Constrained maxima and minima

Let $u = f(x, y, z)$ be a function of x, y, and z which are connected by the relation $g(x, y, z) = 0$.

The problem of finding maximum/ minimum value of $f(x, y, z)$ subject to $g(x, y, z) = 0$ is known as a **constrained optimization** problem. It is denoted as follows:

Maximize/Minimize $f(x, y, z)$

Subject to $g(x, y, z) = 0$.

Lagrange's multiplier method is used to solve the constrained optimization problem.

Working Rule:

1. Construct the **Lagrangian function**
$$L = f(x,y,z) + \lambda g(x,y,z)$$

2. Obtain the equations $L_x = 0$; $L_y = 0$; $L_z = 0$; $L_\lambda = 0$.

3. Obtain the values of x,y and z by solving the above equations. Say a, b and c respectively.

4. The point (a,b,c) is a **stationary point** of $f(x,y,z)$ and satisfies the relation $g(x,y,z) = 0$.

Note: λ is called **a Lagrange's multiplier**

Remark: (Lagrange's multiplier methed)

If the given optimization problem is maximization type and has only one stationary point, the stationary is a **constrained maxima**.

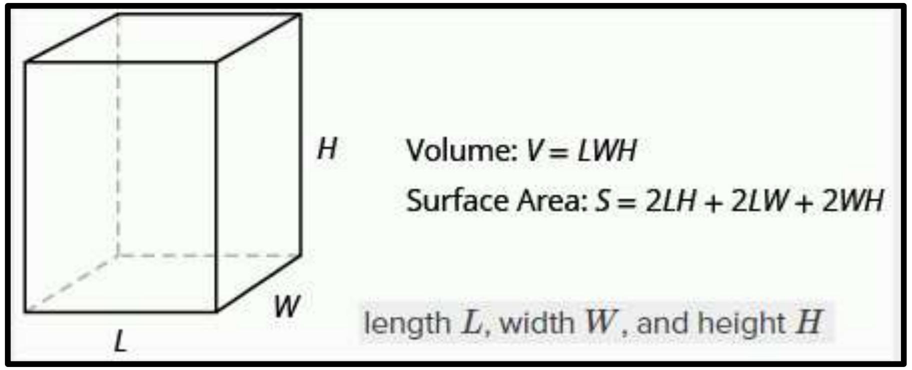
If the given optimization problem is minimization type and has only one stationary point, the stationary is **a constrained minima**.

If the given optimization problem has two or more stationary points, the point giving maximum value of the function is **a constrained maxima** and the point giving minimum value of the function is **a constrained minima**.

Problem:1

Obtain the dimensions of a rectangular box without top of maximum capacity given that the total surface area is 108 m^2.

Soln:



Let the dimensions be x, y and z.

Given total surface area= 108.

Volume , V = xyz

Total surface area, S = 2yz+2zx+xy

Therefore, 2yz+2zx+xy=108

That is, 2yz+2zx+xy-108=0.

Now, the optimization problem is given below:

Maximize xyz

subject to 2yz+2zx+xy-108=0.

Now, Lagrangian function $L = xyz + \lambda(2yz + 2xz + xy - 108)$

Now, $L_x = 0 \; ; L_y = 0 \; ; L_z = 0 \; ; L_\lambda = 0.$

This implies that

$$yz + \lambda(2z + y) = 0 \qquad (1)$$

$$xz + \lambda(2z + x) = 0 \qquad (2)$$

$$xy + \lambda(2y + 2x) = 0 \qquad (3)$$

$$2yz + 2xz + xy - 108 = 0 \qquad (4)$$

From (1) , (2) and (3), we have

$$\lambda = -\frac{yz}{2z+y} \qquad (5)$$

$$\lambda = -\frac{xz}{2z+x} \qquad (6)$$

$$\lambda = -\frac{xy}{2y+2x} \qquad (7)$$

From (5) and (6), we have

$$-\frac{yz}{2z+y} = -\frac{xz}{2z+x}$$

This implies that $x = y$

From (6) and (7), we have

$$-\frac{xz}{2z+x} = -\frac{xy}{2y+2x}$$

This implies that $x = 2z$

Now, we have x = y = 2z.

Now, substituting the values of x and y in (4), we have

$$2(2z)z + 2(2z)z + (2z)(2z) - 108 = 0$$

That is, $12z^2 = 108.$

This implies that $z = 3$.

Therefore, the dimensions of the rectangular box are

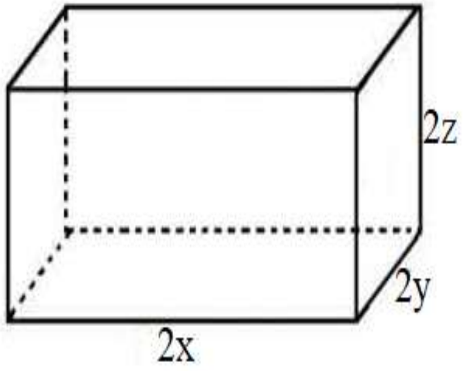
x =6, y =6 and z = 3

and the maximum volume, V= (6)(6)(3)=108.

