Differential Equations & Transforms (BMAT102L)

Module 2
Partial Differential Equations

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Chapter 1

Partial Differential Equations and their Formation

1.1 Introduction

Definition 1.1.1. A relation which binds a function of two or more independent variables, and its partial derivatives upto some order is known as a *partial differential equation*. The order of the highest derivative which appears in a partial differential equation is its *order*.

Linear and Nonlinear Partial Differential Equations

Definition 1.1.2 (Linear Equation). A partial differential equation is said to be *linear*, if

(a) it has no product terms of the dependent variable with its partial derivatives,

(b) the dependent variable and its partial derivatives are not in transcendental form (trigonometric, hyperbolic, exponential, logarithmic etc.),

(c) there are no square roots, cube roots and radicals of some order of the dependent variable and its partial derivatives so that their degree is only one.

Example 1.1.1. The following are linear:

(a) Laplace equation: $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

(b) One dimensional wave equation: $\frac{\partial^2 f}{\partial t^2} = \nu^2 \frac{\partial^2 f}{\partial x^2}$

(c) One dimensional heat equation or Diffusion Equation: $\frac{\partial f}{\partial t} = \nu^2 \frac{\partial^2 f}{\partial x^2}$

(d) Transport equation: $\frac{\partial f}{\partial t} + 2\frac{\partial f}{\partial x} = 0$

Definition 1.1.3 (Quasi-linear and Almost Linear Equations). A partial differential equation is said to be

(a) *quasi-linear*, if it is linear with respect to highest order derivatives and their coefficients are functions of the independent and dependent variables only (that is, the coefficients have no lower order partial derivatives).

(b) *almost linear*, if the degree of highest-order derivatives is one and their coefficients are functions of the independent variables only.

Example 1.1.2. The following are quasi-linear:

(a) Equation for shock waves: $\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 0$

(b) Burger's equation: $\frac{\partial u}{\partial t} + cu\frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2}$

(c) $(1-u^2)\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial t} - (1+u^2)\frac{\partial^2 f}{\partial t^2} = 0$

Example 1.1.3. The following are almost linear:

(a) $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^3 = 0$

(b) $x\frac{\partial^2 u}{\partial x^2} + e^{x+y}\frac{\partial^2 u}{\partial y^2} + u\frac{\partial u}{\partial x} = 0$

Definition 1.1.4 (Fully Noninear Equation). A partial differential equation is said to be *fully nonlinear*, if it is nonlinear with respect to highest order derivatives.

Example 1.1.4. The equations $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$ and $\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} = x^2 + y^2$ are fully nonlinear.

Notation: We regard z as a function of independent variables x and y and employ the notation: $z_x = \frac{\partial z}{\partial x} = p$ and $z_y = \frac{\partial z}{\partial y} = q$, $z_{xx} = \frac{\partial^2 z}{\partial x^2}$, $z_{yy} = \frac{\partial^2 z}{\partial y^2}$, $z_{xy} = \frac{\partial^2 z}{\partial y \partial x}$.

EXERCISE 1.1.1. Classify each of the following partial differential equations with respect to the linearity and identify the order in each case:

(a) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = \cos(ax + bt)$

(b) $x^2 \cdot \frac{\partial^2 u}{\partial x^2} + y^2 \cdot \frac{\partial^2 u}{\partial y^2} = u^{1/3}$

(c) $x^2 \frac{\partial^2 u}{\partial x^2} - y^3 \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 u}{\partial x^3}$

(d) $\frac{\partial^2 f}{\partial x \partial t} + 2u^2 \cdot \frac{\partial f}{\partial y} - 4t = 0$

(e) $5xy \frac{\partial^2 u}{\partial x \partial y} - 3t \frac{\partial u}{\partial y} + 2u = 0$

Ans.

(a) linear, second order

(b) almost linear, second order

(c) linear, third order

(d) quasi-linear, second order

(e) linear, second order

1.2 Formation of a Partial Differential Equation - Elimination of Arbitrary Constants

EXERCISE 1.2.1. Obtain a partial differential equation from each of the following relations, where arbitrary constants are mentioned in braces:

(a) $z = ax^2 - by^2 \qquad (a, b)$

(b) $z = ax + by + a^2 + b^2 \qquad (a, b)$

(c) $z = a(x + y) + b(x - y) + abt + c \qquad (a, b, c)$

(d) $z = ax^2 + bxy + cy^2 \qquad (a, b, c)$

(e) $z = (x - a)^2 + (y - b)^2 + 1 \qquad (a, b)$

(f) $a \sin x + b \cos y = 2z \qquad (a, b)$

(g) $ax^2 + by^2 + z^2 = 1 \qquad (a, b)$

(h) $(x - a)^2 + (y - a)^2 + (z - b)^2 = 1 \qquad (a, b)$

(i) $z = ax + by + cxy \qquad (a, b, c)$

(j) $\log(az - 1) = x + ay + b \qquad (a, b)$

Ans. (a) $2z = px + qy$

(b) $z = px + qy + p^2 + q^2$, other possibility is $z = p^2y^2 + q^2x^2 + 2x^2y^2(px + qy) = 4x^2y^2z$

