Laplace transform The *Laplace transform* of $f(t)$ defined for $t \geq 0$, is given by

$$L\{f(t)\} = \int_0^\infty f(t)\, e^{-st} dt = \bar{F}(s).$$

The function $f(t)$ is called the *inverse* of the transform $\bar{F}(s)$.

Table 1 **Laplace transform – Power Function**

	$f(t)$	$\bar{F}(s) = L\{f(t)\}$
Integer powers	1	$1/s$
	t	$1/s^2$
	t^2	$2!/s^3$
	$\vdots$	$\vdots$
	t^n	$n!/s^{n+1}$
	$t^p, p > 0$	$\Gamma(p+1)/s^{p+1}$, where $\Gamma(p+1) = \int_0^\infty x^p e^{-x} dx$
Positive real powers	$\sqrt{t}$	$\Gamma\left(\frac{3}{2}\right)\frac{1}{s^{3/2}} = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{1}{s^{3/2}} = \frac{\sqrt{\pi}}{2\, s^{3/2}}$

Table 2 **Laplace transform – Exponential and Trigonometric Functions**

	$f(t)$	$\bar{F}(s) = L\{f(t)\}$
Exponential	e^{at}	$\frac{1}{s-a}$
	e^{-at}	$\frac{1}{s+a}$
Trigonometric	$\sin at$	$\frac{a}{s^2+a^2}$
	$\cos at$	$\frac{s}{s^2+a^2}$

Linearity property: $L\{af(t) + bg(t)\} = a\bar{F}(s) + b\bar{G}(s)$, where $\bar{F}(s)$ and $\bar{G}(s)$ are the Laplace transforms of $f(t)$ and $g(t)$.

Table 3 **Laplace transform – Hyperbolic Functions**

Function	Laplace transform
$\sinh at = \frac{e^{at}-e^{-at}}{2}$	$\frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)=\frac{a}{s^2-a^2}$
$\cosh at = \frac{e^{at}+e^{-at}}{2}$	$\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)=\frac{s}{s^2-a^2}$

Table 4 **Examples on Linearity**

Function	Laplace transform
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$
$\frac{ae^{at}-be^{bt}}{a-b}$	$\frac{s}{(s-a)(s-b)}$
$1-\cos at = 2\sin^2\left(\frac{at}{2}\right)$	$\frac{a^2}{s(s^2+a^2)}$
$at-\sin at$	$\frac{a^3}{s^2(s^2+a^2)}$
$\sin^3 at = \frac{3\sin at-\sin 3at}{4}$	$\frac{6a^3}{(s^2+a^2)(s^2+9a^2)}$
$\cosh at-\cos at$	$\frac{2a^2s}{s^4-a^4}$
$\sinh at-\sin at$	$\frac{2a^3}{s^4-a^4}$
$\sin at\sin bt$	$\frac{2abs}{[s^2+(a-b)^2]\,[s^2+(a+b)^2]}$
$\frac{\cos at-\cos bt}{b^2-a^2}$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$

Laplace transform through Multiplication by *t*: If $\bar{F}(s)$ is the Laplace transform of $f(t)$, then

$$L\{t\,f(t)\} = -\frac{d\,\bar{F}(s)}{d\,s}, L\{t^2 f(t)\} = \frac{d^2\,\bar{F}(s)}{d\,s^2}, \dots, L\{t^n f(t)\} = (-1)^n\frac{d^n\,\bar{F}(s)}{d\,s^n}.$$

Table 5 **Examples on Multiplication by t**

Function	Laplace transform
te^{at}	$-\frac{d}{ds}\left(\frac{1}{s-a}\right)=\frac{1}{(s-a)^2}$
$t\cos at$	$-\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)=\frac{s^2-a^2}{(s^2+a^2)^2}$
$L\{t\sin at\}$	$-\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)=\frac{2as}{(s^2+a^2)^2}$
$L\{\sin at-at\cos at\}$	$\frac{a}{s^2+a^2}-\frac{a(s^2-a^2)}{(s^2+a^2)^2}=\frac{2a^3}{(s^2+a^2)^2}$
$\sin at+at\cos at$	$\frac{a}{s^2+a^2}+\frac{a(s^2-a^2)}{(s^2+a^2)^2}=\frac{2as^2}{(s^2+a^2)^2}$

