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Module - 4

Probability Distributions

Theoretical probability distribution gives us a law according to which different values of random variables are distributed with specified probabilities.

1) Binomial distribution:-

(a) Let a random experiment be performed repeatedly, each repitition being called a trial.

(b) The occurrence of an event in a trial is called success and non occurence is called failure

(c) Consider a set of n ~~ide~~ independent trials in which probability of success in p in any trial and $q = 1-p$ is the probability of failure in any trial.

The probability of x successes and consequently $(n-x)$ failures in n independent trial in given by.

$$= \underbrace{p.p.p.p}_{x \text{ times}} \quad \underbrace{q.q.q \cdots q}_{n-x \text{ times}}$$

$$= p^x q^{n-x}$$

in n trials, x successes can occur in ^nC_x ways ie probababitily of x Success in n trials is given by (by addition theo)

$$^nC_x \, p^x \, q^{n-x}$$

The probability distribution of no. of Success, so obtain is called the Binomal probability distribution.

ie prob. of 0 success $= {}^nC_0 p^0 q^n = q^n$

prob. of 1 success $= {}^nC_1 p q^{n-1}$

prob. of 2 success $= {}^nC_2 p^2 q^{n-2}$

" " n success $= {}^nC_n p^n q^0 = p^n$

Def^n

A random variable x is said to follow binomial distribution if its p.m.f is given by

$$\boxed{P(X=x) = {}^nC_x p^x q^{n-x}; \quad x=0,1,2,3,\dots n \quad \& \ q = 1-p}$$

x = no. of success
n = no. of trial

Note:-

$P(X=0) = q^n$
$P(X=1) = {}^nC_1 p q^{n-1} = {}^nC_1 q^{n-1} p$
$P(X=2) = {}^nC_2 q^{n-2} p^2$
$\vdots$
$P(X=n) = {}^nC_n q^0 p^n = p^n$

these all ~~all~~ are successive terms of the bionomial expansion of $(q+p)^n$.

ie $$\boxed{\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} = (q+p)^n = 1}$$ total prob

Mean of Bionomial distribution :-

$\boxed{\text{Mean} = np}$

$$\text{Mean} = \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=0}^{\infty} x\, {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{\infty} x \frac{n!}{x!\, n-x!} p^{x} q^{n-x}$$

$$= \sum_{x=0}^{\infty} \frac{1}{x-1!} \frac{n(n-1)!}{n-x!} p^{x} q^{n-x}$$

$\frac{x}{x!} = \frac{1}{x-1!}$

$n! = n(n-1!)$

$$= np \sum_{x=0}^{\infty} \frac{1}{x-1!} \frac{n-1!}{n-x!} p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{\infty} \frac{1}{x-1!} \frac{n-1!}{n-x!} p^{x-1} q^{n-x}$$

put $x-1 = y$
$n-1 = m$

$$= np \underbrace{\sum_{y=0}^{\infty} \frac{1}{y!} \frac{m!}{m-y!} p^{y} (1-p)^{m-y}}_{} = np \sum_{y=0}^{\infty} {}^{m}C_{y}\, p^{y} q^{m-y}$$

$$= np\, (p+(1-p))^{n} = np$$

standard deviation of Bionomial distribution:

$$\sigma^2 = \sum (x-\bar{x})^2 p(x) \qquad \bar{x} = \text{Mean}$$

$$\sigma = \sqrt{(x-\text{Mean})^2 p(x)}$$

$$\boxed{\sigma = \sqrt{npq}}$$ ← standard deviation

$$\boxed{\text{variance} = npq}$$

Ex:- Find the mean of probability distribution of the no. of heads obtained in three flips of a balanced coin.

Soln: for a bionomial distribution, here

$n = 3$

$p = 1/2$

Mean $= np$

$$\boxed{\therefore \text{Mean} = 3/2}$$

Q:- The mean and variance of a binomial distribution are 4 and $4/3$ respectively. find $p(x \geq 1)$.

