Module 7.3: Inverse Z-Transform

If $Z[u_n] = U(z)$, then $Z^{-1}[U(z)] = u_n$.

eg: (i) $Z^{-1}\left[\frac{z}{z-1}\right] = 1^n = 1$

(ii) $Z^{-1}\left[\frac{z}{z+1}\right] = (-1)^n$

(iii) $Z^{-1}\left[e^{1/z}\right] = \frac{1}{n!}$

(iv) $Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$

(v) $Z^{-1}\left[\log\left(\frac{z}{z-1}\right)\right] = \frac{1}{n}$

(vi) $Z^{-1}[1] = \delta_n = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \neq 0 \end{cases}$

Partial Fractions Method:

Note: In the Z-Transform, we have to resolve $\frac{U(z)}{z}$ into partial fractions

Problems:

1. Find $Z^{-1}\left[\frac{z}{(z-1)(z+1)}\right]$

Sol: Let $U(z) = \frac{z}{(z-1)(z+1)}$ $\boxed{Z[u_n] = U(z)}$

Then, $\frac{U(z)}{z} = \frac{1}{(z-1)(z+1)}$

Resolving $\frac{1}{(z-1)(z+1)}$ into partial fractions,

we have

$$\frac{1}{(z-1)(z+1)} = \frac{A}{z-1} + \frac{B}{z+1}$$

$$\Rightarrow \quad 1 = A(z+1) + B(z-1)$$

which gives $A = \frac{1}{2}$ and $B = -\frac{1}{2}$

Therefore, $\frac{U(z)}{z} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$

$$\Rightarrow \quad Z[u_n] = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z+1}\right]$$

$$\Rightarrow \quad u_n = \frac{1}{2}\left(Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{z+1}\right]\right)$$

Hence, $u_n = \frac{1}{2}\left[1^n - (-1)^n\right]$

(2) Find $Z^{-1}\left[\frac{z^2+z}{(z-2)(z+3)^2}\right]$

Sol: let $U(z)=\frac{z^2+z}{(z-2)(z+3)^2}$

Then $\frac{U(z)}{z}=\frac{z+1}{(z-2)(z+3)^2}$

Now,

$$\frac{z+1}{(z-2)(z+3)^2}=\frac{A}{z-2}+\frac{B}{z+3}+\frac{C}{(z+3)^2}$$

$$\Rightarrow z+1=A(z+3)^2+B(z-2)(z+3)+C(z-2)$$

which gives $A=\frac{3}{25}$, $B=\frac{-3}{25}$ and $C=\frac{2}{5}$

Therefore,

$$\frac{U(z)}{z}=\frac{3}{25}\left[\frac{1}{z-2}\right]-\frac{3}{25}\left[\frac{1}{z+3}\right]+\frac{2}{5}\left[\frac{1}{(z+3)^2}\right]$$

$$\Rightarrow Z[u_n]=\frac{3}{25}\left[\left(\frac{z}{z-2}\right)-\left(\frac{z}{z+3}\right)\right]+\frac{2}{5}\left[\frac{z}{(z+3)^2}\right]$$

$$\Rightarrow u_n = \frac{3}{25}\left(Z^{-1}\left[\frac{z}{z-2}\right] - Z^{-1}\left[\frac{z}{z+3}\right]\right)$$

$$+ \frac{2}{5} Z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

Hence,

$$u_n = \frac{3}{25}\left(2^n - (-3)^n\right) + \frac{2}{5}\left(\frac{-1}{3} n(-3)^n\right)$$

$$= \frac{3}{25}\left(2^n - (-3)^n\right) - \frac{2}{15}\left(n(-3)^n\right)$$

$$Z\left[n(-3)^n\right] = -z \frac{d}{dz} Z\left[(-3)^n\right]$$

$$= -z \frac{d}{dz}\left(\frac{z}{z+3}\right)$$

$$= -z\left[\frac{(z+3)(1) - z(1)}{(z+3)^2}\right]$$

$$= \frac{-3z}{(z+3)^2}$$

and hence

$$Z^{-1}\left[\frac{z}{(z+3)^2}\right] = \frac{-1}{3} \cdot n(-3)^n$$

③ Find $Z^{-1}\left[\frac{z}{(z-1)(z^2+1)}\right]$ using partial fractions method.

Answer: $\frac{1}{2}\left[1^n - \cos n\frac{\pi}{2} - \sin n\frac{\pi}{2}\right]$

④ Find $Z^{-1}\left[\frac{2z(2z-1)}{(z-1)(z-2)^2}\right]$ using Partial fractions method.

Answer: $2(1)^n - 2(2)^n + 3(n2^n)$

⑤ Find $Z^{-1}\left[\frac{z}{z^2+11z+24}\right]$ using Partial fractions method.

