \partial x\right)}{\left(\partial \psi / \partial y\right)}=\dfrac{\partial \phi / \partial y}{\partial \phi / \partial x}=\dfrac{v_y}{v_x}$

This shows that the velocity of the fluid particle is along the tangent to the curve $\psi(x, y) = c^1$, the particle moves along the curve.

$\psi(x, y) = c^1$ - is called stream lines $\phi(x, y) = c$ - called equipotential lines. As the equipotential lines and stream lines cut orthogonally.

$$w(z) = \phi(x, y) + i\psi(x, y)$$

$$\frac{dw}{dz}=\frac{\partial \phi}{\partial x}+i\frac{\partial \psi}{\partial x}=\frac{\partial \phi}{\partial x}-i\frac{\partial \phi}{\partial y}$$

$$= v_x - v_y$$

The magnitude of the fluid velocity $\sqrt{\left(v_x^2 + v_y^2\right)} = \left|\frac{dw}{dz}\right|$

The flow pattern is represented by function w(z) known as complex potential.

Complex potential w(z) can be taken to represent other two-dimensional problems. (steady flow)

1. In electrostatics $\phi(x, y) = c$ --- interpreted as equipotential lines.
   $\psi(x, y) = c^1$ --- interpreted as Lines of force

2. In heat flow problems:
   $\phi(x, y) = c$ --- Interpreted as Isothermal lines
   $\psi(x, y) = c^1$ --- interpreted as heat flow lines.

Cauchy –Riemann equations in polar form:

Let $f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ be analytic at a point z, then there exists four continuous first order partial derivatives ,

$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ and satisfy the equations

$, \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} ; \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$

Proof: The function is analytic at a point $z = re^{i\theta}$.

$f^1(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$ exists and it is unique.

Now $f(z) = u(r, \theta) + iv(r, \theta)$.

Let Δz be the increment in z , corresponding increments are $\Delta r, \Delta\theta$ in r and θ.

$$f'(z) = \lim_{\Delta z \to 0} \frac{\{u(r+\Delta r, \theta+\Delta\theta) + i\, v(r+\Delta r, \theta+\Delta\theta)\} - \{u(r.\theta) + i\, v(r,\theta)\}}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{\{u(r+\Delta r, \theta+\Delta\theta) - u(r.\theta)\}}{\Delta z} + i \lim_{\Delta z \to 0} \frac{\{v(r+\Delta r, \theta+\Delta\theta) - v(r.\theta)\}}{\Delta z}$$

--------------------(I)

Now $z = re^{i\theta}$ and z is a function two variables r and θ, then we have

$$\Delta z = \frac{\partial z}{\partial r}\Delta r + \frac{\partial z}{\partial \theta}\Delta\theta.$$

$$\Delta z = \frac{\partial}{\partial r}(re^{i\theta})\Delta r + \frac{\partial}{\partial \theta}(re^{i\theta})\Delta\theta$$

$$\Delta z = e^{i\theta}\,\Delta r + i\, r\, e^{i\theta}\,\Delta\theta$$

When Δz tends to zero, we have the two following possibilities.

(I). Let $\Delta\theta = 0$, so that $\Delta z = e^{i\theta}\Delta r$

And $\Delta Z \to 0$, implies $\Delta r \to 0$

$$f'(z) = \lim_{\Delta r \to 0} \frac{\{u(r+\Delta r, \theta) - u(r.\theta)\}}{e^{i\theta}\Delta r} + i \lim_{\Delta r \to 0} \frac{\{v(r+\Delta r, \theta) - v(r.\theta)\}}{e^{i\theta}\Delta r}$$

The limit exists,

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \quad \text{-----------------(I)}$$

2. Let $\Delta r = 0$, so that $\Delta z = i\, r\, e^{i\theta}$

And $\Delta z \to 0$, imply $\Delta\theta \to 0$

$$f'(z) = \lim_{\Delta\theta \to 0} \frac{\{u(r, \theta+\Delta\theta) - u(r.\theta)\}}{i\, r\, e^{i\theta}\Delta\theta} + i \lim_{\Delta z \to 0} \frac{\{v(r, \theta+\Delta\theta) - v(r.\theta)\}}{i r\, e^{i\theta}\Delta\theta}$$

$$Tan\phi_1 Tan\phi_2 = \frac{\left(r\,{}^{\partial u}\!/_{\partial r}\right)\left(r\,{}^{\partial v}\!/_{\partial r}\right)}{\left({}^{\partial u}\!/_{\partial\theta}\right)\left({}^{\partial v}\!/_{\partial\theta}\right)}$$

