Finding Linear Transformation from images of basis vectors :-

A linear transformation can be completely determined by images of any set of basis vectors.

If $T: V \to W$ is a linear transformation and if $\{v_1, v_2, -- v_n\}$ is any basis for V then any vector v in V can be expressed as linear combinations of $v_1, v_2, -- v_n$.

ie $$V = K_1 V_1 + K_2 V_2 + -- + K_n V_n$$

$$T(V) = T(K_1 V_1 + K_2 V_2 + -- + K_n V_n)$$

$$\left[T(V) = K_1 T(V_1) + K_2 T(V_2) + -- + K_n T(V_n) \right]$$

Q :- consider the basis S of $\mathbb{R}^2$, $S = \{V_1, V_2\}$ where $V_1 = (1, 1)$, $V_2 = (1, 0)$ and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation s.t $T(V_1) = (1, -2)$, $T(V_2) = (-4, 1)$. Find a formula for $T(x, y)$ and hence find $T(5, 3)$.

basis for $\mathbb{R}^2$ are $v_1 = (1, 1)$, $v_2 = (1, 0)$
ie any vector in $\mathbb{R}^2$ can be expressed as L.C of v_1 & v_2

say $V = (x, y)$

$$V = c_1 v_1 + c_2 v_2$$

$$(x, y) = c_1 (1, 1) + c_2 (1, 0)$$

$$(x, y) = (c_1 + c_2, c_1 + 0)$$

ie $c_1 + c_2 = x$
$c_1 = y$

ie $c_2 = x - y$.

ie $(x, y) = y (1, 1) + (x - y)(1, 0)$

Now

$$T(x, y) = T(y(1, 1)) + T((x - y)(1, 0))$$

$$= y\, T(1, 1) + (x - y)\, T(1, 0)$$

$$= y(1, -2) + (x - y)(-4, 1)$$

$$T(x, y) = (y, -2y) + (-4(x - y), x - y)$$

$$= (y - 4x + 4y, -2y + x - y)$$

$$\boxed{T(x, y) = (-4x + 5y, x - 3y)}$$

L.T

$$T(5, 3) = (-35, 14)$$

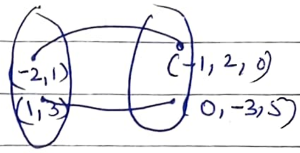
(2) ~~Q03~~ Consider the basis $S = \{v_1, v_2\}$ for $\mathbb{R}^2$, where $v_1 = (-2,1)$ and $v_2 = (1,3)$. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that

$$T(v_1) = (-1,2,0) \qquad T(v_2) = (0,-3,5)$$

Find formula for $T(x_1, x_2)$ and use that to find $T(-2,3)$.

Soln

Let $v = (x_1, x_2) \in \mathbb{R}^2$



as v_1, v_2 are basis for $\mathbb{R}^2$

ie $v = c_1 v_1 + c_2 v_2$

$$(x_1, x_2) = c_1(-2,1) + c_2(1,3)$$

$$-2c_1 + c_2 = x_1$$
$$c_1 + 3c_2 = x_2$$

On solving

$$c_1 = -\frac{3}{7}x_1 + \frac{1}{7}x_2$$

$$c_2 = \frac{1}{7}x_1 + \frac{2}{7}x_2$$

ie $(x_1, x_2) = \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)(-2,1) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)(1,3)$

$$T(x_1, x_2) = \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right) T(-2,1) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right) T(1,3)$$

$$= \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)(-1,2,0) + \left(\frac{x_1}{7} + \frac{2x_2}{7}\right)(0,-3,5)$$

$$T(x_1, x_2) = \left(\frac{3x_1 - x_2}{7}, \frac{-9x_1 - 4x_2}{7}, \frac{5x_1 + 10x_2}{7}\right)$$

$$T(-2,3) = \left(\frac{9}{7}, \frac{-6}{7}, \frac{-20}{7}\right)$$

Ex 3 consider the basis $S = \{v_1, v_2, v_3\}$ for $\mathbb{R}^3$,

$v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 0, 0)$

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be linear transformation

s.t

$T(v_1) = (2, -1, 4)$, $T(v_2) = (3, 0, 1)$, $T(v_3) = (-1, 5, 1)$

Find a formula for $T(x_1, x_2, x_3)$ and use that formula to find $T(2, 4, -1)$.

Soln:- Let $v = K_1 v_1 + K_2 v_2 + K_3 v_3$

$$K_1 = x_3, \quad K_2 = x_2 - x_3, \quad K_3 = x_1 - x_2$$

$$v = x_3 v_1 + (x_2 - x_3) v_2 + (x_1 - x_2) v_3$$

$$T(v) = x_3 T(v_1) + (x_2 - x_3) T(v_2) + (x_1 - x_2) T(v_3)$$

$$T(x_1, x_2, x_3) = x_3 (2, -1, 4) + (x_2 - x_3)(3, 0, 1) + (x_1 - x_2)(-1, 5, 1)$$

$$T(x_1, x_2, x_3) = (-x_1 + 4x_2 - x_3,\ 5x_1 - 5x_2 - x_3,\ x_1 + 3x_3)$$

$$T(2, 4, -1) = (15, -9, -1)$$

Q!- consider $S = \{v_1, v_2\}$ for $\mathbb{R}^2$, where

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Let $T: \mathbb{R}^2 \to P_2$

s.t $T(v_1) = 2 - 3x + x^2$ $\quad T(v_2) = 1 - x^2$

Find $T\begin{bmatrix} a \\ b \end{bmatrix}$ and then find $T\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

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Sol^n.

$$v = k_1 v_1 + k_2 v_2$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$k_1 + 2k_2 = a$$
$$k_1 + 3k_2 = b$$

$$k_1 = (3a - 2b) \qquad k_2 = (b - a)$$

$$v = (3a-2b)\, v_1 + (b-a)\, v_2$$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (3a-2b)\, T(v_1) + (b-a)\, T(v_2)$$

$$= (3a-2b)(2-3x+x^2) + (b-a)(1-x^2)$$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (5a-3b) + (-9a+6b)x + (4a-3b)x^2$$

$$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = -11 + 21x - 10x^2$$