Answer: $\frac{1}{5}\left[(-3)^n - (-8)^n\right]$

Convolution Theorem:

If $Z^{-1}[U(z)] = u_n$ and $Z^{-1}[V(z)] = v_n$

then $Z^{-1}[U(z) \cdot V(z)] = u_n * v_n$

$$= \sum_{m=0}^{n} u_m \cdot v_{n-m}$$

① Find $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$ using Convolution theorem.

Sol: Let $U(z) = \frac{z}{z-1}$ and $V(z) = \frac{z}{z-3}$.

Then $Z^{-1}[U(z)] = 1^n = u_n$

and $Z^{-1}[V(z)] = 3^n = v_n$ (say).

By the convolution theorem, we have

$$Z^{-1}[U(z)\cdot V(z)] = u_n * v_n$$

i.e., $$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \sum_{m=0}^{n} u_m \cdot v_{n-m}$$

$$= \sum_{m=0}^{n} 1^m \, 3^{n-m}$$

$$= 3^n \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m$$

$$\boxed{\begin{array}{l} a + ar + ar^2 + \cdots + ar^n \\ = \dfrac{a(1-r^{n+1})}{1-r} \end{array}}$$

$$= 3^n\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^n\right]$$

$$= 3^n\left[\frac{1\left(1-\left(\frac{1}{3}\right)^{n+1}\right)}{1-\frac{1}{3}}\right]$$

$$= 3^n\left[\frac{3^{n+1}-1}{3^{n+1}\left(\frac{3-1}{3}\right)}\right]$$

$$= \frac{3^{n+1}-1}{2}$$

(2) Find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ using convolution theorem.

Sol: let $U(z) = \frac{z}{z-a}$ and $V(z) = \frac{z}{z-b}$

Then $Z^{-1}[U(z)] = a^n = u_n$

and $Z^{-1}[V(z)] = b^n = v_n$ (say).

By convolution theorem, we have

$$Z^{-1}[U(z).V(z)] = u_n * v_n$$

i.e., $$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \sum_{m=0}^{n} u_m \cdot v_{n-m}$$

$$= \sum_{m=0}^{n} a^m \cdot b^{n-m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n\left[1+\frac{a}{b}+\left(\frac{a}{b}\right)^2+\cdots+\left(\frac{a}{b}\right)^n\right]$$

$$= b^n\left[\frac{1\left(1-\left(\frac{a}{b}\right)^{n+1}\right)}{1-\left(\frac{a}{b}\right)}\right]$$

$$= \frac{b^{n+1}-a^{n+1}}{b-a}.$$

e.g: (i) $$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \frac{3^{n+1}-1^{n+1}}{3-1} = \frac{3^{n+1}-1}{2}.$$

(ii) $$Z^{-1}\left[\frac{z^2}{(z-4)(z-5)}\right] = \frac{5^{n+1}-4^{n+1}}{5-4} = 5^{n+1}-4^{n+1}.$$

3. Find $Z^{-1}[e^{2/z}]$ using convolution theorem.

Sol: Let $U(z) = e^{1/z}$ and $V(z) = e^{1/z}$.

Then $Z^{-1}[U(z)] = \frac{1}{n!} = u_n$ and $Z^{-1}[V(z)] = \frac{1}{n!} = v_n$.

By convolution theorem, we have

$$Z^{-1}[U(z).V(z)] = u_n * v_n$$

i.e., $$Z^{-1}[e^{2/z}] = \sum_{m=0}^{n} u_m \cdot v_{n-m}$$

$$= \sum_{m=0}^{n} \frac{1}{m!} \cdot \frac{1}{(n-m)!}$$

$$= 1 \cdot \frac{1}{n!} + \frac{1}{1!} \frac{1}{(n-1)!} + \frac{1}{2!} \frac{1}{(n-2)!} + \cdots + \frac{1}{n!} \cdot 1$$

$$= \frac{1}{n!} + \frac{1}{1!} \frac{n}{n!} + \frac{1}{2!} \frac{n(n-1)}{n!} + \cdots + \frac{1}{n!}$$

$$= \frac{1}{n!} \left[1 + \frac{n}{1!} + \frac{n(n-1)}{2!} + \cdots + 1\right]$$

$$= \frac{1}{n!} \left[{}^nc_0 + {}^nc_1 + \cdots + {}^nc_n\right]$$

$$= \frac{1}{n!} [1+1]^n = \frac{2^n}{n!}$$

4. Using convolution theorem, find $Z^{-1}\left[\frac{z^2}{z^2-5z+6}\right]$.

<u>Answer:</u> $3^{n+1} - 2^{n+1}$.