(c) $p^2 - q^2 = 4 \cdot \frac{\partial z}{\partial t}$

(d) $2z = px + qy$

(e) $p^2 + q^2 = 4(z - 1)$

(f) $p \tan x - q \cot y = z$

(g) $(px + qy)z = z^2 - 1$, other possibility is $z \cdot \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 0$

(h) $p^2 + q^2 + 1 = \left(\frac{p-q}{y-x}\right)^2$

(i) $\frac{\partial^2 z}{\partial x^2} = 0$, other possibilities are $\frac{\partial^2 z}{\partial y^2} = 0$ and $z = px + qy + \frac{\partial^2 z}{\partial x \partial y}$

(j) $p(q + 1) = qz$

EXERCISE 1.2.2. Obtain a partial differential equation from each of the following families of surfaces:

(a) $z = ae^{bx} \sin by$

(b) family of all planes, which are at a constant distance k units from the origin

(c) family of all spheres with centres lying on the z-axis

(d) family of all spheres with unit radius, and centres lying on the line $y = x$ in the xy-plane

Ans.

(a) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

(b) $z = px + qy - k\sqrt{p^2 + q^2 + 1} = z$

(c) $qx = py$

(d) $(p^2 + q^2 + 1)z^2 = 1$

1.3 Formation of a Partial Differential Equation - Elimination of Arbitrary Functions

Consider the relation

$$\phi(u, v) = 0, \tag{1.3.1}$$

where $u = u(x, y, z)$ and $v = v(x, y, z)$ are functions of x, y and z. We wish to derive a partial differential equation by eliminating the arbitrary function ϕ from the relation (1.3.1). Indeed, differentiating partially with respect to x and y and using chain rule of partial differentiation, (1.3.1) gives

$$\frac{\partial \phi}{\partial u}\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right\} + \frac{\partial \phi}{\partial v}\left\{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right\} = 0 \text{ and } \frac{\partial \phi}{\partial u}\left\{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right\} + \frac{\partial \phi}{\partial v}\left\{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right\} = 0$$

or

$$\frac{\partial \phi}{\partial u}\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right\} + \frac{\partial \phi}{\partial v}\left\{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right\} = 0 \text{ and } \frac{\partial \phi}{\partial u}\left\{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right\} + \frac{\partial \phi}{\partial v}\left\{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right\} = 0.$$

Solving these simultaneous equations for $\partial\phi/\partial u$ and $\partial\phi/\partial v$, we get the determinant relation:

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q \end{vmatrix} = 0. \tag{1.3.2}$$

We realize that u and v are functionally dependent, and hence from the theory of Jacobians, $J\left(\frac{u,v}{x,y}\right) = 0$. Since z is a function of x and y, $J\left(\frac{u,v}{x,y}\right)$ contains the partial derivatives $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Hence, (1.3.2) gives a partial differential equation of first order.

EXERCISE 1.3.1. Obtain a partial differential equation from each of the following relations by eliminating the arbitrary function f:

(a) $f(x^2 + y^2 + z^2, xyz) = 0$

(b) $f(x^2 + y^2) = z - xy$

(c) $f(x^2 + y^2, y^+ z^2) = 0$

(d) $z = f(x^2 + y^2)$

(e) $f(x^2 - y^2) = z/(x + y)$

(f) $f(x^2 + y^2 + z^2) = z$

(g) $z - x - y = f(xy)$

(h) $(x + y)f(xy + yz + zx) = z$

(i) $f(x^2 + y^2 + z^2) = y/x$

Ans.

(a) $px(y^2 - z^2) + qy(z^2 - x^2) = z(x^2 - y^2)$

(b) $py - qx = y^2 - x^2$

(c) $py - qx = xy/z$

(d) $py - qx = 0$

(e) $px + qy = z$

(f) $py - qx = 0$

(g) $px - qy = x - y$

(h) $[p(x + 2z) - q(y + 2z)](x + y) = z(x - y)$

(i) $(px + qy)z + x^2 + y^2 = 0$

EXERCISE 1.3.2. Obtain a partial differential equation from each of the following relations by eliminating the arbitrary functions μ and δ:

(a) $z = x\mu(y/x) + x\delta(y/x)$

(b) $z = y\mu(x) + x\delta(y)$

(c) $z = \mu(x) + e^y\delta(x)$

(d) $z = \mu(x + ct) + \delta(x - ct)$

Ans.

(a) $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$

(b) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy\frac{\partial^2 z}{\partial x \partial y} + z$

(c) $\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}$

(d) $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$

Chapter 2

Partial Differential Equations of First Order

2.1 Solution of a Partial Differential Equation

we begin with

Definition 2.1.1 (General Integral). A solution of a partial differential equation, which has the maximum number of arbitrary functions is called its general integral or general solution.

We shall discuss the solutions of quasi-linear and non-linear partial differential equations of first order.