By C-R Equations $ru_r = v_\theta$, $rv_r = -u_\theta$

The equation reduces to

$$Tan\phi_1 Tan\phi_2 = \frac{\left({}^{\partial v}\!/_{\partial\theta}\right)\left(-{}^{\partial u}\!/_{\partial\theta}\right)}{\left({}^{\partial v}\!/_{\partial\theta}\right)\left({}^{\partial u}\!/_{\partial\theta}\right)} = -1$$

Hence the polar family of curves $u(r,\theta) = c_1$ and $v(r,\theta) = c_2$, intersect orthogonally.

<u>Construction of An Analytic Function When real or Imaginary part is Given(Polar form.)</u>

The method due to Exact differential and Milne-Thomson is explained in earlier section .

Ex: Verify that $u = \frac{1}{r^2}(\cos 2\theta)$ is harmonic . find also an analytic function.

Soln: $\frac{\partial u}{\partial r} = \left(-\frac{2}{r^3}\right)\cos 2\theta$: $\frac{\partial u}{\partial\theta} = \left(-\frac{2}{r^2}\right)\sin 2\theta$

$$\frac{\partial^2 u}{\partial r^2} = \frac{6}{r^4}\cos 2\theta \quad : \quad \frac{\partial^2 u}{\partial\theta^2} = -\frac{4}{r^2}\cos 2\theta.$$

Then the Laplace equation in polar form is given by,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{6}{r^4}\cos 2\theta - \left(\frac{2}{r^4}\right)\cos 2\theta - \frac{4}{r^2}\cos 2\theta = 0$$

Hence u-satisfies the laplace equation and hence is harmonic.

Let us find required analytic function f(z) = u+iv.
We note that from the theory of differentials,

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$$

Using C-R equations $ru_r = v_\theta$, $rv_r = -u_\theta$

$$= \left(-\frac{1}{r}\frac{\partial u}{\partial \theta} \right) dr + \left(r\frac{\partial u}{\partial r} \right) d\theta$$

$$= \left(-\frac{2}{r^3} \right) \sin 2\theta \ dr - \left(\frac{2}{r^3} \cos 2\theta \right) d\theta$$

$$= d\left(-\frac{1}{r^2} \sin 2\theta \right)$$

From this $v = -\frac{1}{r^2} \sin 2\theta + c$

$$f(z) = u + iv = \left(\frac{1}{r^2} \cos 2\theta \right) + i\left(-\frac{1}{r^2} \sin 2\theta \right) + c$$

$$= \frac{1}{r^2} \left[\cos 2\theta - i\sin 2\theta \right] + ic$$

$$= \frac{1}{r^2} e^{-2i\theta} + ic = \frac{1}{(r e^{i\theta})^2} + ic$$

$$f(z) = \frac{1}{z^2} + ic.$$

Ex 2:Find an analytic function f(z)= u+iv given that

$$v = \left(r - \frac{1}{r} \right) \sin \theta \quad r \neq 0$$

Example: Given $u = e^x(x \cos y - y \sin y)$, find $f(z)$ by Milne Thomson method.

Solution: Now $u = e^x(x \cos y - y \sin y)$

$$\therefore \quad \frac{\partial u}{\partial x} = e^x u + e^x \cos y$$

$$\frac{\partial u}{\partial y} = e^x[-x \sin y - \sin y - y \cos y]$$

$$\Rightarrow \quad u_x \text{ at } y = 0 = e^x x + e^x = e^x (x + 1)$$

$$u_y \text{ at } y = 0 = 0$$

$$\therefore \quad \varphi_1(x, 0) = u_x \text{ at } y = 0 = e^x (x + 1)$$

$$\varphi_2(x, 0) = u_y \text{ at } y = 0 = 0.$$

$$\Rightarrow \quad f(z) = \int \{\varphi_1 (z, 0) - i\varphi_2 (z, 0)\} \, dz + c$$

