$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A matrix A is said to be in reduced row echelon form if each column that contains a leading 1 in row echelon form of the matrix A has zeros everywhere else in that column.

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1: In each part determine whether the matrix is in row echelon form, reduced row echelon form, both or neither.

(i) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 7 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -6 & 4 & 3 \\ 0 & 1 & 3 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solution:

(i) The given matrix is in reduced row echelon form and row echelon form since it satisfies properties (i), (ii), (iii) and columns containing leading 1 have zero everywhere else.

(ii) The given matrix is neither in row echelon form nor in reduced row echelon form since it does not satisfy the property (iii).

(iii) The given matrix is in row echelon form since it satisfies properties (i), (ii) and (iii).

(iv) The given matrix is neither in row echelon form nor in reduced row echelon form since it does not satisfy the property (i).

Example 2: Find a row echelon form of the following matrices:

(i) $\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$

Solution: (i)

$$\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

R_{13}

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 3 & 4 & 5 \\ 0 & -1 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

$R_2 - 2R_1,\ R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 6 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & -7 & 7 & -5 \end{bmatrix}$$

R_{23}

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -3 & 6 & 1 \\ 0 & -7 & 7 & -5 \end{bmatrix}$$

$(-1)R_2$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & 6 & 1 \\ 0 & -7 & 7 & -5 \end{bmatrix}$$

$R_3 + 3R_2,\ R_4 + 7R_2$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & -7 & -26 \end{bmatrix}$$

R_{34}

$$\sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -7 & -26 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

$$\overset{\left(-\frac{1}{7}\right)R_3}{\sim}\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & \dfrac{26}{7}\\ 0 & 0 & 0 & -8\end{bmatrix}$$

$$\overset{\left(-\frac{1}{8}\right)R_4}{\sim}\begin{bmatrix}1 & 3 & -1 & 2\\ 0 & 1 & -2 & -3\\ 0 & 0 & 1 & \dfrac{26}{7}\\ 0 & 0 & 0 & 1\end{bmatrix}$$

(ii) $$\begin{bmatrix}1 & 2 & -3 & 1\\ -1 & 0 & 3 & 4\\ 0 & 1 & 2 & -1\\ 2 & 3 & 0 & -3\end{bmatrix}$$

$$\overset{R_2+R_1,\ R_4-2R_1}{\sim}\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 2 & 0 & 5\\ 0 & 1 & 2 & -1\\ 0 & -1 & 6 & -5\end{bmatrix}$$

$$\overset{R_{23}}{\sim}\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & 2 & -1\\ 0 & 2 & 0 & 5\\ 0 & -1 & 6 & -5\end{bmatrix}$$

$$\overset{R_3-2R_2,\ R_4+R_2}{\sim}\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & 2 & -1\\ 0 & 0 & -4 & 7\\ 0 & 0 & 8 & -6\end{bmatrix}$$

$$\overset{\left(-\frac{1}{4}\right)R_3}{\sim}\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 8 & -6 \end{bmatrix}$$

$$\overset{R_4-8R_3}{\sim}\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\overset{\left(\frac{1}{8}\right)R_4}{\sim}\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3: Find the reduced row echelon form of the matrices of Example 2.

(i) $\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$

Solution: (i) The row echelon form of the matrix is

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{26}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Beginning with the last non-zero row and working upward, we add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$R_3 - \frac{26}{7}R_4,\ R_2 + 3R_4,\ R_1 - 2R_4$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 + 2R_3,\ R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) The row echelon form of the matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Beginning with the last non-zero row and working upward, we add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$R_3 + \frac{7}{4}R_4,\ R_2 + R_4,\ R_1 - R_4$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 2R_3,\ R_1 + 3R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.6 SYSTEM OF NON-HOMOGENEOUS LINEAR EQUATIONS

A system of m non-homogeneous linear equations in n variables $x_1, x_2, \ldots x_n$ or simply a linear system, is a set of m linear equations, each in n variables. A linear system is represented by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called coefficient matrix of order $m \times n$,

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any vector of order $n \times 1$.

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is any vector of order $m \times 1$.

1.6.1 *Solutions of System of Linear Equations: Gaussian Elimination and Gauss–Jordan Elimination Method*

For a system of m linear equations in n variables, there are three possibilities of the solutions to the system:

(i) The system has unique solution.
(ii) The system has infinite solutions.
(iii) The system has no solution.

When the system of linear equations has one or more solutions, the system is said to be consistent, otherwise it is inconsistent.

The matrix
$$[A:B]=\left[\begin{array}{cccc:c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

is called the augmented matrix of the given system of linear equations.

To solve a system of linear equations, elementary transformations are used to reduce the augmented matrix to either row echelon form or reduced row echelon form.

Reducing the augmented matrix to row echelon form is called Gaussian elimination method. Reducing the augmented matrix to reduced row echelon form is called Gauss–Jordan elimination method.

The Gaussian elimination method for solving the linear system is as follows:

Step 1: Write the augmented matrix.
Step 2: Obtain the row echelon form of the augmented matrix by using elementary row operations.
Step 3: Write the corresponding linear system of equations from row echelon form.
Step 4: Solve the corresponding linear system of equations by back substitution.

The Gauss–Jordan elimination method for solving the linear system is as follows:

Step 1: Write the augmented matrix.
Step 2: Obtain the reduced row echelon form of the augmented matrix by using elementary row operations.
Step 3: For each non-zero row of the matrix, solve the corresponding system of equations for the variables associated with the leading one in that row.

