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Beta and Gamma Functions

The betā function is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad m,n>0$$

If we put $(1-x) = y \Rightarrow x = 1-y$

$\Rightarrow -dx = dy$

x	0	1
y	1	0

Hence,

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}\, y^{n-1}\,(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m)$$

Now, if we put $x = \sin^2\theta$. $(1-x) = \cos^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}(2\sin\theta\cos\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta\,\cos^{2n-2+1}\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

Gamma Functions:

The gamma function is denotēd as $\Gamma(n)$, and is defined by

$$\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}\,dx \qquad n>0$$

In particular

$$\Gamma(1) = \int_0^\infty e^{-x} x^0 \, dx$$

$$= \int_0^\infty e^{-x} \, dx = 1$$

Reduction Formula:

$$\Gamma(n+1) = n\,\Gamma(n)$$

particularly, if $n \in \mathbb{N}$, then

$$\Gamma(n+1) = n!$$

For an example, $\Gamma(8) = 7\,\Gamma(7) = 7\cdot 6\;\Gamma(6)$

$$= 7!$$

$n+1 = \frac{7}{2}$

$n = \frac{5}{2}$

$$\Gamma\left(\tfrac{7}{2}\right) = \frac{5}{2}\,\Gamma\left(\tfrac{5}{2}\right)$$

$$= \frac{5}{2}\cdot\frac{3}{2}\;\Gamma\left(\tfrac{3}{2}\right)$$

$$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\;\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{15}{8}\;\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{15\sqrt{\pi}}{8}$$

Relation between gamma and beta functions:

$$\beta(m,n) = \frac{\Gamma(m)\;\Gamma(n)}{\Gamma(m+n)}$$

Find the value for $\Gamma\left(\tfrac{1}{2}\right)$.

Soln: Let us consider, $\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \dfrac{\Gamma\left(\tfrac{1}{2}\right)\;\Gamma\left(\tfrac{1}{2}\right)}{\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\right)} = \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2$

Now,

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{2}-1}\theta\;\cos^{2\cdot\frac{1}{2}-1}\theta\;d\theta$$

$$= 2\int_0^{\pi/2} d\theta = \frac{\pi}{2}\cdot 2$$

$$\Rightarrow \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \pi$$

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Prob: Simplify $\int^{1} \; dx$

Prob: Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$

Soln: Consider $x^2 = \sin\theta$

x	0	1
θ	0	$\pi/2$

$\Rightarrow x = \sqrt{\sin\theta}$

$\Rightarrow dx = \frac{1}{2}\sin^{-\frac{1}{2}}\theta \cos\theta\, d\theta$

$1-x^4$

$= 1-\sin^2\theta$

$= \cos^2\theta$

Then, $$\int_0^{\pi/2} \frac{\frac{1}{2}\sin^{-\frac{1}{2}}\theta \cos\theta\, d\theta}{\cos\theta}$$

$2m-1 = -\frac{1}{2}$

$\Rightarrow m = \frac{1}{4}$

$2n-1 = 0$

$\Rightarrow n = \frac{1}{2}$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-\frac{1}{2}}\theta\, d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{2}\,\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{4}\cdot\frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})}$$

$$= \frac{\sqrt{\pi}}{4}\,\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$$

Prob: Evaluate: $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

Soln: $$\int_0^{\pi/2} \sin^{\frac{1}{2}}\theta\, \cos^{-\frac{1}{2}}\theta\, d\theta$$

$2m-1 = \frac{1}{2}$

$\Rightarrow m = \frac{3}{4}$

$2n-1 = -\frac{1}{2}$

$\Rightarrow n = \frac{1}{4}$

$$= \frac{1}{2}\,\beta\left(\frac{3}{4},\frac{1}{4}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4}+\frac{1}{4})} = \frac{1}{2}\,\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$