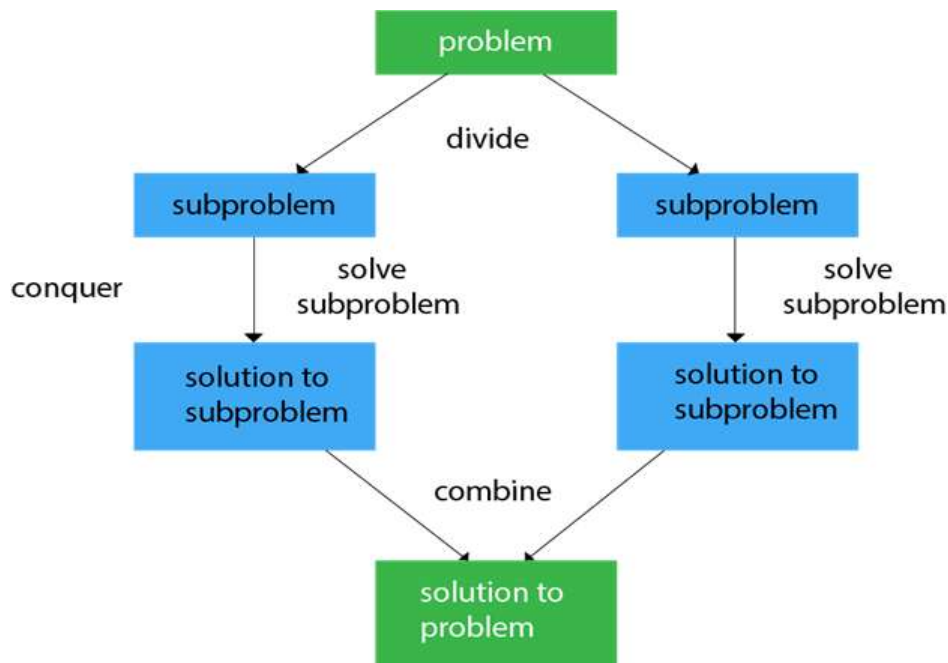


Divide And Conquer

- In divide and conquer approach, the problem in hand, is divided into smaller sub-problems and then each problem is solved independently.
- When we keep on dividing the subproblems into even smaller sub-problems, we may eventually reach a stage where no more division is possible.
- Those "atomic" smallest possible sub-problem (fractions) are solved.
- The solution of all sub-problems is finally merged in order to obtain the solution of an original problem.
- **Divide-and-conquer** approach is a three-step process:
 - **Divide (Break):** This involves dividing the problem into smaller sub-problems.
 - **Conquer (Solve):** Solve sub-problems by calling recursively until solved.
 - **Combine (Merge):** Combine the sub-problems to get the final solution of the whole problem.



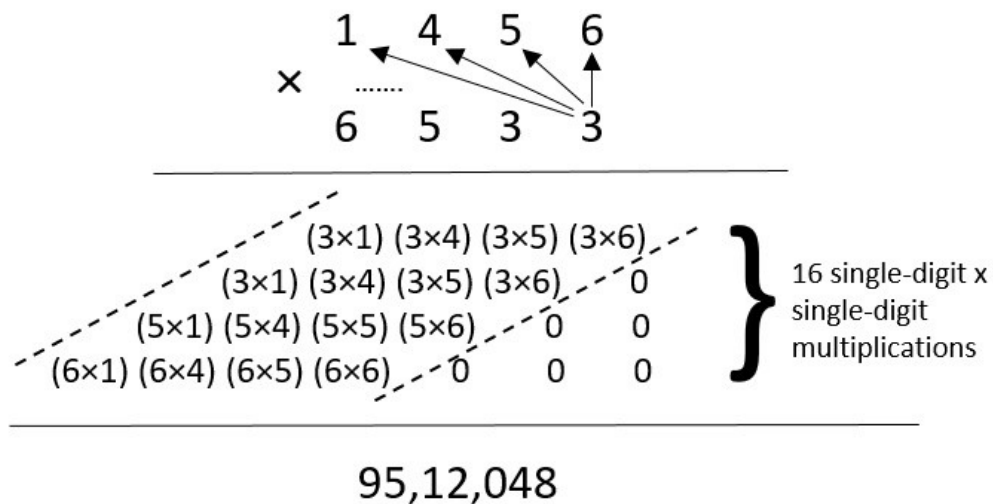
Divide and Conquer – Strategy

```
Algorithm DAndC( $P$ )
{
    if Small( $P$ ) then return  $S(P)$ ;
    else
    {
        divide  $P$  into smaller instances  $P_1, P_2, \dots, P_k, k \geq 1$ ;
        Apply DAndC to each of these subproblems;
        return Combine(DAndC( $P_1$ ), DAndC( $P_2$ ), ..., DAndC( $P_k$ ));
    }
}
```