Notation: We regard z as a function of independent variables x and y and employ the notation:

$$z_x = \frac{\partial z}{\partial x} = p,\ z_y = \frac{\partial z}{\partial y} = q,$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2},\ z_{yy} = \frac{\partial^2 z}{\partial y^2},\ z_{xy} = \frac{\partial^2 z}{\partial y \partial x}.$$

2.2 Quasi-linear Partial Differential Equations of First Order

Consider the Lagrange's quasi-linear equation of first order:

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z) \text{ or } Pp + Qq - R = 0. \tag{2.2.1}$$

Let $z = z(x,y)$ be a solution of (2.2.1). Then

$$f(x,y,z) \equiv z(x,y) - z = 0 \tag{2.2.2}$$

represents an *integral surface*. We recall from vector calculus that the normal at a point $\mathscr{P}(x,y,z)$ on the integral surface (2.2.2) is given by its gradient function:

$$\nabla f = (z_x, z_y, -1) = (p, q, -1). \tag{2.2.3}$$

Note that the left hand side of (2.2.1) is written as

$$Pp + Qq - R = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot (p\mathbf{i} + q\mathbf{j} + \mathbf{k}) = \mathbf{V} \cdot \nabla f,$$

where

$$\mathbf{V} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}. \tag{2.2.4}$$

Inserting (2.2.4) in (2.2.1), we see that

$$\mathbf{V} \cdot \nabla f = 0.$$

Thus $\mathbf{V}$ is perpendicular ∇f. Since ∇f is normal at $\mathscr{P}$, $\mathbf{V} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is tangent at $\mathscr{P}$. Geometrically, $\mathbf{V}$ defines a direction field, called *the characteristic field*.

Again, let $\mathscr{C}$ be a space curve on the surface (2.2.2), parametrized as

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ t \in [t_1, t_2] \tag{2.2.5}$$

then $\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}$ represents its tangent. Comparing this with (2.2.4), we find that

$$\frac{dx/dt}{P}=\frac{dy/dt}{Q}=\frac{dz/dt}{R} \quad \text{or} \quad \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=dt. \tag{2.2.6}$$

Solving any pair of the auxiliary equations (2.2.6), we obtain two linearly independent solutions

$$u(x,y,z)=a \text{ and } v(x,y,z)=b, \tag{2.2.7}$$

where a and b are arbitrary constants. Each pair of the level surfaces $u=a$ and $v=b$ represents a unique integral curve, called the *characteristic*. The locus of all characteristics (2.2.7), obtained by assigning an arbitrary functional relation $\phi(a,b)=0$ between a and b, that is

$$\phi(a,b)=0 \quad \text{or} \quad \phi\big(u(x,y,z),v(x,y,z)\big)=0 \tag{2.2.8}$$

is also an integral surface, and gives the *general integral* or *general solution* of the partial differential equation (2.2.1).

2.3 Solution of Lagrange's Equation

We follow the working rule, given below for finding the general integral of (2.2.1):

STEP 1. Write the auxiliary equations (2.2.6)

STEP 2. Find any two linearly independent solutions (2.2.7) of (2.2.6)

STEP 3. The general integral of (2.2.1) is given by (2.2.8).

Linearly independent solutions in STEP 2 are obtained either in two ways:

(a) *Method of Grouping*: any pair of fractions in (2.2.6)

(b) Method of multipliers: Let l, m, n be one set of multipliers in (2.2.6). Then each ratio in it equals the pooled ratio

$$\frac{l\,dx+m\,dy+n\,dz}{lP+mQ+nR}. \tag{2.3.1}$$

The multipliers l, m and n are chosen such that

$$lP+mQ+nR=0 \tag{2.3.2}$$

and the numerator of the pooled ratio can be grouped as the total differential of some function $u(x,y,z)$. That is

$$d[u(x,y,z)]=0, \tag{2.3.3}$$

which on integration yields a solution $u(x,y,z)=a$. Similarly, choose another set of multipliers to get a second solution $v(x,y,z)=b$.

EXERCISE 2.3.1. Find the general integral of each of the following first order partial differential equations:

(a) $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$

(b) $pyz+qzx=xy$

(c) $py^2z+qx^2z=y^2x$

(d) $py^2 - qxy = x(z - 2y)$

Ans.

(a) Auxiliary equations are $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$. Grouping the first two ratios and solving, we get $\sqrt{x} - \sqrt{y} = a$. Grouping the second and the third ratios and solving, $\sqrt{y} - \sqrt{z} = b$. Therefore, the general integral is $f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$. Other form of general integral is $g(\sqrt{x} - \sqrt{y}, \sqrt{x} - \sqrt{z}) = 0$.

(b) Auxiliary equations are $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$. Grouping the first two ratios and solving, we get $x^2 - y^2 = a$. Grouping the second and the third ratios and solving, $y^2 - z^2 = b$. Therefore, the general integral is $f(x^2 - y^2, y^2 - z^2) = 0$. Other form of general integral is $g(y^2 - z^2, x^2 - z^2) = 0$.

(c) Auxiliary equations are $\frac{dx}{y^2 z} = \frac{dy}{zx^2} = \frac{dz}{xy^2}$. Grouping the first two ratios and solving, $x^3 - y^3 = a$. Grouping the first and the third ratios and solving, $x^2 - z^2 = b$. Therefore, the general integral is $f(x^3 - y^3, x^2 - z^2) = 0$.

(d) Auxiliary equations are $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$. Grouping the first two ratios and solving, $x^2 + y^2 = a$. From the second and the third ratios, $y\,dz + z\,dy - 2y\,dy = 0$. This on integration gives the second solution $yz - y^2 = b$. Therefore, the general integral is $f(x^2 + y^2, yz - y^2) = 0$.