Note: The linear system has a unique solution if $\det(A) \neq 0$

Example 1: Solve each of the following systems by Gaussian elimination method.

(i)
$$\begin{aligned} x+y+2z&=9 \\ 2x+4y-3z&=1 \\ 3x+6y-5z&=0 \end{aligned}$$

(ii)
$$\begin{aligned} 4x-2y+6z&=8 \\ x+y-3z&=-1 \\ 15x-3y+9z&=21 \end{aligned}$$

(iii)
$$\begin{aligned} 3x+y-3z&=13 \\ 2x-3y+7z&=5 \\ 2x+19y-47z&=32 \end{aligned}$$

Solution: (i) The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 2R_1,\ R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$$\left(\frac{1}{2}\right) R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$$R_3 - 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

$$(-2)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x + y + 2z &= 9 \\ y - \frac{7}{2}z &= -\frac{17}{2} \\ z &= 3 \end{aligned}$$

Solving these equations,

$$x = 1, y = 2$$

Hence, $x = 1, y = 2, z = 3$ is the solution of the system.

(ii) The matrix form of the system is

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_{12}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$R_2 - 4R_1,\ R_3 - 15R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$$

$$\left(-\frac{1}{6}\right)R_2,\ \left(-\frac{1}{18}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x + y - 3z = -1$$

$$y - 3z = -2$$

The leading ones are in columns 1 and 2. Hence, the variables x and y are called leading variables whereas the variable z is called a free variable. Assigning the free variable z an arbitrary value t,

$$y = 3t - 2$$
$$x = -1 - 3t + 2 + 3t = 1$$

Hence, $x = 1$, $y = 3t - 2$, $z = t$ is the solution of the system where t is a parameter.

(iii) The matrix form of the system is

$$\begin{bmatrix} 3 & 1 & -3 \\ 2 & -3 & 7 \\ 2 & 19 & -47 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 5 \\ 32 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 3 & 1 & -3 & 13 \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{3}\right)R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 2 & -3 & 7 & 5 \\ 2 & 19 & -47 & 32 \end{array}\right]$$

$$R_2 - 2R_1,\ R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 0 & -\frac{11}{3} & 9 & -\frac{11}{3} \\ 0 & \frac{55}{3} & -45 & \frac{70}{3} \end{array}\right]$$

$$\left(-\frac{3}{11}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 0 & 1 & -\frac{27}{11} & 1 \\ 0 & \frac{55}{3} & -45 & \frac{70}{3} \end{array}\right]$$

$$\underset{\sim}{R_3 - \frac{55}{3}R_2} \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -1 & \frac{13}{3} \\ 0 & 1 & -\frac{27}{11} & 1 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

From the last row of the augmented matrix,

$$0x + 0y + 0z = 5$$

Hence, the system is inconsistent and has no solution.

Example 2: Solve the following system for x, y and z.

$$-\frac{1}{x} + \frac{3}{y} + \frac{4}{z} = 30$$
$$\frac{3}{x} + \frac{2}{y} - \frac{1}{z} = 9$$
$$\frac{2}{x} - \frac{1}{y} + \frac{2}{z} = 10$$

Solution: The matrix form of the system is

$$\begin{bmatrix} -1 & 3 & 4 \\ 3 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 30 \\ 9 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} -1 & 3 & 4 & 30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\underset{\sim}{(-1)R_1} \left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right]$$

$$R_2 - 3R_1, \; R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 70 \end{array}\right]$$

$$\left(\frac{1}{11}\right)R_2, \; \left(\frac{1}{5}\right)R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 1 & 2 & 14 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The corresponding system of equations is

$$\frac{1}{x} - \frac{3}{y} - \frac{4}{z} = -30$$

$$\frac{1}{y} + \frac{1}{z} = 9$$

$$\frac{1}{z} = 5$$

Solving these equations,

$$x = \frac{1}{2}, \; y = \frac{1}{4}, \; z = \frac{1}{5}$$

Hence, $x = \frac{1}{2}, y = \frac{1}{4}, z = \frac{1}{5}$ is the solution of the system.

Example 3: Solve the following system of non-linear equations for the unknown angles α, β and γ, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$ and $0 \le \gamma < \pi$.

$$2\sin\alpha - \cos\beta + 3\tan\gamma = 3$$
$$4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2$$
$$6\sin\alpha - 3\cos\beta + \tan\gamma = 9$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & -2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} \sin\alpha \\ \cos\beta \\ \tan\gamma \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{2}\right)R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 4 & 2 & -2 & 2 \\ 6 & -3 & 1 & 9 \end{array}\right]$$

$$R_2 - 4R_1,\ \ R_3 - 6R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 4 & -8 & -4 \\ 0 & 0 & -8 & 0 \end{array}\right]$$

$$\left(\frac{1}{4}\right)R_2,\ \left(-\frac{1}{8}\right)R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} \sin\alpha - \frac{1}{2}\cos\beta + \frac{3}{2}\tan\gamma &= \frac{3}{2} \\ \cos\beta - 2\tan\gamma &= -1 \\ \tan\gamma &= 0 \end{aligned}$$

Solving these equations,

$$\begin{aligned} \gamma &= 0 \\ \cos\beta &= -1 \quad \Rightarrow \quad \beta = \pi \end{aligned}$$

$$\begin{aligned}\sin\alpha &= \frac{1}{2}\cos\beta - \frac{3}{2}\tan\gamma + \frac{3}{2}\\ &= \frac{1}{2}(-1) - \frac{3}{2}(0) + \frac{3}{2} = 1\\ \alpha &= \frac{\pi}{2}\end{aligned}$$

Hence, $\alpha = \frac{\pi}{2}, \beta = \pi, \gamma = 0$ is the solution of the system.