$$f(z) = \int \{e^2 (z + 1) - \text{i}.0\} \, dz + c$$

$$\therefore \quad f(z) = ze^2 + c$$

Exercise

Q. 1 Find most general analytic function corresponding to

$$u(x, y) = y^3 - 3x^2 y$$

Q. 2 Prove that each of these functions is entire

(a) $f(z) = 3x + y + i(3y - x)$

(b) $f(z) = e^{-y} (\cos x + i \sin x)$

Q. 3 Show why each of these functions is nowhere analytic.

(a) $f(z) = xy + iy$

(b) $f(z) = e^y (\cos x + i \sin x)$

Q. 4 If in some domain $f = u + iv$ and its complex conjugate $\bar{f} = u - iv$ are both analytic, then prove that f is constant.

Q. 5 In the domain $r > 0$, $0 < \theta < 2\pi$, show that $u = \log r$ is harmonic and find its harmonic conjugate.

Q. 6 Determine a and b such that the given functions are harmonic and find a conjugate harmonic.

(a) $u = ax^3 + by^3$

(b) $u = e^{ax} \cos 5y$

Q. 7 Find analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that

$$v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$$

Q. 8 Show that $f(z) = \sqrt{|xy|}$ satisfies Cauchy-Riemann equation at the origin but is not analytic at that point.

Q. 9 Prove that $\dfrac{x - iy}{x^2 + y^2}$ is analytic or not.

Q. 10 For what values of z do the function defined by the following equation ceases to be analytic. $f(z) = \dfrac{1}{z^2 - 1}$.

Q. 11 Show that $u = \dfrac{1}{2} \log (x^2 + y^2)$ is harmonic. Find its harmonic conjugate v.

Answers

Q. 1 $f(z) = i(z^3 + C)$ **Q. 5** $v = \theta + C$

Q. 6 (a) $a = b = 0$, $v = C$

(b) $a = \pm 5$, $v = \pm e^{\pm 5x} \sin 5y + C$

Q. 7 $u = -r^2$, $\sin 2\theta + r \sin \theta + C$

$f(z) = u + iv = i(r^2 e^{2i\theta} - re^{i\theta}) + 2i + C.$

Q. 9 Not analytic **Q. 10** $z = \pm 1$

Q. 11 $v(x, y) = \tan^{-1} y/x + C.$

Miscellaneous Solved Exercise 2

Q. 1 Prove that $\bar{z}$ is not differentiable

Sol. Let $f(z) = \bar{z} = x - iy$

$\Delta z = \Delta x + i\Delta y$, so we have

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{(\overline{z + \Delta z}) - \bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta 2} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim \cdot \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$ and $\Delta \to 0$, we get from above we have Limit al +1 and when $\Delta x = 0$ and $\Delta y \to 0$, we get Limit al –1. Hence the Limit comes out different along different paths. Thus Limit does not exist and so $\bar{z}$ is not differentiable.

Q. 2 Determine whether the following functions are continuous inside a unit circle

(a) $\frac{1}{1+z^2}$ (b) $\frac{1}{z-1}$.

Sol: (a) $f(z) = \frac{1}{1+z^2}$ is continuous except at where $1 + z^2$ is zero, that is, a point $z = \pm i$, for unit circle $|z| < 1$, $z = \pm i$ are excluded. Thus the given function is continuous inside $|z| < 1$.

(b) Similar method is applied for $f(z) = \frac{1}{z-1}$. This one is also continuous except at $z = 1$ but $|z| < 1$ as the domain. Thus $\frac{1}{z-1}$ is continuous in $|z| < 1$.

Q. 3 Is $f(z) = z/|z|$ continuous at origin, where $f(z)$ is defined for $z \neq 0$ and $f(0) = 0$.

Sol: Now, $\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{z}{|z|}$

$$= \lim_{z \to 0} \frac{x+iy}{\sqrt{x^2+y^2}} = \lim_{x \to iy} \frac{x+iy}{\sqrt{x^2+y^2}} +$$

Let first $y = 0$ and $x \to 0$, then $\lim_{x \to iy} \frac{x}{x} = 1$

Again when $x = 0$ and $y \to 0$ $\lim \frac{z}{|z|} = i$

As the Limit along different paths are 1 and i, that is, limit does not exist. $f(z)$ is dis continuous at $z_0 = 0$.