EXERCISE 2.3.2. Find the general integral of each of the following first order partial differential equations, using appropriate multipliers:

(a) $x(y - z)p + y(z - x)q = z(x - y)$

(b) $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$

(c) $x(y^2 - z^2)p + y(z^2 - x^2)q = z(y^2 - x^2)$

(d) $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \left(\frac{x-y}{xy}\right)$

Ans.

(a) Auxiliary equations are $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$. Choosing $(1, 1, 1)$ as multipliers, the numerator of the pooled ratio gives $dx + dy + dz = 0$ or $d(x + y + z) = 0$. Integrating this total differential, one solution is $x+y+z = a$. Choosing $(1/x, 1/y, 1/z)$ as multipliers, the numerator of the pooled ratio gives $x^{-1}\,dx + y^{-1}\,dy + z^{-1}\,dz = 0$. Integrating this, one more solution is $\log(xyz) = \log b$ or $xyz = b$. This can be achieved with multipliers (yz, zx, xy) also. Therefore, the general integral is $f(x+y+z, xyz) = 0$.

(b) Auxiliary equations are $\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$. Choosing $(1/x^2, 1/y^2, 1/z^2)$ as multipliers, the numerator of the pooled ratio gives $x^{-2}\,dx + y^{-2}\,dy + z^{-2}\,dz = 0$. Integrating this, one solution is $-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -a$ or $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a$. Choosing (yz, zx, xy) as multipliers, the numerator of the pooled ratio gives $yz\,dx + zx\,dy + xy\,dz = 0$. Integrating this, the second solution is $xyz = b$. Therefore, the general integral is $f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$.

(c) Auxiliary equations are $\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(y^2-x^2)}$. With one set of multipliers $(1/x, 1/y, -1/z)$, one solution is $\frac{xy}{z} = a$. Choosing $(x, y, -z)$ as multipliers, the second solution is $x^2 + y^2 - z^2 = b$. Therefore, the general integral is $f\left(\frac{xy}{z}, x^2 + y^2 - z^2\right) = 0$.

(d) Auxiliary equations are $\frac{dx}{1/z-1/y} = \frac{dy}{1/x-1/z} = \frac{dz}{1/y-1/x}$. With one set of multipliers $(1, 1, 1)$, one solution is $x + y + z = a$. Choosing (yz, zx, xy) as multipliers, the second solution is $xyz = b$. Therefore, the general integral is $f\big(x + y + z, xyz\big) = 0$.

EXERCISE 2.3.3. Find the general integral of each of the following first order partial differential equations:

(a) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

(b) $p\cos(x+y)+q\sin(x+y)=z$

(c) $(z^2-2yz-y^2)p+x(y+z)q=x(y-z)$

Ans.

(a) Auxiliary equations are $\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy}$. With two sets of multipliers $(1,-1,0)$ and $(0,1,-1)$, we get the combined fractions: $\frac{dx-dy}{x^2-yz-y^2+zx}=\frac{dy-dz}{y^2-zx-z^2+xy}$. Canceling the common factor $x+y+z$ from the denominators, we get $\frac{dx-dy}{x-y}=\frac{dy-dz}{y-z}$. This gives one solution: $\frac{x-y}{y-z}=a$. Similarly, by symmetry, the second solution will be $\frac{y-z}{z-x}=b$. Therefore, the general solution is $f\left(\frac{x-y}{y-z},\frac{y-z}{z-x}\right)=0$.

(b) Auxiliary equations are $\frac{dx}{\cos(x+y)}=\frac{dy}{\sin(x+y)}=\frac{dz}{z}$. With two sets of multipliers $(1,1,0)$ and $(1,-1,0)$, we get the combined fractions:

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dx-dy}{\cos(x+y)-\sin(x+y)},$$

which can be written as
$$\left[\frac{\cos(x+y)-\sin(x+y)}{\sin(x+y)+\cos(x+y)}\right]d(x+y)+d(y-x)=0.$$

Integrating this, one solution is

$$\log[\sin(x+y)+\cos(x+y)]+y-x=a.$$

Now from the first and third ratios,

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dz}{z}\quad\text{or}\quad\frac{dx+dy}{\cos(\pi/4)\cos(x+y)+\sin(\pi/4)\sin(x+y)}=\sqrt{2}\frac{dz}{z}$$

so that $\sec(x+y-\pi/4)\,d(x+y)-\sqrt{2}\,dz/z=0$. Integrating this, the second solution is

$$\log\left\{\tan\left(\frac{x+y-\pi/4}{2}+\frac{\pi}{4}\right)\right\}-\sqrt{2}\log z=b.$$

Then the general integral is

$$f\left(\log[\sin(x+y)+\cos(x+y)]+y-x,\log\left\{\tan\left(\frac{x+y}{2}+\frac{\pi}{8}\right)\right\}-\sqrt{2}\log z\right)=0.$$

(c) Auxiliary equations are $\frac{dx}{z^2-2yz-y^2}=\frac{dy}{x(y+z)}=\frac{dz}{x(y-z)}$. From the last two fractions: $\frac{dy}{y+z}=\frac{dz}{y-z}$, which on rearranging $2y\,dy-2(z\,dy+y\,dz)-2z\,dz=0$. The first solution is $y^2-2yz-z^2=a$. Now, with multipliers (x,y,z), the second solution is $x^2+y^2+z^2=b$. Hence the general integral is $f(y^2-2yz-z^2,x^2+y^2+z^2)=0$.