Example 4: Investigate for what values of λ and μ the equations

$$\begin{aligned} x + 2y + z &= 8\\ 2x + 2y + 2z &= 13\\ 3x + 4y + \lambda z &= \mu \end{aligned}$$

have (i) no solution, (ii) a unique solution, and (iii) many solutions.

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 1\\ 2 & 2 & 2\\ 3 & 4 & \lambda \end{bmatrix}\begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 8\\ 13\\ \mu \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 8\\ 2 & 2 & 2 & 13\\ 3 & 4 & \lambda & \mu \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 2R_1,\ R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 8\\ 0 & -2 & 0 & -3\\ 0 & -2 & \lambda - 3 & \mu - 24 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 8\\ 0 & 1 & 0 & \frac{3}{2}\\ 0 & -2 & \lambda - 3 & \mu - 24 \end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & \lambda-3 & \mu-21 \end{array}\right]$$

(i) If $\lambda = 3$ and $\mu \neq 21$, the system is inconsistent and has no solution.
(ii) If $\lambda \neq 3$ and μ has any value, the system is consistent and has a unique solution.
(iii) If $\lambda = 3$ and $\mu = 21$, the system is consistent and has infinite (many) solutions.

Example 5: Determine the values of λ for which the following equations are consistent. Also, solve the system for these values of λ.

$$\begin{aligned} x+2y+z &= 3 \\ x+y+z &= \lambda \\ 3x+y+3z &= \lambda^2 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ \lambda \\ \lambda^2 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & \lambda \\ 3 & 1 & 3 & \lambda^2 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\xrightarrow{R_2-R_1,\ R_3-3R_1} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda-3 \\ 0 & -5 & 0 & \lambda^2-9 \end{array}\right]$$

$$\xrightarrow{(-1)R_2} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3-\lambda \\ 0 & -5 & 0 & \lambda^2-9 \end{array}\right]$$

$$\xrightarrow{R_3+5R_2} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 0 & 3-\lambda \\ 0 & 0 & 0 & \lambda^2-5\lambda+6 \end{array}\right]$$

The equations will be consistent if $\lambda^2 - 5\lambda + 6 = 0$, i.e. $\lambda = 3$ or $\lambda = 2$.

Case I: When $\lambda = 3$,

$$\begin{aligned} x + 2y + z &= 3 \\ y \quad &= 0 \end{aligned}$$

Assigning the free variable z any arbitrary value t,

$$x = 3 - 2(0) - t = 3 - t$$

Hence, $x = 3 - t$, $y = 0$, $z = t$ is the solution of the system where t is a parameter.

Case II: When $\lambda = 2$,

$$\begin{aligned} x + 2y + z &= 3 \\ y \quad &= 1 \end{aligned}$$

Assigning the free variable z any arbitrary value t,

$$x = 3 - 2(1) - t = 1 - t$$

Hence, $x = 1 - t$, $y = 1$, $z = t$ is the solution of the system where t is a parameter.

Example 6: Show that the system of equations

$$\begin{aligned} 3x + 4y + 5z &= \alpha \\ 4x + 5y + 6z &= \beta \\ 5x + 6y + 7z &= \gamma \end{aligned}$$

is consistent only if α, β and λ are in arithmetic progression (A.P.)

Solution: The matrix form of the system is

$$\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 3 & 4 & 5 & \alpha \\ 4 & 5 & 6 & \beta \\ 5 & 6 & 7 & \gamma \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - R_1, \ R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 3 & 4 & 5 & \alpha \\ 1 & 1 & 1 & \beta - \alpha \\ 2 & 2 & 2 & \gamma - \alpha \end{array}\right]$$

$$\underset{\sim}{R_{12}}\left[\begin{array}{ccc|c} 1 & 1 & 1 & \beta-\alpha \\ 3 & 4 & 5 & \alpha \\ 2 & 2 & 2 & \gamma-\alpha \end{array}\right]$$

$$\underset{\sim}{R_2-3R_1,\ R_3-2R_1}\left[\begin{array}{ccc|c} 1 & 1 & 1 & \beta-\alpha \\ 0 & 1 & 2 & 4\alpha-3\beta \\ 0 & 0 & 0 & \alpha-2\beta+\gamma \end{array}\right]$$

The system of equations is consistent if,

$$\alpha-2\beta+\gamma=0$$

$$\beta=\frac{\alpha+\gamma}{2}$$

i.e. α, β and γ are in arithmetic progression (A.P.)

Example 7: Show that if $\lambda \neq 0$, the system of equations

$$\begin{aligned} 2x+\ \ y \qquad &= a \\ x+\lambda y-\ z &= b \\ y+2z &= c \end{aligned}$$

has a unique solution for every value of a, b, c. If $\lambda = 0$, determine the relation satisfied by a, b, c such that the system is consistent. Find the solution by taking $\lambda = 0$, $a = 1$, $b = 1$, $c = -1$.

Solution: The matrix form of the system is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & \lambda & -1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The system has a unique solution if $\det(A) \neq 0$

$$\det(A)=2(2\lambda+1)-1\ (2+0)\neq 0$$

$$4\lambda \neq 0$$

$$\lambda \neq 0$$

Hence, the system of equations has a unique solution if $\lambda \neq 0$ for any value of a, b, c.

