18 October 2021 05:01 PM

CAT- I

Syllabus: Mod-1 and Mod-2.

Pattern: Three Question of 10 marks each. Two from Mod-1 and One from Mod-2.

Timing: 50 mins for writing and 10 mins for uploading.

Partial Derivatives

$y = f(x)$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$z = f(x,y)$

The partial derivatives of $f(x,y)$ with respect to x at any (x_0,y_0) is denoted as, $f_x(x_0,y_0)$ or $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ or $\left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}$ and defined as

$$\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

provided the limit exists.

The partial derivative of $f(x,y)$ with respect to y at any pt. (x_0,y_0) is denoted by $f_y(x_0,y_0)$, $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$, $\left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}$ and is defined as

$$\lim_{k\to 0} \frac{f(x_0, y_0+k) - f(x_0,y_0)}{k}$$

provided the limit exists.

Prob: Find f_x and f_y at the point $(4,-5)$ if

$$f(x,y) = x^2 + 3xy + y - 1$$

Soln:

$$f_x = 2x + 3y$$

$$f_x \Big|_{(4,-5)} = 8 - 15 = -7$$

$$f_y \Big|_{(4,-5)} = (1 + 3x) \Big|_{(4,-5)} = 13$$

Prob: Find $\frac{\partial z}{\partial x}$ or z_x from the equation:

$$yz - \ln z = x + y$$

Soln:

$$\frac{\partial}{\partial x}(yz) - \underline{\frac{\partial}{\partial x}(\ln z)} = 1 + 0$$

$$\frac{\partial}{\partial x}(\ln z)$$

$$= \frac{\partial}{\partial z}(\ln z) \cdot \frac{\partial z}{\partial x}$$

$$= \frac{1}{z} \frac{\partial z}{\partial x}$$

$$\Rightarrow y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1$$

$$\Rightarrow \frac{\partial z}{\partial x}\left(y - \frac{1}{z}\right) = 1$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{z}{yz - 1}$$

Exercise: Find f_x and f_y from $f(x,y) = \dfrac{2y}{y + \cos x}$

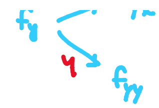
Second Order Partial Derivatives

$z = f(x,y)$

a. $\dfrac{\partial^2 z}{\partial x^2}$ or $\dfrac{\partial^2 f}{\partial x^2}$ or f_{xx}

b. $\dfrac{\partial^2 z}{\partial y^2}$ or $\dfrac{\partial^2 f}{\partial y^2}$ or f_{yy}

$$\begin{tikzcd} & f_{xx} \\ f_x \arrow[ur, "x"] \arrow[dr, "y"'] & \\ & f_{xy} \\ & f_{yx} \\ f_y \arrow[ur, "x"] \arrow[dr, "y"'] & \\ & f_{yy} \end{tikzcd}$$



b. $\dfrac{\partial^2 z}{\partial y^2}$ or $\dfrac{\partial^2 f}{\partial y^2}$ or f_{yy}

c. $\dfrac{\partial^2 z}{\partial x \partial y}$ or $\dfrac{\partial^2 f}{\partial x \partial y}$ or f_{yx} (First w.r.t. y, then x)

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$

d. $\dfrac{\partial^2 z}{\partial y \partial x}$ or $\dfrac{\partial^2 f}{\partial y \partial x}$ or f_{xy} (First w.r.t. x, then y)

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$

Prob: Find f_{yxyz} for $f(x,y,z) = 1 - 2xy^2 z + x^2 y$

Soln:

$f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$

Chain Rule

$w = f(x,y)$ is differentiable and $x(t) = x$ and $y(t) = y$ are differentiable with respect to t, then for $w = f(x(t), y(t))$, the differentiation w.r.t. t is

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$