2.4 Nonlinear Partial Differential Equations of First Order

Definition 2.4.1 (Complete Integral). A solution of a partial differential equation, in which the number of arbitrary constants equals the number of independent variables is called its complete integral or complete solution.

For a nonlinear partial differential equation of first order:

$$f(x,y,z,p,q)=0, \tag{2.4.1}$$

the complete integral is of the form

$$\omega(x,y,z,a,b)=0. \tag{2.4.2}$$

The complete solution (2.4.2) represents a two-parameter family of surfaces. If the envelope of the system

(2.4.2) exsts, it is also a solution called the *singular integral* of the equation (2.4.1). Note that an envelope of the system (2.4.2) touches a member-surface of the system. The singular solution of (2.4.1) is obtained by eliminating the arbitrary constants a and b from the relations: $\omega \equiv 0$, $\frac{\partial\omega}{\partial a} \equiv 0$ and $\frac{\partial\omega}{\partial b} \equiv 0$.

Charpit's Auxiliary Equations

Given below are the Charpit's auxiliary equations, which are employed to get the complete solution (2.4.2) of (2.4.1):

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}. \tag{2.4.3}$$

In fact, we solve equations (2.4.1) and (2.4.3) for p and q, and then substitute these expressions in the total differential

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial x}\,dy = p\,dx + q\,dy. \tag{2.4.4}$$

Integrating (2.4.4), we get the complete integral of the form (2.4.2).

EXERCISE 2.4.1. Find the complete integrals of the following equations:

(a) $p^2x + q^2y = z$

(b) $(p^2 + q^2)y = qz$

(c) $p = (z + qy)^2$

(d) $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$

(e) $2(z + xp + yq) = yp^2$

Ans.

(a) $\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b$

(b) $(x + b)^2 + y^2 = az^2$

(c) $z = bx^a y^{1/a}$

(d) $z = \frac{2}{3}(y + a)^{3/2} + be^{3/x^2} + \frac{1}{3x^2} + \frac{1}{9}$

(e) $z = \frac{ax}{y^2} + \frac{b}{y} - \frac{a^2}{4y^3}$

2.5 Special forms of Nonlinear First Order Partial Differential Equations

TYPE 1: Equations involving only p and q

Consider an equation of the form

$$f(p, q) = 0. \tag{2.5.1}$$

For this, the Charpit's auxiliary equations reduce to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}.$$

Note that $p = a$ or $q = a$ is an obvious solution of these relations. Inserting $p = a$ in (2.5.1),

$$f(a, q) = 0 \text{ or } q = \phi(a). \tag{2.5.2}$$

Using $p = a$ and (2.5.2) in (2.4.4),

$$dz = a\,dx + \phi(a)\,dy,$$

which on integration gives the complete integral of (2.5.1) as

$$z = ax + \phi(a)y + b. \tag{2.5.3}$$

Sometimes the substitution $q = a$ reduces computations in obtaining the complete solution.

Non-existence of Singular Integral: Partially differentiating (2.5.3) with respect to b, we het a contradiction that $0 = 1$. Therefore, the envelope of the 2-parameter family of surfaces represented by the complete integral (2.5.3) does not exist. Thus Type 1 equations do not have singular integrals.

EXERCISE 2.5.1. Find the complete integrals of the following equations:

(a) $\sqrt{p} + \sqrt{q} = 1$

(b) $p + q + pq = 0$

(c) $p^2 + q^2 = 4$

(d) $p(p + 1) = q^2$

Ans.

(a) $z = ax + (1 - \sqrt{a})^2 y + b$ or $z = (1 - \sqrt{a})^2 x + ay + b$

(b) $z = ax + b - ay/(a + 1)$

(c) $z = ax + \sqrt{4 - a^2}\,y + b$

(d) $z = ax + \sqrt{a(a + 1)}\,y + b$

TYPE 2: Equations involving p and q and only one of the variables x, y and z

(a) For an equation involving p, q and x:

$$f(p, q, x) = 0. \tag{2.5.4}$$

Charpit's auxiliary equations become

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{0}.$$

Note that $q = a$ is a solution of these ratios. Inserting $q = a$ in (2.5.4),

$$f(p, a, x) = 0 \text{ or } p = \phi(a, x). \tag{2.5.5}$$

Using $q = a$ and (2.5.5) in (2.4.4),

$$dz = \int \phi(a, x)\,dx + a\,dy,$$

which on integration gives the complete integral of (2.5.7) as

$$z = \int \phi(a, x)\,dx + ay + b. \tag{2.5.6}$$

Non-existence of Singular Integral: Type 2(a) equations do not have singular integrals.