If $\lambda = 0$, the system is either inconsistent or has an infinite number of solutions.

For $\lambda = 0$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ 1 & 0 & -1 & b \\ 0 & 1 & 2 & c \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\overset{R_{12}}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & -1 & b \\ 2 & 1 & 0 & a \\ 0 & 1 & 2 & c \end{array}\right]$$

$$\overset{R_2 - 2R_1}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & -1 & b \\ 0 & 1 & 2 & a-2b \\ 0 & 1 & 2 & c \end{array}\right]$$

$$\overset{R_3 - R_2}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & -1 & b \\ 0 & 1 & 2 & a-2b \\ 0 & 0 & 0 & c-a+2b \end{array}\right]$$

The system is consistent if $c - a + 2b = 0$
The corresponding system of equations is

$$\begin{aligned} x - \quad z &= b \\ y + 2z &= a - 2b \end{aligned}$$

Assigning the free variable z any arbitrary value t,

$$\begin{aligned} y &= a - 2b - 2t \\ x &= b + t \end{aligned}$$

Hence, $x = b + t$, $y = a - 2b - 2t$, $z = t$ is the solution of the system where t is a parameter.

When $a = 1$, $b = 1$, $c = -1$

$$\begin{aligned} x &= 1 + t \\ y &= -1 - 2t \\ z &= t \end{aligned}$$

Example 8: Solve each of the following systems by Gauss–Jordan elimination method:

(i) $\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$

(ii) $\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$

(iii) $\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x - 3w &= -3 \end{aligned}$

(iv) $\begin{aligned} -2y + 3z &= 1 \\ 3x + 6y - 3z &= -2 \\ 6x + 6y + 3z &= 5 \end{aligned}$

(v) $\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\ 2x_1 - 4x_2 + x_3 &= 5 \\ x_1 - 2x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$

(vi) $\begin{aligned} 2x - y + z &= 9 \\ 3x - y + z &= 6 \\ 4x - y + 2z &= 7 \\ -x + y - z &= 4 \end{aligned}$

Solution: (i) The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 + R_1,\ R_3 - 3R_1$$
$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$(-1)R_2$$
$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array}\right]$$

$$R_3 + 10R_2$$
$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$$

$$\left(-\frac{1}{52}\right)R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_2 + 5R_3,\ R_1 - 2R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 - R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system of equations is

$$x_1 = 3$$
$$x_2 = 1$$
$$x_3 = 2$$

Hence, $x_1 = 3, x_2 = 1, x_3 = 2$ is the solution of the system.

(ii) The matrix form of the system is

$$\begin{bmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\left(\frac{1}{2}\right)R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 1 & 4 & -1 \end{array}\right]$$

$$R_2 + 2R_1,\ R_3 - 8R_1$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -7 & -4 & -1 \end{array}\right]$$

$$R_3 + R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{7}\right)R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} & \frac{1}{7} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 + \frac{3}{7}x_3 = -\frac{1}{7}$$
$$x_2 + \frac{4}{7}x_3 = \frac{1}{7}$$

Since leading ones are in columns 1 and 2, x_1 and x_2 are called leading variables whereas x_3 is a free variable. Assigning the free variable x_3 any arbitrary value t,

$$x_1 = -\frac{1}{7} - \frac{3}{7}t$$
$$x_2 = \frac{1}{7} - \frac{4}{7}t$$

Hence, $x_1 = -\frac{1}{7} - \frac{3}{7}t,\ x_2 = \frac{1}{7} - \frac{4}{7}t,\ x_3 = t$ is the solution of the system where t is a parameter.

(iii) The matrix form of the system is

$$\begin{bmatrix} 1 & -1 & 2 & -1 \\ 2 & 1 & -2 & -2 \\ -1 & 2 & -4 & 1 \\ 3 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \\ -3 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - 2R_1,\ R_3 + R_1,\ R_4 - 3R_1$$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{3}\right)R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right]$$

$$R_3 - R_2,\ R_4 - 3R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 + R_2$$

$$\sim \left[\begin{array}{cccc:c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x - \qquad\quad w &= -1 \\ y - 2z \qquad &= 0 \end{aligned}$$

The leading ones are in columns 1 and 2. Hence, the variables x and y are called leading variables whereas the variables z and w are called free variables. Assigning the free variables z and w any arbitrary values t_1 and t_2 respectively,

$$x = -1 + t_2$$

and

$$y = 2t_1$$

Hence, $x = -1 + t_2,\ y = 2t_1,\ z = t_1,\ w = t_2$ is the solution of the system where t_1 and t_2 are parameters.

(iv) The matrix form of the system is

$$\begin{bmatrix} 0 & -2 & 3 \\ 3 & 6 & -3 \\ 6 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\overset{R_{12}}{\sim} \left[\begin{array}{ccc:c} 3 & 6 & -3 & -2 \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$\overset{\left(\frac{1}{3}\right)R_1}{\sim} \left[\begin{array}{ccc:c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 6 & 6 & 3 & 5 \end{array}\right]$$

$$R_3 - 6R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & -2 & 3 & 1 \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$$\left(-\frac{1}{2}\right)R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & -6 & 9 & 9 \end{array}\right]$$

$$R_3 + 6R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 6 \end{array}\right]$$

From the last row of the augmented matrix,

$$0x + 0y + 0z = 6$$

Hence, the system is inconsistent and has no solution.