Q. 4 Show that $f(z) = \text{Re}z = x$ is continuous but not differentiable

Sol: As $\lim_{z \to z0} f(z) = \lim_{z \to z0} = \lim_{\substack{x \to x_0 \\ y \to y_0}} x = x_0 = f(z_0)$

the function is continuous at $z = z_0$.

$$\text{Now} \qquad f'(z) = \lim_{\Delta_z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta_x \to 0 \\ \Delta_y \to 0}} \frac{\Delta x}{\Delta x + i\Delta y} = 0$$

$$\text{while} \qquad f'(z) = \lim_{\substack{\Delta y \to 0 \\ \Delta_x \to 0}} \frac{\Delta x}{\Delta x + i\Delta y} = 1$$

Hence Limit does not exist, that $f'(z) = 0$ and 1 along different paths. Thus $f(z)$ is not differentiable.

Q. 5 If $\lim_{z \to z0} f(z)$ exist, prove that it must be unique

Sol: Let $\lim_{z \to z0} f(z) = l_1$ and $\lim_{z \to z0} f(z) = l_2$

then for uniqueness we must show that $l_1 = l_2$. Now for given $\in > 0$, we can find a number $\delta > 0$ such that

$$|f(z) - l_1| < \in/2 \text{ when } 0 < |z - z_0| < \delta$$

$$\text{and} \quad |f(z) - l_2| < \in/2 \text{ when } 0 < |z - z_0| < \delta$$

$$\begin{aligned} \text{Then} \quad |l_1 - l_2| &= |l_1 - f(z) + f(z) - l_2| \\ &\le |l_1 - f(z)| + |-l_2 + f(z)| \\ &\le \in/2 + \in/2 \\ &= \in \end{aligned}$$

$\Rightarrow |l_1 - l_2| < \in$ (very-very small number) and so must be zero. Thus we have $l_1 = l_2$ Hence we get the derived result.

Q. 6 If $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$, Prove that $\lim_{n \to \infty} (a_n + b_n) = A + B$.

Sol: In view of the definition and for given $\in$ we can find N. such that

$$|a_n - A) < \in/2, \ |b_n - B| < \in/2 \text{ for } n > N$$

Thus for $n > N$

$|a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B) \leq |a_n - A| + |b_n - B|$
$\leq \in/2 + \in/2 = \in$

which gives the desired result.

Q. 7 Is $f(z) = z^3$ analytic?

Sol: We have $f(z) = z^3 = (x + iy)^3$
$= x^2 - 3xy^2 + i(3x^2y - y^3)$

Thus $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

Also $u_x = 3x^2 - 3y^2 = v_y$ and $u_y = -6xy = -v_x$

Thus Cauchy-Riemann equation is satisfied for every z. Hence $f(z) = z^3$ analytic for all z.

Q. 8 Prove that an analytic function of constant absolute value is constant.

Sol: Given $f(z)$ is analytic and $|f(z)| = k$ (constant). Now we have to show that $f(z)$ is also constant. since $|f(z)| = k \Rightarrow u^2 + v^2 = k^2$

Thus we have from $u^2 + v^2 = k^2$, $uu_x + vv_x = 0$ and $uu_y + vv_y = 0$.

As $u_x = v_y$ and $u_y = -v_x$. We have from these equations and above equations

$$(u^2 + v^2)u_x = 0 \text{ and } (u^2 + v^2)u_y = 0$$

If $k^2 = u^2 + v^2 = 0$ then $u = v = 0 \Rightarrow (f(z) = 0$. and if $k \neq 0$, then $u_x = u_y = 0$. Hence by $C - R$ equations also $v_x = v_y = 0$. This gives us that u = const. and v = cons. Hence $f(z)$ is constant.

Q. 9 Prove that e^z is and entire function.

Sol: Now $f(z) = e^z = e^{x + iy}$
$= e^x [\cos y + i \sin y]$

Thus we have $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$

$\Rightarrow \qquad u_x = e^x \cos y = u_y$

and $\qquad u_y = -e^x \sin y = -u_x$.

That is, $C - R$ equations are satisfied for all x and y. Therefore e^z is analytic everywhere. Thus e^z is an entire function.