EXERCISE 2.5.2. Find the complete integrals of the following equations:

(a) $p^2 + px = q$
(b) $\sqrt{p} + \sqrt{q} = x$

Ans.
(a) $z = -\frac{x^2}{4} + \frac{x}{4}\sqrt{x^2 + 4a^2}) + a^2 \sinh^{-1}(x/2a) + a^2 y + b$
(b) $z - (x-a)^3/3 = a^2 y + b$

(b) For an equation involving p, q and y:

$$f(p, q, y) = 0. \tag{2.5.7}$$

Charpit's auxiliary equations become

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{-(f_y + qf_z)}.$$

Since $p = a$ is a solution, inserting $p = a$ in (2.5.7),

$$f(a, q, y) = 0 \text{ or } q = \phi(a, y). \tag{2.5.8}$$

Using $p = a$ and (2.5.8) in (2.4.4),

$$dz = a\,dx + \phi(a, y)\,dy,$$

which on integration gives the complete integral of (2.5.7) as

$$z = ax + \int \phi(a, y)\,dy + b. \tag{2.5.9}$$

Non-existence of Singular Integral: Type 2(b) equations do not have singular integrals.

EXERCISE 2.5.3. Find the complete integrals of the following equations:

(a) $q^2 = yp^4$
(b) $\sqrt{p} + \sqrt{q} = y$

Ans.
(a) $z = ax + \frac{2a^2y^{3/2}}{3} + b$
(b) $z = a^2x + \frac{(y-a)^3}{3} + b$

(c) For an equation involving p, q and z:

$$f(p, q, z) = 0. \tag{2.5.10}$$

Charpit's auxiliary equations become

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}.$$

From the last two ratios, we see that $p = aq$, Inserting this in (2.5.10),

$$f(aq, q, z) = 0 \text{ or } q = \phi(z, a). \tag{2.5.11}$$

Using $p = aq$ and (2.5.11) in (2.4.4),

$$dz = q(a\,dx + dy) \quad \text{or} \quad \{\phi(z, a)\}^{-1}\,dz = a\,dx + dy$$

which on integration gives the complete integral of (2.5.10):

$$\phi(z, a) = (ax + y) + b. \tag{2.5.12}$$

We may employ the substitution $q = ap$ also, and obtain the complete integral.

Non-existence of Singular Integral: Type 2(c) equations do not have singular integrals.

EXERCISE 2.5.4. Find the complete integrals of the following equations:

(a) $z = p^2 + q^2$

(b) $p^3 = qz$

(c) $z = p^2 + q^2 + 1$

(d) $z(p^2 + q^2 + 1) = 1$

(e) $4(z^3 + 1) = 9z^4 pq$

(f) $q^2 = p^2 z^2 (1 - p^2)$

Ans.

(a) $4(a^2 + 1)z = (x + ay + b)^2$

(b) $2\sqrt{z} = \sqrt{a}(x + ay + b)$

(c) $\sqrt{a^2 + 1}\cosh^{-1} z = x + ay + b$

(d) $(a^2 + 1)(1 - z^2) = (x + ay + b)^2$

(e) $a^2(z^3 + 1) = (x + a^2 y + b)^2$

(f) $z = a^2 + (x + a^2 y + b)^2$

TYPE 3: Separable Equations of the form

Consider an equation of the form

$$f(p, x) = g(q, y). \tag{2.5.13}$$

For this, the Charpit's auxiliary equations reduce to

$$\frac{\mathrm{d}x}{f_p} = \frac{\mathrm{d}y}{-g_q} = \frac{\mathrm{d}z}{pf_p - qg_q} = \frac{\mathrm{d}p}{-f_x} = \frac{\mathrm{d}q}{-g_y}.$$

From these, we have an ordinary differential equation: $\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{f_x}{f_p} = 0$, which can be written as

$$f_p \,\mathrm{d}p + f_x \,\mathrm{d}x = 0 \quad \text{or} \quad \mathrm{d}f(p, x) = 0.$$

Integrating this total differential, we get the solution

$$f(p, x) = a.$$

Using this in (2.5.13),

$$f(p, x) = a, g(q, y) = a.$$

Solving these for p and q,

$$p = \mu(x, a), q = \nu(y, a). \tag{2.5.14}$$

Using (2.5.14) in (2.4.4),

$$\mathrm{d}z = \mu(x, a)\,\mathrm{d}x + \nu(y, a)\,\mathrm{d}y,$$

which on integration gives the complete integral of (2.5.13) as

$$z = \int \mu(x, a)\,\mathrm{d}x + \int \nu(y, a)\,\mathrm{d}y + b. \tag{2.5.15}$$

Non-existence of Singular Integral: Type 3 equations also do not have singular integrals.

EXERCISE 2.5.5. Find the complete integrals of the following equations:

(a) $px^2 = qy^2$

(b) $pq + qx = y$

(c) $\frac{p^2}{x^2} - \frac{q^2}{y^2} = 1$

(d) $p^2 - q^2 = x - y$

(e) $\sqrt{p} + \sqrt{q} = x + y$

(f) $(p + q)x + pq = 0$

Ans.