(v) The matrix form of the system is

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & 1 & 0 \\ 1 & -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc:c} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - 2R_1,\ R_3 - R_1$$

$$\sim\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -6 & 3 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(\frac{1}{3}\right)R_2$$

$$\sim\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 + R_2$$

$$\sim\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 - 2x_2 + x_4 &= 2 \\ x_3 - 2x_4 &= 1 \end{aligned}$$

The leading ones are in columns 1 and 3. Hence, the variables x_1 and x_3 are called leading variables whereas the variables x_2 and x_4 are called free variables. Assigning the free variables x_2 and x_4 any arbitrary values t_1 and t_2 respectively,

$$\begin{aligned} x_1 &= 2 + 2t_1 - t_2 \\ x_3 &= 1 + 2t_2 \end{aligned}$$

Hence, $x_1 = 2 + 2t_1 - t_2,\ x_2 = t_1,\ x_3 = 1 + 2t_2,\ x_4 = t_2$ is the solution of the system where t_1 and t_2 are the parameters.

(vi) The matrix form of the system is

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 7 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 9 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ -1 & 1 & -1 & 4 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_{14}$$

$$\sim \left[\begin{array}{rrr|r} -1 & 1 & -1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

$$(-1)R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & -4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

$$R_2 - 3R_1,\ R_3 - 4R_1,\ R_4 - 2R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & -4 \\ 0 & 2 & -2 & 18 \\ 0 & 3 & -2 & 23 \\ 0 & 1 & -1 & 17 \end{array}\right]$$

$$R_{24}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & 17 \\ 0 & 3 & -2 & 23 \\ 0 & 2 & -2 & 18 \end{array}\right]$$

$$R_3 - 3R_2,\ R_4 - 2R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & 17 \\ 0 & 0 & 1 & -28 \\ 0 & 0 & 0 & -16 \end{array}\right]$$

From the last row of the augmented matrix,

$$0x + 0y + 0z = -16$$

Hence, the system is inconsistent and has no solution.

Exercise 1.2

1. Solve the following systems of equations by Gaussian elimination method:

(i) $$\begin{aligned} 2x-3y-z&=3\\ x+2y-z&=4\\ 5x-4y-3z&=-2 \end{aligned}$$

(ii) $$\begin{aligned} x+2y-z&=1\\ x+y+2z&=9\\ 2x+y-z&=2 \end{aligned}$$

(iii) $$\begin{aligned} 6x+y+z&=-4\\ 2x-3y-z&=0\\ -x-7y-2z&=7 \end{aligned}$$

(iv) $$\begin{aligned} 2x-y-z&=2\\ x+2y+z&=2\\ 4x-7y-5z&=2 \end{aligned}$$

(v) $$\begin{aligned} 2x_1+x_2+2x_3+x_4&=6\\ 6x_1-6x_2+6x_3+12x_4&=36\\ 4x_1+3x_2+3x_3-3x_4&=1\\ 2x_1+2x_2-x_3+x_4&=10 \end{aligned}$$

Ans.:
(i) inconistent
(ii) consistent
$x=2,\ y=1, z=3$
(iii) consistent
$x=-1,\ y=-2,\ z=-4$
(iv) consistent
$x=\dfrac{6+t}{5},\ y=\dfrac{2-3t}{5},\ z=t$
(v) consistent
$x_1=2,\ x_2=1,\ x_3=-1,\ x_4=3$

2. Solve the following system of equations by Gauss–Jordan elimination method:

(i) $$\begin{aligned} x+2y+z&=-1\\ 6x+y+z&=-4\\ 2x-3y-z&=0\\ -x-7y-2z&=7\\ x-y&=1 \end{aligned}$$

(ii) $$\begin{aligned} x+y+z&=6\\ x-2y+2z&=5\\ 3x+y+z&=8\\ 2x-2y+3z&=7 \end{aligned}$$

(iii) $$\begin{aligned} 2x_1+x_2+5x_4&=4\\ 3x_1-2x_2+2x_3&=2\\ 5x_1-8x_2-4x_3&=1 \end{aligned}$$

Ans.:
(i) consistent
$x=-1,\ y=-2,\ z=4$
(ii) consistent
$x=-1,\ y=-2,\ z=3$
(iii) inconsistent

3. Investigate for what values of λ and μ, the system of simultaneous equations

$$\begin{aligned} x+y+z&=6\\ x+2y+3z&=10\\ x+2y+\lambda z&=\mu \end{aligned}$$

have (i) no solution, (ii) a unique solution, and (iii) infinite number of solutions.

Ans.:
(i) $\lambda=3, \mu\neq 10$
(ii) $\lambda\neq 3$, any value of μ
(iii) $\lambda=3, \mu=10$

4. Investigate for what values of k the equations

$$\begin{aligned} x+y+\ \ z &= 1 \\ 2x+y+\ 4z &= k \\ 4x+y+10z &= k^2 \end{aligned}$$

have infinite number of solutions.

[**Ans.:** $k = 1, 2$]

5. Determine the values of λ for which the following system of equations.

$$\begin{aligned} 3x-\ \ y+\lambda z &= 0 \\ 2x+\ \ y+\ \ z &= 2 \\ x-2y-\lambda z &= -1 \end{aligned}$$

will fail to have a unique solution. For this value of λ, are the equations consistent?