Q. 10 If $u(r, \theta) = \left(r - \frac{1}{r}\right) \sin \theta$, $r \neq 0$, find an analytic function $f(z) = u + iv$.

sol: In view of $C - R$ equation in polar form, $u_r = u_\theta$ and $u_\theta = -r\, u_r$, we have from the given $u(r, \theta)$,

$$u_\theta = -ru_r = -r\left(1 + \frac{1}{r^2}\right)\sin \theta = -\left(1 + \frac{1}{r^2}\right)\sin \theta$$

Integrating above w.r.t. θ, we get

$$u(r, \theta) = \left(r + \frac{1}{r}\right)\cos \theta + c(r)$$

Now differentiating above w.r.t. r and using $u_r = \frac{1}{r} u_\theta$, we get

$$\frac{dc}{dr} = 0 \Rightarrow c(r) = \text{constant} = c_1$$

Hence $\qquad u(r, \theta) = \left(r + \frac{1}{r}\right)\cos \theta + c_1$

Thus $\qquad f(z) = u\,(r, \theta) + iv(r, \theta) + c$

Q. 11 For what values of z the function w defined by the equation ceases to be analytic?

$$z = \log r + i\,\theta,\ w = re^{i\theta}$$

Sol: Given $\qquad z = \log r + i\,\theta \qquad$ ...(1)

$\qquad w = re^{i\theta} = r\,(\cos \theta + i \sin \theta) \qquad$ ...(2)

from the above equations (1) we have

$$z = \log re^{i\theta} \Rightarrow re^{i\theta} = e^z$$

$\Rightarrow \qquad e^z = w$ from (2)

$$\Rightarrow \qquad \frac{dw}{dz} = e^z = re^{i\theta}$$

This explains that w will be analytic of z if r is finite. Hence w is analytic function in a finite domain. Thus the given function ceases to be analytic if $r = \infty$.

Q. 12 For what values of z the function $z = \sin h \cos v + i \cos hu \sin v$, $w = u + iv$ ceases to be analytic.

Sol: Now as $\cos ix = \cos hx$ and $\sin ix = i \sin hx$

$$\sin h\,(u + iv) = \frac{1}{i} \sin i\,(u + iv) = i \sin (iu - v)$$

$$= -i\,(\sin \text{hu} \cos v - \cos u \sin v)$$

$$\therefore \qquad z = \sin h\,(u + iv) \sin hw$$

$$\Rightarrow \qquad w = \sin h^{-1} z$$

$$\Rightarrow \qquad \frac{dw}{dz} = \frac{1}{\sqrt{1 + z^2}}$$

$$\Rightarrow \qquad \frac{dw}{dz} = \infty \text{ at } z = \pm i$$

$\Rightarrow$ w is not analytic at $z = \pm i$

Q. 13 Prove that an analytic function with constant real part is constant.

Sol: Given $f(z) = u + iv$ is analytic

also $u = \text{constant} = c$ (given)

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}$$

but $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and

Thus from the above relations, we have

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0$$

$\Rightarrow v = \text{constant}$

Hence the result.

Q. 14 Show that the function $f(z) = \sqrt{(|xy|)}$ is not regular at the origin, although $C - R$ equations are satisfied at that point.

Sol: Let $$f(z) = u + iv = \sqrt{(|xy|)}$$

$$\Rightarrow \quad u = \sqrt{(|xy|)} \text{ and } v = 0$$

A the origin $$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0$$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - v(0,0)}{x} = 0$$

and $$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - v(0,0)}{y} = 0$$

$\Rightarrow C - R$ equations are satisfied at (0, 0).

Again $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\Rightarrow \quad f(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - z0} = \lim_{z \to 0} \frac{\sqrt{(|xy|)}}{x + iy}$$

$$\Rightarrow \quad f'(0) = \lim_{x \to 0} \frac{\sqrt{(|mm^2|)}}{x + imm} \text{ along } y = mn$$

$$= \frac{\sqrt{|m|}}{1 + im}$$

$\Rightarrow f'$ depends on m, hence it is not unique

Thus $f(z)$ is not analytic at the origin although $C - R$ equations are satisfied at the origin.