Soln:

$$\text{Mean} = np$$

$$\boxed{np = 4}$$

$$\text{Variance} = 4/3$$

$$npq = 4/3$$

$$\Rightarrow \boxed{q = 1/3}$$

using $p + q = 1$

$$p = 1 - q = 1 - 1/3$$

$$\boxed{p = \frac{2}{3}} \qquad \boxed{n = 6}$$

Now $P(X = x) = \cancel{\sum_x} \; {}^{n}C_{x}\, p^{x} q^{n-x} = {}^{6}C_{x} \left(\frac{2}{3}\right)^{x} \left(\frac{1}{3}\right)^{6-x}$

0 1 2 3 4 X

Now $P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^{n}C_{0}\, p^{0} q^{n-0}$$

$$= 1 - 1 \cdot 1 \cdot \left(\frac{1}{3}\right)^{6}$$

$$\boxed{P(X \geq 1) = 0.998}$$

Q:- A coin is tossed 6 times. what is the probability of obtaining four or more heads.

Soln:- $n = 6$

probability of getting head in one trial $= \frac{1}{2}$

$$p = 1/2 \quad q = 1/2$$

$$P(X = x) = {}^{6}C_{x}\, p^{x} q^{n-x}$$

$$\left[P(X=x) = {}^{6}C_{x} \left(\tfrac{1}{2}\right)^{x} \left(\tfrac{1}{2}\right)^{6-x} \right]$$

$$P(X \geq 4) = P(X=4) + P(X=5) + P(X=6)$$

$$= {}^{6}C_{4} \left(\tfrac{1}{2}\right)^{4} \left(\tfrac{1}{2}\right)^{2} + {}^{6}C_{5} \left(\tfrac{1}{2}\right)^{5} \left(\tfrac{1}{2}\right)^{1} + {}^{6}C_{6} \left(\tfrac{1}{2}\right)^{6} \left(\tfrac{1}{2}\right)^{0}$$

$$= 15 \times \left(\tfrac{1}{2}\right)^{2} \times \left(\tfrac{1}{2}\right)^{4} + 6 \times \left(\tfrac{1}{2}\right)^{5} \times \left(\tfrac{1}{2}\right)^{1} + \left(\tfrac{1}{2}\right)^{6}$$

$$P(X \geq 4) = 0.234 + 0.094 + 0.016$$

$$\boxed{P(X \geq 4) = 0.349}$$

Note:- Probability of all the events are the terms of the expansion of $(q+p)^6$

$$(q+p)^6 = q^6 + 6q^5 p + 15q^4 p^2 + 20q^3 p^3 + 15q^2 p^4 + 6q p^5 + p^6 = 1$$

ie

$$\sum_{x=0}^{6} {}^{6}C_{x}\, p^{x} q^{n-x} = {}^{6}C_{0}\, q^6 p^0 + {}^{6}C_{1}\, q^5 p^1 + {}^{6}C_{2}\, q^4 p^2 + {}^{6}C_{3}\, q^3 p^3 + {}^{6}C_{4}\, q^2 p^4 + {}^{6}C_{5}\, q^1 p^5 + {}^{6}C_{6}\, q^0 p^6$$

$= 1$

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Ex: 20% of items produced by a factory are defective. Find the probability that in a sample of 5 chosen at random

1. None is defective.
2. one is defective.
3. $P(1<X<4)$

Sol^n:- Probability of one defective item

$$p = \frac{20}{100} = \frac{1}{5}$$

$$q = 1 - \frac{1}{5} = \frac{4}{5}$$

$$n = 5$$

Probability distribution:- out of 5, prob. of x defective item

$$P(X=x) = {}^nC_x \, p^x q^{n-x}, \quad n = 1,2,3,4,5$$

$$P(X=x) = {}^5C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{n-x}$$

1. $P(X=0) = {}^5C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^5 = \frac{1024}{3121} = .3277$

2. $P(X=1) = {}^5C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^4 = .4096$

3. $P(1<X<4) = P(X=2) + P(X=3)$

$$= {}^5C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3 + {}^5C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2$$

$$= .2566$$

Q:- In a bionomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are ·4096 and ·2048 respectively. Find the parameter 'p' of the distribution.