$\left[\textbf{Ans.: } \lambda = -\frac{7}{2}, \text{no solution}\right]$

6. Find for what values λ, the set of equations

$$\begin{aligned} 2x-3y+6z-5t &= 3 \\ y-4z+\ \ t &= 1 \\ 4x-5y+8z-9t &= \lambda \end{aligned}$$

has (i) no solution, and (ii) infinite number of solutions and find the solutions of the equations when they are consistent.

$$\left[\begin{aligned} &\textbf{Ans.: } \text{(i) } \lambda \neq 7, \\ &\quad \text{(ii) } \lambda = 7, x = 3k_1 + k_2 + 3, \\ &\quad\quad y = 4k_1 - k_2 + 1, z = k_1, \\ &\quad\quad t = k_2 \end{aligned}\right]$$

1.7 SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS

A system of m homogeneous linear equations in n variables $x_1, x_2, \ldots x_n$ or simply a linear system, is a set of m linear equations each in n variables. A linear system is represented by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = \mathbf{0}$$

where A is any matrix of order $m \times n$, $\mathbf{x}$ is a vector of order $n \times 1$ and $\mathbf{0}$ is a null vector of order $m \times 1$. The matrix A is called coefficient matrix of the system of equations.

1.7.1 *Solutions of a System of Linear Equations*

For a system of m linear equations in n variables, there are two possibilities of the solutions to the system.

(i) The system has exactly one solution, i.e. $x_1 = 0, x_2 = 0\ldots, x_n = 0$. This solution is called the trivial solution.

(ii) The system has infinite solutions.

Note: The system of equations has a non-trivial solution if $\det(A) = 0$.

Example 1: Solve the following systems of equations by the Gauss–Jordan elimination method.

(i) $\begin{aligned} 3x - y - z &= 0 \\ x + y + 2z &= 0 \\ 5x + y + 3z &= 0 \end{aligned}$

(ii) $\begin{aligned} x + y - z + w &= 0 \\ x - y + 2z - w &= 0 \\ 3x + y \quad + w &= 0 \end{aligned}$

(iii) $\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 + 2x_2 \quad &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$

Solution: (i) The matrix form of the system is

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 2 \\ 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 3 & -1 & -1 & 0 \\ 1 & 1 & 2 & 0 \\ 5 & 1 & 3 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\overset{R_{12}}{\sim} \left[\begin{array}{ccc:c} 1 & 1 & 2 & 0 \\ 3 & -1 & -1 & 0 \\ 5 & 1 & 3 & 0 \end{array}\right]$$

$$\overset{R_2 - 3R_1,\ R_3 - 5R_1}{\sim} \left[\begin{array}{ccc:c} 1 & 1 & 2 & 0 \\ 0 & -4 & -7 & 0 \\ 0 & -4 & -7 & 0 \end{array}\right]$$

$$\overset{\left(-\frac{1}{4}\right)R_2,\ \left(-\frac{1}{4}\right)R_3}{\sim} \left[\begin{array}{ccc:c} 1 & 1 & 2 & 0 \\ 0 & 1 & \frac{7}{4} & 0 \\ 0 & 1 & \frac{7}{4} & 0 \end{array}\right]$$

$$\overset{R_3 - R_2}{\sim} \left[\begin{array}{ccc:c} 1 & 1 & 2 & 0 \\ 0 & 1 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x + \frac{1}{4}z = 0$$
$$y + \frac{7}{4}z = 0$$

Solving for the leading variables,

$$x = -\frac{1}{4}z$$
$$y = -\frac{7}{4}z$$

Assigning the free variable z an arbitrary value t,

$$x = -\frac{1}{4}t$$
$$y = -\frac{7}{4}t$$

Hence, $x = -\frac{1}{4}t,\ y = -\frac{7}{4}t$ is the non-trivial solution of the system where t is a parameter.

(ii) The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ 3 & 1 & 0 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to the reduced row echelon form,

$$\begin{array}{l} R_2 - R_1,\ R_3 - 3R_1 \\ \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 3 & -2 & 0 \\ 0 & -2 & 3 & -2 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} \left(-\frac{1}{2}\right)R_2,\ \left(-\frac{1}{2}\right)R_3 \\ \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} R_3 - R_2 \\ \sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} R_1 - R_2 \\ \sim \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

The corresponding system of equations is

$$\begin{aligned} x + \frac{1}{2}z \qquad &= 0 \\ y - \frac{3}{2}z + w &= 0 \end{aligned}$$

Solving for the leading variables,

$$\begin{aligned} x &= -\frac{1}{2}z \\ y &= \frac{3}{2}z - w \end{aligned}$$

Assigning the free variables z and w arbitrary values t_1 and t_2 respectively,

$$x = -\frac{1}{2}t_1$$

$$y = \frac{3}{2}t_1 - t_2$$

Hence, $x = -\frac{1}{2}t_1,\ y = \frac{3}{2}t_1 - t_2,\ z = t_1,\ w = t_2$ is the non-trivial solution of the system where t_1 and t_2 are parameters.

(iii) The matrix form of the system is

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\underset{\sim}{R_{12}}\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{R_2 - 2R_1}\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{\left(-\frac{1}{3}\right)R_2}\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\underset{\sim}{R_3 - R_2}\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\left(\frac{1}{2}\right)R_3$$
$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 + R_3$$
$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_1 - 2R_2$$
$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x = 0$$
$$y = 0$$
$$z = 0$$

Hence, the system has a trivial solution, i.e. $x = 0, y = 0, z = 0$.