(a) $xyz + a(x + y) = bxy$

(b) $z = ax - x^2 + \frac{y^2}{2a} + b$

(c) $z = \frac{x^3a^2}{3} + 2\sqrt{a^2 - 1}y + b$

(d) $3z = 2(x + a)^{3/2} + 2(y + a)^{3/2} + \kappa$

(e) $3z = (x + a)^3 + (y - a)^3 + \kappa$

(f) $2a(a + 1)z = -(a + 1)x^2 + a^y 2 + b$

EXERCISE 2.5.6. Find the general and complete integrals of the following equations:

(a) $px - qy = y^2 - x^2$

(b) $p + q = \sin x + \sin y$

(c) $px^2 - 2y^3q = 1$

(d) $2p - 3q = z$

(e) $p + q = 1$

Ans.

(a) Auxiliary equations are $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$. Grouping the first two ratios, we get $xy = a$. Choosing $(x, y, 1)$ as multipliers, the second solution is $x^2 + y^2 + 2z = b$. Therefore, the general integral is $f\left(xy, x^2 + y^2 + 2z\right) = 0$. Also, the complete integral is $x^2 + y^2 + 2z = 2a \log xy + b$.

(b) The general and complete integrals are $f(z - \cos x - \cos y, x - y) = 0$ and $z = a(x - y) - (\cos x + \cos y) + b$ respectively.

(c) The general and complete integrals are $f\left(\frac{1}{x} + \frac{1}{4y^2}, z + \frac{1}{x}\right) = 0$ and $z = -\frac{a}{x} + \frac{a-1}{4y^2} + b$ respectively.

(d) The general and complete integrals are $f\left(2x + 3y, x - \log\sqrt{z}\right) = 0$ and $(2 - 3a)\log z = 4(ax + y) + b$ respectively.

(e) The general and complete integrals are $f\left(x - y, y - z\right) = 0$ and $z = ax + (1 - a)y + b$ respectively.

TYPE 4: Clairaut's equation

Consider an equation of the form

$$z = px + qy + f(p, q). \tag{2.5.16}$$

For this, the Charpit's auxiliary equations reduce to

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}.$$

From the last two ratios, obviously $p = a$ and $q = b$ are solutions. Using these in (2.4.4) and then integrating, the complete integral of (2.5.16) is

$$z = ax + by + f(a, b). \tag{2.5.17}$$

Singular Integral: Partially differentiating (2.5.17) with respect to a and b,

$$0 = x + f_a \text{ and } 0 = y + f_b. \tag{2.5.18}$$

Eliminating a and b from (2.5.17) and (2.5.18), we obtain the singular solution of (2.5.16).

EXERCISE 2.5.7. Find the complete and singular integrals of the following equations:

(a) $z = px + qy + p^2q^2$

(b) $z = px + qy - 2\sqrt{pq}$

(c) $z = px + qy + pq$

(d) $q(z - px - qy) = p(1 - q)$

Ans.

(a) The complete integral is $z = ax + by + a^2b^2$; the singular integral is $16z^3 + 27x^2y^2 = 0$

(b) The complete integral is $z = ax + by - 2\sqrt{ab}$; the singular integral is $xy = 1$

(c) The complete integral is $z = ax + by + ab$; the singular integral is $z = xy$

(d) The complete integral is $z = ax + by + \frac{a}{b} - a$; the singular integral is $(1 - x) = y$

EXERCISE 2.5.8. Find the complete, singular and general integrals of $(1 - x)p + (2 - y)q = 3 - z$.

Ans.
The complete integral is $z = ax + by + (3 - a - 2b)$; the singular integral is $z = 3$; the general integral is $f\left(\frac{y-2}{x-1}, \frac{z-3}{y-2}\right) = 0$

Table 2.1: Reference Table

Type	Description	Finding the Complete and Singular Integrals
Type I: $f(p,q)=0$	Equations having only p and q	Substitute $p=a$ or $q=a$. Write $p=a$ in the given equation and solve it for q, say $q=\phi(a)$. Then the total differential $\mathrm{d}z = a\,\mathrm{d}x + \phi(a)\,\mathrm{d}y$ on integration gives the complete integral $z = ax+\phi(a)y+b$.
Type II(a): $f(p,q,x)=0$	Equations with p, q and x	Substitute $q=a$ in the given equation and solve it for p, say $p=\phi(a)$. Then $\mathrm{d}z = \phi(a)\,\mathrm{d}x + a\,\mathrm{d}y$ gives the complete integral $z=\phi(a)x+ay+b$.
Type II(b): $f(p,q,y)=0$	Equations having p, q and y	Write $p=a$ in the given equation and solve it for q, say $q=\phi(a)$. Then $\mathrm{d}z = a\,\mathrm{d}x + \phi(a)\,\mathrm{d}y$ gives the complete integral $z=ax+\phi(a)y+b$.
Type II(c): $f(p,q,z)=0$	Equations with p, q and z	Insert $p=aq$ in the given equation, and then solve it for q and z, say $q=\phi(z,a)$. With these substitutions, the total differential $\{\phi(z,a)\}^{-1}\,\mathrm{d}z = a\,\mathrm{d}x+\mathrm{d}y$ is integrated to get the complete integral $\phi(z,a)=(ax+y)+b$.
Type III $f(p,x)=g(q,y)$	Separables form containing p and x on one side, and q and y on the other side	Equate each side to a constant a and solve for p and q, say $p=\mu(x,a)$, $q=\nu(y,a)$. Use these in the total differential $$\mathrm{d}z = \mu(x,a)\,\mathrm{d}x + \nu(y,a)\,\mathrm{d}y,$$ which on integration then gives the complete integral $z = \int \mu(x,a)\,\mathrm{d}x + \int \nu(y,a)\,\mathrm{d}y + b$.
Type IV $z=px+qy+f(p,q)$	Clairaut's equation	Substituting $p=a$ and $q=b$ in the given equation, its complete integral is $$\sigma = z-ax-by-f(a,b)=0.$$ Elimination of the arbitrary constants a and b from the relations $\sigma\equiv 0$, $\partial\sigma/\partial a\equiv 0$ and $\partial\sigma/\partial b\equiv 0$ results in the singular solution.