soln:- $n = 5$

$P(X=1) = \cdot 4096, \quad P(X=2) = \cdot 2048.$

$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$P(X=x) = {}^{5}C_{x}\, p^{x} q^{5-x}$

ie $\quad {}^{5}C_{1}\, p\, q^{4} = \cdot 4096, \quad {}^{5}C_{2}\, p^{2} q^{3} = \cdot 2048$

dividing

$$\frac{{}^{5}C_{1}\, p\, q^{4}}{{}^{5}C_{2}\, p^{2} q^{3}} = \frac{\cdot 4096}{\cdot 2048}$$

$$\frac{\frac{5!}{1!\,4!}\, q}{\frac{5!}{2!\,3!}\, p} = \frac{\cdot 4096}{\cdot 2048}.$$

$$\frac{2!\,3!}{4!}\,\frac{(1-p)}{p} = 2$$

$$\frac{2}{4}\,\frac{1-p}{p} = 2$$

$$\frac{1-p}{p} = 4 \Rightarrow 5p = 1 \Rightarrow \boxed{p = \frac{1}{5}}$$

Q:- The probability of a man hitting a target is $\frac{1}{4}$.

(a) If he fires 7 times what is the probability of his hitting the target at least twice.

(b) How many times he must be fire so that the probability of his hitting the target at least once is greater than $\frac{2}{3}$.

Solution:- Probability of hitting a target $= \frac{1}{4}$

(a) $n = 7$

$p = \frac{1}{4}$ $\quad q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

$$P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$P(X = x) = {}^{7}C_{x} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{7-x}$$

Probability of hitting the target at least twice

ie $P(X \geq 2) = 1 - P(X \leq 1)$

$= 1 - (P(X=0) + P(X=1))$

$$P(X \geq 2) = 1 - \left({}^{7}C_{0} \left(\frac{1}{4}\right)^{0} \left(\frac{3}{4}\right)^{7-0} + {}^{7}C_{1} \left(\frac{1}{4}\right)^{1} \left(\frac{3}{4}\right)^{6}\right)$$

$$\boxed{P(X \geq 2) = \frac{4547}{8192}}$$

(b) Let he should fire n time

ie $P(X \geq 1) > \frac{2}{3}$ ie $1 - P(X=0) > \frac{2}{3}$

ie $\cancel{{}^{n}C_{x}\, p^{x} q^{n-x} > 2/3}$

$$1 - {}^{n}C_0 \, p^0 q^n > \frac{2}{3}$$

$$1 - \left(\frac{3}{4}\right)^n > \frac{2}{3}$$

$$1 - \frac{2}{3} > \left(\frac{3}{4}\right)^n$$

$$\left(\frac{3}{4}\right)^n < \frac{1}{3}$$

$$n(\log 3 - \log 4) < \log 1 - \log 3$$

$$n(\cdot 4771 - \cdot 6020) < 0 - \cdot 4771$$

$$n \times - \cdot 1250 < - \cdot 4771$$

$$n > \frac{\cdot 4771}{\cdot 1250}$$

$$n > 3.8$$

as trial can'nt be fractional ie

$$\boxed{n = 4}$$

Q:- A coffee connoisseur claims that he can distinguish between a cup of instant coffee and a cup of percolator coffee 75% of the time. It is agreed that his claim will be accepted if he correctly identifies at least 5 of the 6 cups. Find his chances of having the claims (a) accepted (b) rejected, when he does have the ability he claims.

Solⁿ:- Let p denotes the probability of correct distinction b/w a cup of instant coffee and cup of percolator coffee

$$p = \frac{75}{100} = \frac{3}{4}, \quad q = \frac{1}{4}, \quad n = 6$$

Let X be the random variable, denotes the no. of correct distinction, then by Binomial distribution

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}.$$

$$P(X=x) = {}^{6}C_{x} \left(\frac{3}{4}\right)^{x} \left(\frac{1}{4}\right)^{6-x}$$

$x = 0, 1, 2, 3, 4, 5, 6$

(a) The prob. of the claim being accepted

$$P(X \geq 5) = P(5) + P(6)$$

$$= {}^{6}C_{5} \left(\frac{3}{4}\right)^{5} \left(\frac{1}{4}\right)^{6-5} + {}^{6}C_{6} \left(\frac{3}{4}\right)^{6}$$

$$\boxed{P(X \geq 5) = 0.534}$$

(b) The prob. of the claim being rejected is

$$P(X \leq 4) = 1 - P(X \geq 5)$$

$$= 1 - 0.534$$

$$\boxed{P(X \leq 4) = 0.466}$$

Q:- In a precision bombing attack there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target.