Laplace transform through Division by t: If $\bar{F}(s)$ is the Laplace transform of $f(t)$, then

$$L\left\{\frac{f(t)}{t}\right\}=\int_{u=s}^{\infty}\bar{F}(u)\,\mathrm{d}u=\bar{G}(s).$$

Hence

$$\int_0^{\infty}\left\{\frac{f(t)}{t}\right\}e^{-st}\,\mathrm{d}t=\bar{G}(s)\quad\Rightarrow\quad\int_0^{\infty}\frac{f(t)}{t}\,\mathrm{d}t=\lim_{s\to 0}\bar{G}(s).$$

Table 6 **Examples on Division by t**

Function	Laplace transform
$\frac{1-e^{-at}}{t}$	$\int_{u=s}^{\infty}\left(\frac{1}{u}-\frac{1}{u+a}\right)\mathrm{d}u=\left\lvert\log\left(\frac{u}{u+a}\right)\right\rvert_{u=s}^{\infty}=\log\left(\frac{s+a}{s}\right)$
$\frac{e^{-at}-e^{-bt}}{t}$	$\int_{u=s}^{\infty}\left(\frac{1}{u+a}-\frac{1}{u+b}\right)\mathrm{d}u=\left\lvert\log\left(\frac{u+a}{u+b}\right)\right\rvert_{u=s}^{\infty}=\log\left(\frac{s+b}{s+a}\right)$ $\Rightarrow\int_0^{\infty}\left(\frac{e^{-at}-e^{-at}}{t}\right)\mathrm{d}t=\lim_{s\to 0}\log\left(\frac{s+b}{s+a}\right)=\log\left(\frac{b}{a}\right).$
$\frac{\cos at-\cos bt}{t}$	$\int_{u=s}^{\infty}\left(\frac{u}{u^2+a^2}-\frac{u}{u^2+b^2}\right)\mathrm{d}u=\frac{1}{2}\left\lvert\log\left(\frac{u^2+a^2}{u^2+b^2}\right)\right\rvert_{s}^{\infty}$ $=\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$ $\Rightarrow\int_0^{\infty}\left(\frac{\cos at-\cos bt}{t}\right)\mathrm{d}t=\frac{1}{2}\lim_{s\to 0}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)=\log\left(\frac{b}{a}\right).$

$\frac{1-\cos at}{t}$	$\int_{u=s}^{\infty}\left(\frac{1}{u}-\frac{u}{u^2+a^2}\right)du = \left\lvert\log\left(\frac{u}{\sqrt{u^2+a^2}}\right)\right\rvert_s^{\infty} = \log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$ $\Rightarrow \int_0^{\infty}\left(\frac{1-\cos at}{t}\right)e^{-bt}dt = \lim_{s\to b}\log\left(\frac{\sqrt{s^2+a^2}}{s}\right) = \log\left(\frac{\sqrt{b^2+a^2}}{b}\right)$

Table 6 **Examples on Division by *t* (Continued)**

Function	Laplace transform
$\frac{\sin^2 at}{t} = \frac{1-\cos 2at}{2t}$	$\int_0^{\infty}\left(\frac{\sin^2 at}{t}\right)e^{-bt}dt = \frac{1}{2}\log\left(\frac{\sqrt{b^2+4a^2}}{b}\right)$
$\frac{\sin at}{t}$	$\int_{u=s}^{\infty}\left(\frac{a}{u^2+a^2}\right)du = \left\lvert\tan^{-1}\left(\frac{u}{a}\right)\right\rvert_{u=s}^{\infty} = \frac{\pi}{2}-\tan^{-1}\left(\frac{s}{a}\right)$ $\Rightarrow \int_0^{\infty}\frac{\sin at}{t}dt = \lim_{s\to 0}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{a}\right)\right] = \frac{\pi}{2}$
$\frac{1-\cosh at}{t}$	$\int_{u=s}^{\infty}\left(\frac{1}{u}-\frac{u}{u^2-a^2}\right)du = \left\lvert\log\left(\frac{u}{\sqrt{u^2-a^2}}\right)\right\rvert_s^{\infty}$ $= \log\left(\frac{\sqrt{s^2-a^2}}{s}\right)$
$\frac{1}{\sqrt{t}} = \frac{\sqrt{t}}{t}$	$\frac{\sqrt{\pi}}{2}\int_{u=s}^{\infty}\left(\frac{1}{u^{3/2}}\right)du = \frac{\sqrt{\pi}}{2}\left\lvert-\frac{2}{\sqrt{u}}\right\rvert_s^{\infty} = \sqrt{\frac{\pi}{s}}$