Karatsuba Fast Integer Multiplication Algorithm

- The **Karatsuba algorithm** is used to perform fast multiplication on two n -digit numbers, i.e. the system compiler takes lesser time to compute the product than the time-taken by a normal multiplication.
- The usual multiplication approach takes n^2 computations to achieve the final product, since the multiplication has to be performed between all digit combinations in both the numbers and then the sub-products are added to obtain the final product. This approach of multiplication is known as **Naive Multiplication**.
- Example: Let us consider two 4-digit integers: **1456** and **6533**, and find the product using Naive approach.

$$1456 \times 6533 = ?$$



- In this method of naive multiplication, given the number of digits (n) in both numbers is 4, there are 16 single-digit $\times$ single-digit multiplications being performed.
- Thus, the time complexity of this approach is $O(n^2)$ since it takes 4^2 steps to calculate the final product.
- But when the value of n keeps increasing, the time complexity of the problem also keeps increasing.
- Hence, Karatsuba algorithm is adopted to perform faster multiplications.

Karatsuba Algorithm

- The main idea of the Karatsuba Algorithm is to reduce multiplication of multiple sub problems to multiplication of **three sub problems**.
- Arithmetic operations like additions and subtractions are performed for other computations.
- For this algorithm, two n -digit numbers are taken as the input and the product of the two number is obtained as the output.
- In this algorithm we assume that n is a power of 2.

- Three subproblems:
 - $a = x_H y_H$
 - $d = x_L y_L$
 - $e = (x_H + x_L) (y_H + y_L) - a - d$
- Then $xy = a r^n + e r^{n/2} + d$

Time complexity:

- $T(n) = 3 T(n/2) + O(n)$
- $T(n) = O(n^{\log 3}) = O(n^{1.584...})$

Example:

- Compute $1234 * 4321$.
- Subproblems:
 - $a_1 = 12 * 43$
 - $d_1 = 34 * 21$
 - $e_1 = (12 + 34) * (43 + 21) - a_1 - d_1$
 $= 46 * 64 - a_1 - d_1$
 - Need to recurse...

- First subproblem:

$$a_1 = 12 * 43$$

- Subproblems:

$$- a_2 = 1 * 4 = 4$$

$$- d_2 = 2 * 3 = 6$$

$$\begin{aligned} - e_2 &= (1+2)(4+3) - a_2 - d_2 \\ &= 11 \end{aligned}$$

- Answer: $4 * 10^2 + 11 * 10 + 6 = 516$

- Second subproblem

$$d_1 = 34 * 21$$

- Subproblems:

$$- a_2 = 3 * 2 = 6$$

$$- d_2 = 4 * 1 = 4$$

$$\begin{aligned} - e_2 &= (3+4)(2+1) - a_2 - d_2 \\ &= 11 \end{aligned}$$

- Answer: $6 * 10^2 + 11 * 10 + 4 = 714$

- Third subproblem:

$$e_1 = 46 * 64 - a_1 - d_1$$

- Subproblems:

$$- a_2 = 4 * 6 = 24$$

$$- d_2 = 6 * 4 = 24$$

$$\begin{aligned} - e_2 &= (4+6)(6+4) - a_2 - d_2 \\ &= 52 \end{aligned}$$

- Answer: $24 * 10^2 + 52 * 10 + 24 - 714 - 516$
 $= 1714$

- Final Answer:

$$\begin{aligned} 1234 * 4321 &= 516 * 10^4 + 1714 * 10^2 + 714 \\ &= 5,332,114 \end{aligned}$$

Pseudocode:

Karatsuba (X, Y)

if X and Y has one digit then

Return $X * Y$

else

Let n = number of digits

Let X_H and X_L be higher and lower order $n/2$ digits of X

Let Y_H and Y_L be higher and lower order $n/2$ digits of Y

$a = \text{Karatsuba}(X_H, Y_H)$

$d = \text{Karatsuba}(X_L, Y_L)$

$e = \text{Karatsuba}(X_H + X_L, Y_H + Y_L) - a - d$

$XY = a 10^n + e 10^{n/2} + d$

Return XY