Module 6.4: Parseval's Identity

If $F\{f(x)\} = F(p)$ and $F\{g(x)\} = G(p)$,

then (i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\cdot\overline{G(p)}\,dp = \int_{-\infty}^{\infty} f(x)\cdot\overline{g(x)}\,dx$

and (ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(p)|^2\,dp = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

Parseval's Identities for Fourier Sine and Cosine transforms

(i) $\frac{2}{\pi}\int_{0}^{\infty} F_s(p)\cdot G_s(p)\,dp = \int_{0}^{\infty} f(x)\,g(x)\,dx$

(ii) $\frac{2}{\pi}\int_{0}^{\infty} |F_s(p)|^2\,dp = \int_{0}^{\infty} |f(x)|^2\,dx$

(iii) $\frac{2}{\pi}\int_{0}^{\infty} F_c(p)\,G_c(p)\,dp = \int_{0}^{\infty} f(x)\,g(x)\,dx$

(iv) $\frac{2}{\pi}\int_{0}^{\infty} |F_c(p)|^2\,dp = \int_{0}^{\infty} |f(x)|^2\,dx$

Problems:

1. Evaluate (i) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$

and (ii) $\int_0^\infty \frac{1}{(a^2+x^2)^2}\,dx$ $(a>0)$,

using Parseval's Identity

Sol: Let $f(x) = e^{-ax}$ $(0<x<\infty)$.

Then $F_s\{f(x)\} = F_s(p) = \frac{p}{a^2+p^2}$

and $F_c\{f(x)\} = F_c(p) = \frac{a}{a^2+p^2}$

(i) From the Parseval's Identity of Fourier sine transform, we have

$$\int_0^\infty |f(x)|^2\,dx = \frac{2}{\pi}\int_0^\infty |F_s(p)|^2\,dp$$

i.e., $$\int_0^\infty \left|\frac{p}{a^2+p^2}\right|^2 dp = \frac{\pi}{2}\int_0^\infty |e^{-ax}|^2\,dx$$

$$\Rightarrow \int_0^\infty \frac{p^2}{(a^2+p^2)^2}\,dp = \frac{\pi}{2}\int_0^\infty e^{-2ax}\,dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{\pi}{2}\left(0+\frac{1}{2a}\right) = \frac{\pi}{4a}$$

Hence $\displaystyle\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a}$

(ii) Again we have,

$$\frac{2}{\pi}\int_0^\infty |F_c(p)|^2\,dp = \int_0^\infty |f(x)|^2\,dx$$

$$\Rightarrow \int_0^\infty \frac{a^2}{(a^2+p^2)^2}\,dp = \frac{\pi}{2}\int_0^\infty e^{-2ax}\,dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$= \frac{\pi}{4a}$$

Hence $\displaystyle\int_0^\infty \frac{1}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a^3}$

2. Find the Fourier transform of the function

$$f(x) = \begin{cases} 1-x^2 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

and hence evaluate $\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx$

using Parseval's Identity.

<u>Sol:</u> we have

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x)\, e^{ipx}\, dx$$

$$= \int_{-1}^{1} (1-x^2)\,(\cos px + i\sin px)\, dx$$

$$= 2\int_0^1 (1-x^2)\cos px\, dx$$

$$\left(\because \int_{-1}^{1} (1-x^2)\sin px\, dx = 0\right)$$

$$= 2\left\{\left[(1-x^2)\left(\frac{\sin px}{p}\right)\right]_0^1 - \left[(-2x)\left(-\frac{\cos px}{p^2}\right)\right]_0^1 + \left[(-2)\,\frac{-\sin px}{p^3}\right]_0^1\right\}$$

$$= 2\left\{0 - \frac{1}{p^2}(2\cos p - 0) + \frac{2}{p^3}(\sin p - 0)\right\}$$

$$= \frac{4}{p^3}\,(\sin p - p\cos p)$$

By the Parseval's Identity, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(p)|^2\,dp = \int_{-\infty}^{\infty}|f(x)|^2\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\left[4\frac{(\sin p - p\cos p)}{p^3}\right]^2 dp = 2\pi\int_{-1}^{1}(1-x^2)^2\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \frac{\pi}{8}\left(2\int_{0}^{1}(1-x^2)^2\,dx\right)$$

$$\Rightarrow 2\int_{0}^{\infty}\left(\frac{\sin p - p\cos p}{p^3}\right)^2 dp = \frac{2\pi}{8}\left(\frac{8}{15}\right)$$

Hence $\displaystyle\int_{0}^{\infty}\left(\frac{\sin x - x\cos x}{x^3}\right)^2 dx = \frac{\pi}{15}.$

3. Evaluate $\displaystyle\int_{0}^{\infty}\frac{1}{(a^2+x^2)(b^2+x^2)}\,dx$ using Parseval's Identity.

Hint: Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$ ($a>0$ and $b>0$)

Then $\underbrace{F_c\{f(x)\}}_{\boxed{F_c(p)}} = \dfrac{a}{a^2+x^2}$ and $G_c(p) = \dfrac{b}{b^2+x^2}$

By the Parseval's Identity for Fourier cosine transform, we have

$$\frac{2}{\pi}\int_0^\infty F_c(p)\cdot G_c(p)\,dp = \int_0^\infty f(x)\cdot g(x)\,dx$$

$$\Rightarrow \frac{2ab}{\pi}\int_0^\infty \frac{1}{(a^2+p^2)(b^2+p^2)}\,dp = \int_0^\infty e^{-(a+b)x}\,dx$$

$$\Rightarrow \int_0^\infty \frac{1}{(a^2+p^2)(b^2+p^2)}\,dp = \frac{\pi}{2ab}\left(\frac{1}{a+b}\right)$$

Hence $\displaystyle\int_0^\infty \frac{1}{(a^2+x^2)(b^2+x^2)}\,dx = \frac{\pi}{2ab(a+b)}$

4. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{if } |x|<a \\ 0 & \text{if } |x|>a \end{cases} \quad (a>0)$$ and hence evaluate $\displaystyle\int_0^\infty \frac{\sin^2 ax}{x^2}\,dx$ using parseval's Identity.

Hint: $\displaystyle F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x)\cdot e^{ipx}\,dx$

$$= \int_{-a}^{a} 1\cdot e^{ipx}\,dx$$

$$= \frac{2}{p}(\sin ap).$$

By the Parseval's Identity, we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(p)|^2\,dp = \int_{-\infty}^{\infty}|f(x)|^2\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{4}{p^2}\sin^2 ap\,dp = 2\pi\int_{-a}^{a}1^2\,dx$$

$$\Rightarrow 8\cdot\int_0^\infty \frac{\sin^2 ap}{p^2}\,dp = 2\pi(2a).$$ Hence $\displaystyle\int_0^\infty \frac{\sin^2 ax}{x^2}\,dx = \frac{\pi a}{2}.$