First Shifting Property (*s*-Shifting): If $\bar{F}(s)$ is the Laplace transform of $f(t)$, then

$$L\{e^{at}f(t)\} = \bar{F}(s-a).$$

Table 7 **Examples on First Shifting Property**

Function	Laplace transform
$t^n e^{at}$	$n!/(s-a)^{n+1}$
$t^p e^{at}, p>0$	$\Gamma(p+1)/(s-a)^{p+1}$
$\sqrt{t}\,e^{at}$	$\frac{\sqrt{\pi}}{2(s-a)^{3/2}}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$

$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sinh bt$	$\frac{b}{(s-a)^2-b^2}$
$e^{at}\cosh bt$	$\frac{s-a}{(s-a)^2-b^2}$

Heaviside Unit Step Function: For $a \geq 0$, $H(t-a) = U_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$

Note: $1 - H(t-a) = U_a(t) = \begin{cases} 1 & t < a \\ 0 & t \geq a \end{cases}$

Second Shifting Property (t-shifting): If $\bar{F}(s)$ is the Laplace transform of $f(t)$, then

$$L\{H(t-a)\, f(t-a)\} = e^{-as}\bar{F}(s).$$

Table 8 **Examples on Second Shifting Property**

Function	Laplace transform
$H(t-a) = H(t-a).1$	$e^{-as}L\{1\} = \frac{e^{-as}}{s}$
$1 - H(t-a)$	$\frac{1-e^{-as}}{s}$
$H\left(t-\frac{\pi}{2}\right)\cos t = H\left(t-\frac{\pi}{2}\right)\sin\left(t-\frac{\pi}{2}\right)$	$\frac{\pi}{2}\cdot\frac{1}{s^2+(\pi/2)^2}\cdot e^{-\pi s/2} = \frac{2\pi\, e^{-\pi s/2}}{s^2+\pi^2}$
Rectangular Pulse $R(t;a,b) = H(t-a) - H(t-b)$	$\frac{e^{-as}-e^{-bs}}{s}$

Think about It Suppose that

$$f(t) = \begin{cases} g(t), & a \leq t < b \\ 0, & \text{elsewhere.} \end{cases}$$

What is the Laplace transform of $f(t)$?

Think about It Suppose that

$$f(t) = \begin{cases} g(t) & 0 \leq t < a \\ h(t) & t \geq a. \end{cases}$$

What is the Laplace transform of $f(t)$?

Dirac Delta Function: For arbitrarily small $\varepsilon > 0$, and $a \geq 0$, we have the pulse function

$$\delta_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & a \leq t < a+\varepsilon \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\delta_\varepsilon(t-a) = \frac{1}{\varepsilon}[H(t-a) - H(t-a-\varepsilon)]$

$\Rightarrow \qquad L\{\delta_\varepsilon(t-a)\} = \frac{e^{-as} - e^{-(a-\varepsilon)s}}{s} = \frac{e^{-as}(1-e^{-\varepsilon s})}{s}$.

The Dirac Delta function is defined by $\delta(t-a) = \lim_{\varepsilon\to 0} \delta_\varepsilon(t-a)$.

Thus $\quad \delta(t-a) = \begin{cases} +\infty & t = a \\ 0 & t \neq a. \end{cases}$

Now,

$\mathscr{L}\{\delta(t-a)\} = \lim_{\varepsilon\to 0} \mathscr{L}\{\delta_\varepsilon(t-a)\} = \lim_{\varepsilon\to 0} \frac{e^{-as}(1-e^{-\varepsilon s})}{s} = e^{-as}$ for $a \geq 0$.

In particular, $\mathscr{L}\{\delta(t)\} = 1$.