

4.2: Solutions of PDEs Using Laplace Transform

Given a function $u(x, t)$ defined for all $t > 0$ and assumed to be bounded, we can apply Laplace transform in t considering x as a parameter.

$$L\{u(x, t)\} = \int_0^{\infty} e^{-st} u(x, t) dt \equiv U(x, s).$$

$$\begin{aligned} \text{Also } L\{u_t(x, t)\} &= \int_0^{\infty} e^{-st} u_t(x, t) dt \\ &= \left[e^{-st} u(x, t) \right]_0^{\infty} + s \int_0^{\infty} e^{-st} u(x, t) dt \\ &= 0 - u(x, 0) + s U(x, s) \end{aligned}$$

$$\therefore L\{u_t(x, t)\} = s U(x, s) - u(x, 0)$$

$$\text{and } L\{u_x(x, t)\} = \int_0^{\infty} e^{-st} u_x(x, t) dt \equiv U_x(x, s)$$

Solved Problems:

1. solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x \quad (x > 0, t > 0)$

with $u(0, t) = 0 \quad (t > 0)$ and $u(x, 0) = 0 \quad (x > 0)$

Sol: Given pde is $u_x(x, t) + u_t(x, t) = x \quad \rightarrow \textcircled{1}$

Taking Laplace transform on both sides, we get

$$L\{u_x(x, t)\} + L\{u_t(x, t)\} = L\{x\}$$

$$\Rightarrow U_x(x, s) + s U(x, s) - u(x, 0) = x L\{1\}$$

$$\Rightarrow \frac{dU(x, s)}{dx} + s \cdot U(x, s) = \frac{x}{s} \longrightarrow \textcircled{2}$$

This is a linear differential equation with constant coefficients.

operator form of $\textcircled{2}$ is

$$(D + s) U(x, s) = \frac{x}{s}, \quad D \equiv \frac{d}{dx}$$

The A.E. is $m + s = 0 \Rightarrow m = -s$

$$\therefore \text{C.F.} = C e^{-sx}$$

Let $U^*(x, s) = A + Bx$ be the trial solution of

P.I. of $\textcircled{2}$.

$$\text{Then } (D + s)(A + Bx) = \frac{x}{s}$$

$$\Rightarrow s[B + s(A + Bx)] = x$$

$$\Rightarrow s^2 A + Bs = 0 \text{ and } s^2 B = 1$$

$$\Rightarrow A = -\frac{1}{s^3} \text{ and } B = \frac{1}{s^2}$$

$$\therefore \text{P.I} = -\frac{1}{s^3} + \frac{x}{s^2}$$

$$\text{and hence } U(x, s) = c e^{-sx} + \frac{x}{s^2} - \frac{1}{s^3}$$

$$\text{Finally, } 0 = U(0, s) \left(\begin{array}{l} U(0, s) = L\{u(0, t)\} \\ = L\{0\} = 0 \end{array} \right)$$

$$\Rightarrow 0 = c - \frac{1}{s^3} \Rightarrow c = \frac{1}{s^3}$$

$$\text{Hence } U(x, s) = \frac{e^{-sx}}{s^3} + \frac{x}{s^2} - \frac{1}{s^3}$$

and hence

$$L^{-1}(U(x, s)) = L^{-1}\left\{e^{-sx} \cdot \frac{1}{s^3}\right\} + L^{-1}\left\{\frac{x}{s^2}\right\} - L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$\Rightarrow u(x, t) = \frac{1}{2} H(t-x) (t-x)^2 + x t - \frac{t^2}{2}$$