Let X be a random variable, representing no. of bombs striking the target.

given prob. that bomb strike the target $= \frac{50}{100}$

$= \frac{1}{2}$

$$P(X=x) = {}^{n}C_{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{n-x}$$

Let n bombs must be dropped such that

$$P(X \geq 2) \geq 0.99$$

$$1 - (P(X=0) + P(X=1)) \geq 0.99$$

$$1 - \left[{}^{n}C_{0} \left(\frac{1}{2}\right)^{0} \left(\frac{1}{2}\right)^{n} + {}^{n}C_{1} \left(\frac{1}{2}\right)^{1} \left(\frac{1}{2}\right)^{n-1} \right] \geq 0.99$$

$$1 - 0.99 \geq 1 \times \left(\frac{1}{2}\right)^{n} + n \times \left(\frac{1}{2}\right)^{n}$$

$$0.01 \geq \frac{n+1}{2^{n}}$$

$$2^{n} \geq 100(n+1)$$

$$2^{n} \geq 100n + 100.$$

$\cancel{n \log 2 \geq n \log 100 + \log}$

by trial method

$$\boxed{n = 11}$$

ie Min no. of bombs needed to destroy the target is 11.

Q1 A irregular 6 faced die is such that the prob that it gives 3 even number in 5 throws is twice the prob. that it gives 2 even nos in 5 throws. How many sets of exactly 5 trial can be expected to give no even number out of 2500 sets.

Solⁿ:-

$n = 5$

$P(X=3) = 2\,P(X=2)$

$^5C_3\, p^3 q^2 = 2 \times {}^5C_2\, p^2 q^3$

$\Rightarrow \boxed{p = 2q}$

$p = 2(1-p)$

$\Rightarrow \boxed{p = \frac{2}{3}}$ and $\boxed{q = \frac{1}{3}}$

Prob. of getting no even no

$P(X=0) = {}^5C_0\, p^0 q^5 = \left(\frac{1}{3}\right)^5$

$\boxed{P(X=0) = \frac{1}{243}}$

required no. of sets $= N \times P(X=0)$

$= 2500 \times \frac{1}{243}$

$= 10$ (approx)

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Moment generating function of bionomial distribution :-

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{\infty} {}^nC_x (e^t p)^x q^{n-x}$$

$$\boxed{M_x(t) = (e^t p + q)^n} = \mu_r'$$

$$\boxed{(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_n b^n}$$

Mean and variance of Bionomial distribution

$$\text{Mean} = \mu_1' = \left[\frac{d}{dt}(M_x(t))\right]_{t=0}$$

$$\mu_1' = \left[n(e^t p + q)^{n-1}(e^t \cdot p)\right]_{t=0}$$

$$= n(p+q)^{n-1} p$$

$$\boxed{\mu_1' = np} \qquad p+q=1$$

Varimce of Binomial distribution

$$\mu_2' = \left[\frac{d^2}{dt^2}\{M_x(t)\}\right]_{t=0}$$

$$\mu_2' = \left[\frac{d}{dt}\left(n(pe^t+q)^{n-1}\, pe^t\right)\right]_{t=0}$$

$$\mu_2' = \Big[n(n-1)(pe^t+q)^{n-2}(pe^t+0)\,pe^t + n(pe^t+q)^{n-1}\,pe^t\Big]_{t=0}$$

$$\mu_2' = n\left[(n-1)(p+q)^{n-2}p^2 + (p+q)^{n-1}p\right]$$

$$\mu_2' = np\left[(n-1)p+1\right]$$

$$\mu_2' = n(n-1)p^2 + np$$

$$\text{variance} = \mu_2' - (\mu_1')^2$$

$$= (n(n-1)p^2 + np) - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$\boxed{\text{variance} = npq}$$