Example 2: Show that the following non-linear system has 18 solutions if $0 \le \alpha \le 2\pi, 0 \le \beta \le 2\pi$ and $0 \le \gamma < 2\pi$.

$$\begin{aligned} \sin\alpha + 2\cos\beta + 3\tan\gamma &= 0 \\ 2\sin\alpha + 5\cos\beta + 3\tan\gamma &= 0 \\ -\sin\alpha - 5\cos\beta + 5\tan\gamma &= 0 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ -1 & -5 & 5 \end{bmatrix}\begin{bmatrix} \sin\alpha \\ \cos\beta \\ \tan\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & -5 & 5 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - 2R_1,\ R_3 + R_1$$
$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 8 & 0 \end{array}\right]$$

$$R_3 + 3R_2$$
$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$(-1)R_3$$
$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 + 3R_3,\ R_1 - 3R_3$$
$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$R_1 - 2R_2$$
$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\sin\alpha = 0$$
$$\cos\beta = 0$$
$$\tan\gamma = 0$$

From these equations,

$$\alpha = 0, \pi, 2\pi$$
$$\beta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad [\because \alpha, \beta \text{ and } \gamma \text{ lie between 0 and } 2\pi]$$
$$\gamma = 0, \pi, 2\pi$$

Hence, there are $3\cdot 2\cdot 3 = 18$ possible solutions which satisfy the system of equations.

Example 3: For what value of λ does the following system of equations possess a non-trivial solution? Obtain the solution for real values of λ.

$$\begin{aligned} x+2y+3z &= \lambda x \\ 3x+\ y+2z &= \lambda y \\ 2x+3y+\ z &= \lambda z \end{aligned}$$

Solution: The system of equations is

$$\begin{aligned} (1-\lambda)x+2y+3z &= 0 \\ 3x+(1-\lambda)y+2z &= 0 \\ 2x+3y+(1-\lambda)z &= 0 \end{aligned}$$

The matrix form of the system is

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system will possess a non-trivial solution if $\det(A) = 0$.

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (1-\lambda)[(1-\lambda)^2-6]-2(3-3\lambda-4)+3(9-2+2\lambda) &= 0 \\ (1-\lambda)(\lambda^2-2\lambda-5)+2+6\lambda+21+6\lambda &= 0 \\ \lambda^2-2\lambda-5-\lambda^3+2\lambda^2+5\lambda+12\lambda+23 &= 0 \\ -\lambda^3+3\lambda^2+15\lambda+18 &= 0 \end{aligned}$$

$$\lambda = 6, \quad \lambda = -1.5 \pm 0.866\, i$$

For real value of λ, i.e. $\lambda = 6$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 3 & -5 & 2 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|c} -5 & 2 & 3 & 0 \\ 1 & -8 & 7 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

$$R_{12}$$

$$\sim \left[\begin{array}{rrr|r} 1 & -8 & 7 & 0 \\ -5 & 2 & 3 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

$$R_2 + 5R_1,\ R_3 - 2R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & -8 & 7 & 0 \\ 0 & -38 & 38 & 0 \\ 0 & 19 & -19 & 0 \end{array}\right]$$

$$\left(-\frac{1}{38}\right)R_2,\ \left(\frac{1}{19}\right)R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & -8 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & -8 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 + 8R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x - z = 0$$
$$y - z = 0$$

Solving for the leading variables,

$$x = z$$
$$y = z$$

Assigning the free variable z an arbitrary value t,

$$x = t$$
$$y = t$$

Hence, $x = t, y = t, z = t$ is the non-trivial solution of the system where t is a parameter.

Example 4: If the following system has a non-trivial solution, then prove that $a+b+c=0$ or $a=b=c$ and hence find the solution in each case.

$$ax+by+cz=0$$
$$bx+cy+az=0$$
$$cx+ay+bz=0$$

Solution: The matrix form of the system is

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has a non-trivial solution if det $(A) = 0$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$$

$$a(bc-a^2)-b(b^2-ac)+c(ab-c^2)=0$$
$$-a^3+b^3+c^3-3abc=0$$
$$-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$$
$$a+b+c=0$$

or
$$a^2+b^2+c^2-ab-bc-ca=0$$
$$\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$$
$$a-b=0,\ b-c=0,\ c-a=0$$
$$a=b,\ b=c,\ c=a$$
$$a=b=c$$

Hence, the system has a non-trivial solution if $a+b+c=0$ or $a=b=c$.
The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} a & b & c & 0 \\ b & c & a & 0 \\ c & a & b & 0 \end{array}\right]$$

$R_3+R_1+R_2$

$$\sim\left[\begin{array}{ccc:c} a & b & c & 0 \\ b & c & a & 0 \\ a+b+c & a+b+c & a+b+c & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} ax + \quad by \quad + \quad cz &= 0 \\ bx + \quad cy \quad + \quad az &= 0 \\ (a+b+c)x + (a+b+c)y + (a+b+c)z &= 0 \end{aligned}$$

(i) When $a+b+c=0,$ we have only two equations.

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$

$$\frac{x}{\begin{vmatrix} b & c \\ c & a \end{vmatrix}} = -\frac{y}{\begin{vmatrix} a & c \\ b & a \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & b \\ b & c \end{vmatrix}} = t$$

$$\frac{x}{ab-c^2} = -\frac{y}{a^2-bc} = \frac{z}{ac-b^2} = t$$

Hence, $x=(ab-c^2)t, y=(bc-a^2)t, z=(ac-b^2)t$ is the solution of the system where t is a parameter.

(ii) When $a = b = c$, we have only one equation.

$$x + y + z = 0$$

Let
$$y = t_1$$
$$z = t_2$$

Then
$$x = -t_1 - t_2$$

Hence, $x=-t_1-t_2, y=t_1, z=t_2$ is the solution of the system where t_1 and t_2 are parameters.