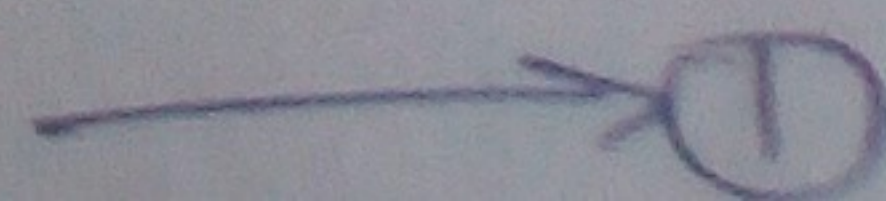
2. Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + u = 0$

with $u(0, t) = 0 \quad (t > 0)$

and $u(x, 0) = \sin x \quad (x > 0).$

Sol: Given pde is

$$u_x(x, t) + u_t(x, t) + u(x, t) = 0$$



Taking Laplace Transform on both sides, we get

$$L\{u_x(x,t)\} + L\{u_t(x,t)\} + L\{u(x,t)\} = L\{0\}$$

$$\Rightarrow U_x(x,s) + (sU(x,s) - u(x,0)) + U(x,s) = 0$$

$$\Rightarrow \frac{d}{dx} U(x,s) + (s+1) U(x,s) = \sin x \quad \rightarrow (2)$$

This is a linear differential equation with constant coefficients.

operator form of (2) is

$$(D + (s+1)) U(x,s) = \sin x, \quad (D \equiv \frac{d}{dx})$$

The A.E. is $m + (s+1) = 0$

$$\Rightarrow m = -(s+1)$$

$$\therefore C.F. = C e^{-(s+1)x}$$

Let $U^*(x,s) = A \cos x + B \sin x$ be the

trial solution of P.I of (2).

$$\text{Then } D(A \cos x + B \sin x) + (s+1)(A \cos x + B \sin x) = \sin x$$

$$\Rightarrow -A \sin x + B \cos x + (s+1)A \cos x + (s+1)B \sin x = \sin x$$

$$\Rightarrow -A + (s+1)B = 1 \quad \text{and} \quad B + (s+1)A = 0$$

$$\Rightarrow A = \frac{-1}{s^2+2s+2} \quad \text{and} \quad B = \frac{s+1}{s^2+2s+2}$$

$$\therefore \text{PI} = \frac{-\cos x}{s^2+2s+2} + \frac{(s+1) \sin x}{s^2+2s+2}$$

and hence

$$U(x, s) = C e^{-(s+1)x} + \frac{(s+1) \sin x}{s^2+2s+2} - \frac{\cos x}{s^2+2s+2}$$

Now, $U(0, s) = 0$

$$\Rightarrow C + 0 - \frac{1}{s^2+2s+2} = 0 \Rightarrow C = \frac{1}{s^2+2s+2}$$

$$\text{So, } U(x, s) = \frac{1}{s^2+2s+2} e^{-(s+1)x} + \frac{(s+1) \sin x}{s^2+2s+2}$$

$$- \frac{\cos x}{s^2+2s+2}$$

Taking the inverse Laplace transform,
we get

$$L^{-1} \{ U(x, s) \} = L^{-1} \left\{ e^{-(s+1)x} \cdot \frac{1}{s^2+2s+2} \right\}$$

$$+ L^{-1} \left\{ (\sin x) \frac{(s+1)}{s^2+2s+2} \right\}$$

$$- L^{-1} \left\{ (\cos x) \frac{1}{s^2+2s+2} \right\}$$

$$\Rightarrow u(x,t) = L^{-1} \left\{ e^{-(s+1)x} \frac{1}{(s+1)^2 + 1} \right\} \\ + (\sin x) L^{-1} \left\{ \frac{s+1}{(s+1)^2 + 1} \right\} \\ - (\cos x) L^{-1} \left\{ \frac{1}{(s+1)^2 + 1} \right\}$$

Thus,

$$u(x,t) = e^{-x} L^{-1} \left\{ e^{-sx} L \{ e^{-t} \sin t \} \right\} \\ + (\sin x) L^{-1} \left\{ L \{ e^{-t} \cos t \} \right\} \\ - (\cos x) L^{-1} \left\{ L \{ e^{-t} \sin t \} \right\}$$

$$= e^{-x} H(t-x) e^{-(t-x)} \sin(t-x) \\ + (\sin x) e^{-t} \cos t - (\cos x) e^{-t} \sin t$$

$$= e^{-t} \left[H(t-x) \sin(t-x) + \sin x \cdot \cos t - \cos x \cdot \sin t \right]$$

$$= e^{-t} \left[H(t-x) \sin(t-x) + \sin(x-t) \right]$$

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