REMARK 2.5.1. The singular solutions do not exist for the equations of Types I, II and III.

Chapter 3

Solution of Partial Differential Equations by Separation of Variables

3.1 Separation of Variables, Product Method or Fourier's Method

Assume that the solution of a given *pde* is of the product-form: $u(x,t) = X(x)T(t)$, neither of the factors $X(x)$ and $T(t)$ being identically 0. Then find the partial derivatives of u with respect to x and t involved, and then substitute all these expressions in in the *pde*. This reduces to a pair of ordinary differential equations in x and t. Solve these for $X(x)$ and $T(t)$. In addition, if the initial and boundary conditions are provided, find the arbitrary constants involved.

Example 3.1.1. Solve

$$3\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0, \tag{3.1.1}$$

with

$$u(0,t) = 5e^{-t}. \tag{3.1.2}$$

Solution. Write $u(x,t) = X(x)T(t)$. Then $\frac{\partial u}{\partial x} = X'(x)T(t)$ and $\frac{\partial u}{\partial t} = X(x)T'(t)$. Substituting these in (3.1.1),

$$3X'(x)T(t) - X(x)T'(t) = 0 \text{ or } \frac{1}{4}\frac{X'(x)}{X(x)} = \frac{1}{3}\frac{T'(t)}{T(t)} \cdot . \tag{3.1.3}$$

Note that the left hand side fraction is independent of t and the right hand side fraction is independent of x. Therefore, the two fractions in (3.1.3) are equal only if each of it equals some constant, say λ. Thus we get two ordinary differential equations

$$\frac{X'(x)}{X(x)} = 4\lambda \text{ and } \frac{T'(t)}{T(t)} = 3\lambda. \tag{3.1.4}$$

Solving these, $X(x) = ae^{4\lambda x}$ and $T(t) = be^{3\lambda t}$. Thus

$$u(x,t) = ae^{4\lambda x}be^{3\lambda t} \text{ or } u(x,t) = ce^{(4x+3t)\lambda}, \tag{3.1.5}$$

where $c = ab$. Employing the condition (3.1.10), $ce^{3t\lambda} = 5e^{-t}$. On comparison, we find that $c = 5$ and $\lambda = -1/3$. Thus the solution we need is $u(x,t) = 5e^{-(4x+3t)/3}$ or $u(x,t) = 5e^{-t-4x/3}$.

Example 3.1.2. Solve

$$2\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = 0, \tag{3.1.6}$$

with

$$u(x,1) = \frac{1}{3}e^{-x}. \tag{3.1.7}$$

Solution. Write $u(x,t) = X(x)T(t)$. Then $\frac{\partial u}{\partial x} = X'(x)T(t)$ and $\frac{\partial u}{\partial t} = X(x)T'(t)$. Substituting these in (3.1.1),

$$2X'(x)T(t) + tX(x)T'(t) = 0 \text{ or } \frac{X'(x)}{X(x)} = -\frac{t}{2}\frac{T'(t)}{T(t)} \cdot . \tag{3.1.8}$$

Note that the left hand side fraction is independent of t and the right hand side fraction is independent of x.

Therefore, the two fractions in (3.1.8) are equal only if each of it equals some constant, say λ. Thus we get two ordinary differential equations

$$\frac{X'(x)}{X(x)} = \lambda \text{ and } \frac{T'(t)}{T(t)} = -\frac{2\lambda}{t}\cdot \tag{3.1.9}$$

Solving these, $X(x) = ae^{\lambda x}$ and $T(t) = b/t^{2\lambda}$. Thus

$$u(x,t) = ae^{4\lambda x}b/t^{2\lambda} \text{ or } u(x,t) = c\left(e^x/t^2\right)^{\lambda}, \tag{3.1.10}$$

where $c = ab$. Employing the condition (3.1.7), $ce^{\lambda x} = \frac{1}{3}e^{-x}$. On comparison, we find that $c = 1/3$ and $\lambda = -1$. Thus the solution we need is $u(x,t) = \frac{1}{3}\left(e^x/t^2\right)^{-1}$ or $u(x,t) = t^2/3e^x$.

Self-check Exercises

EXERCISE 3.1.1. Solve the following using the method of separation of variables:

(a) $4x\frac{\partial u}{\partial x} - 5\frac{\partial u}{\partial t} = 0$

(b) $2x\frac{\partial u}{\partial x} + 3t\frac{\partial u}{\partial t} = 0$

(c) $t\frac{\partial u}{\partial x} - 2x\frac{\partial u}{\partial t} = 0$

(d) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 2u$

(e) $\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 3u$