Q. 15 Prove that $e^{\bar{z}}$ is nowhere analytic

Sol: $f(z) = u + iv = e^{\bar{z}} = e^{x-iy} = e^x[\cos y - i \sin y]$

$\Rightarrow$ $u = e^x \cos y$ and $v = -e^x \sin y$

Now $u_x = e^x \cos y,\ u_y = -e^x \sin y$

$u_x = -e^x \sin y$ and $u_y = -e^x \sin y$

As $u_x \neq u_y$ and $u_y \neq -u_x$

This gives us that $e^{\bar{z}}$ is no where analytic.

Miscellaneous Exercise 2 (Unsolved)

Q. 1 Find out whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$, the function $f(z)$ is equal to

(a) $(Imz)/|z|$ (b) $(Rez^2)/|z|$

(c) $(Rez)/(1 + |z|)$ (d) $(Rez - Imz)/|z|^2$

Q. 2 Find an analytic function $w = u + iv$ where $u = x^3 - 3x^2y + 3x^2 - 3y^2 + 1$

Q. 3 Show that $f(z) = \sin x \cos hy + i \cos x \sin hy$ is analytic everywhere.

Q. 4 Show that $u = \cos x \cos y$ is harmonic and find its harmonic conjugate.

Q. 5 Prove that $u\ e^x(x \cos y - y \sin y)$ satisfies Laplace equation. Find $f(z) = u + iv$.

Q. 6 Find an analytic function $w = u + iv$, if

(a) $u = x^3 - 3xy^2$ (b) $u = e^x \cos y$

Q.7 If $f(z)$ is analytic show that $\left\{\frac{\partial}{\partial x}|f|\right\}^2 + \left\{\frac{\partial}{\partial y}|f|\right\}^2 = |f'|^2$

Q. 8 Determine whether $C - R$ conditions are satisfied for the given function:

(a) $f(z) = \frac{1}{2}\ln(x^2 + y^2) + i \tan^{-1} y/x$

(b) $f(z) = x + ay + i\,(bx + cy)$

(c) $f(z) = xy + iy$

(d) $f(z) = z\bar{z}$

Q. 9 Show that $u(r, \theta) = e^{-\theta} \cos (\ln r)$ is harmonic. Find its conjugate harmonic.

Q. 10 Find the conjugate harmonic function of $u(r, \theta) = -r^3 \sin 3\theta$. Also show that u is harmonic.

Q. 11 If $f'(z) = 0$, then show that $f(z)$ is constant.

Q. 12 If both $f(z)$ and $\overline{f(z)}$ are analytic, show that $f(z)$ is constant.

Q. 13 Show that if u is harmonic and v is conjugate harmonic of u, then u in conjugate harmonic of $-v$.

Q. 14 Verify if $f(z) = \dfrac{xy^2 \cdot z}{x^2 + y^4}, z \neq 0$

$$f(0) = 0,$$

is analytic or not?

Q. 15 Given $u(x, y) = 2xy + 2x$, construct harmonic function. Also find $f(z)$ using Milne-Thomson method.

Q. 16 Construct an analytic function where the imaginary part is $v(x, y) = 2x(y + 1) - 4$ under the condition $f(1 + i) = 2$. Also write function in terms of z.

Q. 17 Let $f(z) = u(x, y) + iv(x, y)$ be analytic.

Prove that

(a) $f(z) = 2u\left(\dfrac{z}{2}, -\dfrac{iz}{2}\right) + c$ (b) $f(z) = 2iv\left(\dfrac{z}{2}, \dfrac{-iz}{2}\right) + c_1$

Q. 18 Given $f(z)$ is an analytic function. Prove that Ln $f(z)|$ is also analytic.

Miscellaneous Answers 2

Q. 1 (a) No (b) Yes (c) No (d) No

Q. 2 $f(z) = z^3 + 3z^2 + c$

Q. 4 $v = -\sin x \sin hy + c$

Q. 5 $f(z) = ze^z + c$

Q. 6 (a) $w = z^3 + c$ (b) $w = z^3 + c$

Q. 8 (a) for all z (b) $a = -b$, $c = 1$ (c) No where (d) only at the origin.

Q. 9 $v(r, \theta) = e^{-\theta} \sin(\ln r) + c$

Q. 10 $v = r^3 \cos 3\theta + c$

Q. 14 No

Q. 15 $-iz^2 + 2z + c$

Q. 16 $z^2 + 2iz + 4 - 4i$