Example 5: Discuss for all values of k, the system of equations

$$\begin{aligned} 2x + \quad 3ky + (3k+4)z &= 0 \\ x + (k+4)y + (4k+2)z &= 0 \\ x + 2(k+1)y + (3k+4)z &= 0 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{12}

$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 - R_1$$
$$\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{vmatrix} \\ &= (k-8)(-k+2) + 5k(k-2) \\ &= (k-2)(-k+8+5k) \\ &= 4(k-2)(k+2) \end{aligned}$$

(i) When $k \neq \pm 2$, $\det(A) \neq 0$, the system has a trivial solution, i.e. $x = 0, y = 0, z = 0$.

(ii) When $k = \pm 2$, $\det(A) = 0$, the system has non-trivial solutions.

Case I: When $k = 2$, the augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 6 & 10 & 0 \\ 0 & -6 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left(-\frac{1}{6}\right)R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 6 & 10 & 0 \\ 0 & 1 & \frac{10}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - 6R_2$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{10}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x = 0$$
$$y + \frac{10}{6}z = 0$$

Solving for the leading variables,

$$x = 0$$
$$y = -\frac{10}{6}z$$

Assigning the free variable z any arbitrary value t,

$$y = -\frac{10}{6}t = -\frac{5}{3}t$$

Hence, $x = 0,\ y = -\frac{5}{3}t,\ z = t$ is the solution of the system where t is a parameter.

Case II: When $k = -2$, the augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & 2 & -6 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right]$$

Reducing the augmented matrix to reduced row echelon form,

$$\left(-\frac{1}{10}\right)R_2, \left(-\frac{1}{4}\right)R_3$$

$$\sim\left[\begin{array}{ccc:c} 1 & 2 & -6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_3 - R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 2 & -6 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1 - 2R_2$$

$$\sim\left[\begin{array}{ccc:c} 1 & 0 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x - 4z = 0$$
$$y - \quad z = 0$$

Solving for the leading variables,

$$x = 4z$$
$$y = z$$

Assigning the free variable z any arbitrary value t,

$$x = 4t$$
$$y = t$$

Hence, $x = 4t,\ y = t,\ z = t$ is the solution of the system where t is a parameter.

Exercise 1.3

1. Solve the following equations:

(i) $$\begin{aligned} x - y + z &= 0 \\ x + 2y + z &= 0 \\ 2x + y + 3z &= 0 \end{aligned}$$

(ii) $$\begin{aligned} x - 2y + 3z &= 0 \\ 2x + 5y + 6z &= 0 \end{aligned}$$

(iii) $$\begin{aligned} 2x - 2y + 5z + 3w &= 0 \\ 4x - y + z + w &= 0 \\ 3x - 2y + 3z + 4w &= 0 \\ x - 3y + 7z + 6w &= 0 \end{aligned}$$

(iv) $$\begin{aligned} 2x - y + 3z &= 0 \\ 3x + 2y + z &= 0 \\ x - 4y + 5z &= 0 \end{aligned}$$

(v) $$\begin{aligned} 7x + y - 2z &= 0 \\ x + 5y - 4z &= 0 \\ 3x - 2y + z &= 0 \\ 2x - 7y + 5z &= 0 \end{aligned}$$

(vi) $$\begin{aligned} 3x + 4y - z - 9w &= 0 \\ 2x + 3y + 2z - 3w &= 0 \\ 2x + y - 14z - 12w &= 0 \\ x + 3y + 13z + 3w &= 0 \end{aligned}$$

(vii) $$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 - x_4 &= 0 \\ 3x_1 - x_2 + 2x_3 + 3x_4 &= 0 \end{aligned}$$

(viii) $$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 0 \\ 3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 4x_2 + 5x_3 &= 0 \end{aligned}$$

[**Ans.:** (*i*) $x = 0, y = 0, z = 0$

(*ii*) $x = -3t, y = 0, z = t$

(*iii*) $x = \dfrac{211}{9}t, y = 4t, z = \dfrac{7}{9}t,$ $w = t$

(*iv*) $x = -t, y = t, z = t$

(*v*) $x = \dfrac{3}{17}t, y = \dfrac{13}{17}t, z = t$

(*vi*) $x = 11t, y = -8t, z = t,$ $w = 0$

(*vii*) $x_1 = -\dfrac{1}{3}t, x_2 = \dfrac{2}{3}t,$ $x_3 = -\dfrac{2}{3}t, x_4 = t$

(*viii*) $x_1 = -x_2 = -x_3 = t$]

2. For what value of λ does the following system of equations possess a non-trivial solution? Obtain the solution for real values of λ.

(i) $$\begin{aligned} 3x + y - \lambda z &= 0 \\ 4x - 2y - 3z &= 0 \\ 2\lambda x + 4y - \lambda z &= 0 \end{aligned}$$

(ii) $$\begin{aligned} (1-\lambda)x_1 + 2x_2 + 3x_3 &= 0 \\ 3x_1 + (1-\lambda)x_2 + 2x_3 &= 0 \\ 2x_1 + 3x_2 + (1-\lambda)x_3 &= 0 \end{aligned}$$

[**Ans.:**

(i) Non-trivial solution $\lambda = 1, -9$

For $\lambda = 1, x = -t, y = -t, z = -2t$

For $\lambda = -9, x = -3t, y = -9t, z = 2t$

(ii) $\lambda = 6, x = y = z = t$]