Problem 2:

Find the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Soln:

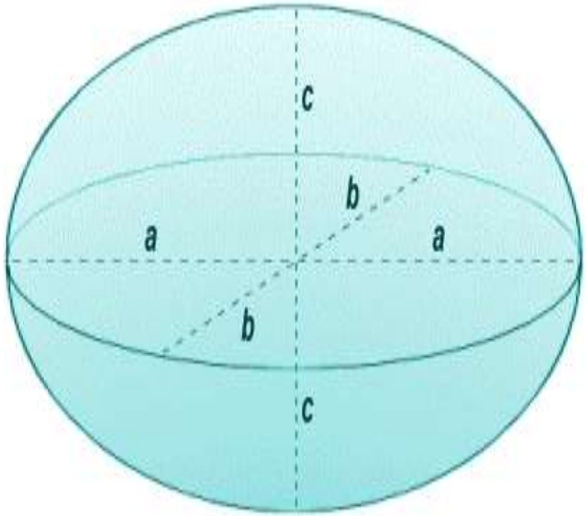


Let (x,y,z) be a vertex point of the parallelepiped (box).

Then, all other vertices are $(\pm x, \pm y, \pm z)$

Therefore , the dimensions of the box are 2x, 2y and 2z.

Then, the volume, V = 8xyz.



Given surface is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Optimization problem

Max.V = 8xyz

subject $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$

Now, Lagrangian function $L = 8xyz + \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1)$

Now,

$L_x = 0 \; ; L_y = 0 \; ; L_z = 0 \; ; L_\lambda = 0.$

This implies that

$$8yz + \frac{2\lambda x}{a^2} = 0 \qquad (1)$$

$$8xz + \frac{2\lambda y}{b^2} = 0 \qquad (2)$$

$$8xy + \frac{2\lambda z}{c^2} = 0 \qquad (3)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \qquad (4)$$

From (1) , (2) and (3), we have

$$\lambda = -\frac{4yza^2}{x} \qquad (5)$$

$$\lambda = -\frac{4xzb^2}{y} \qquad (6)$$

$$\lambda = -\frac{4xyc^2}{z} \qquad (7)$$

From (5) and (6), we have

$$-\frac{4yza^2}{x} = -\frac{4xzb^2}{y}$$

This implies that $\frac{x^2}{a^2} = \frac{y^2}{b^2}$

From (6) and (7), we have

$$-\frac{4xzb^2}{y} = -\frac{4xyc^2}{z}$$

This implies that $\dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$

Now, we have $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$ (8)

Now, from (4) and (8) , we have

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

That is, $\dfrac{x^2}{a^2} = \dfrac{1}{3}$

This implies that $x = \dfrac{a}{\sqrt{3}}$.

From (8), we have

$$\frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{1}{3}.$$

This implies that $y = \frac{b}{\sqrt{3}}$ and $z = \frac{c}{\sqrt{3}}$

Therefore, the dimensions of the rectangular box are

$$2x = \frac{2a}{\sqrt{3}} \quad , \quad 2y = \frac{2b}{\sqrt{3}} \quad \text{and} \quad 2z = \frac{2c}{\sqrt{3}} \quad .$$

Max. Volume, $V = = \frac{8abc}{3\sqrt{3}}$.

Problem 3:

A rectangular box open at the top is to have volume of 32 m^3 . Find the dimensions of the box requiring least material for its construction.

Hint: Let the dimensions be x, y and z.

Volume , V: xyz =32

Surface area, S : 2yz+2zx+xy

Optimization problem:

Minimize 2yz+2zx+xy

subject to xyz -32=0.

Now, Lagrangian function $L = 2yz + 2xz + xy + \lambda(xyz - 32)$

Problem:4

The temperature u(x,y,z) at any point in a space is $u = 400xyz^2$. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = 1.$

Hint: Optimization problem:

Max. $u = 400xyz^2$

subject to $x^2 + y^2 + z^2 - 1 = 0.$

The Lagrangian function $L = 400xyz^2 + \lambda(x^2 + y^2 + z^2 - 1).$

Problem :5

Find the maximum and minimum distances from the origin to the curve

$$5x^2+6xy+5y^2-8=0.$$

Hint:

Let P(x,y) be a point on the curve.

Now, the distance between P and the origin, $D=\sqrt{x^2+y^2}$

Let $S=D^2=x^2+y^2$

Now, the optimization problem

Optimize $S=x^2+y^2.$

subject to $5x^2+6xy+5y^2-8=0.$

The Lagrangian function $L=x^2+y^2+\lambda(5x^2+6xy+5y^2-8)$

EXAMPLE Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

HINTS:

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that x and y have the same sign. With these values for x and y, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

Thus,

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

Minimizing a sum of squares Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

Ans: 3,3,3

Find all the local maxima, local minima, and saddle points of the functions

1. $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$
2. $f(x, y) = x^2 + xy + 3x + 2y + 5$
3. $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$

Ans:

$f(-3, 3) = -5$, local minimum

$f(-2, 1)$, saddle point

$f\left(3, \frac{3}{2}\right) = \frac{17}{2}$, local maximum