Change of basis

The matrix representation $[Id]_\beta^\alpha$ of the identity transformation $Id: V \to V$ with respect to any two bases α and β is called the transition matrix (or) the co-ordinate change matrix from β to α.

$$[Id]_\beta^\alpha = \left([Id]_\alpha^\beta\right)^{-1}.$$

1) Find the basic change matrix (transition matrix) from $(2,3,1), (1,2,0), (2,0,3)$ to $(1,0,1), (1,1,0), (0,1,1)$.

Solution:

Let $\alpha = \{\underset{v_1}{(2,3,1)}, \underset{v_2}{(1,2,0)}, \underset{v_3}{(2,0,3)}\}$

$\beta = \{\underset{w_1}{(1,0,1)}, \underset{w_2}{(1,1,0)}, \underset{w_3}{(0,1,1)}\}$

$$v_1 = a_1 w_1 + a_2 w_2 + a_3 w_3$$

$$(2,3,1) = a_1(1,0,1) + a_2(1,1,0) + a_3(0,1,1)$$
$$= (a_1, 0, a_1) + (a_2, a_2, 0) + (0, a_3, a_3)$$
$$= (a_1 + a_2, a_2 + a_3, a_1 + a_3)$$

$$\Rightarrow \left.\begin{aligned} a_1 + a_2 &= 2 \\ a_2 + a_3 &= 3 \\ a_1 + a_3 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} a_1 &= 0 \\ a_2 &= 2 \\ a_3 &= 1. \end{aligned}$$

$$v_2 = b_1 w_1 + b_2 w_2 + b_3 w_3 = b_1(1,0,1) + b_2(1,1,0) + b_3(0,1,1)$$
$$(1,2,0) = (b_1, 0, b_1) + (b_2, b_2, 0) + (0, b_3, b_3)$$
$$= (b_1 + b_2, b_2 + b_3, b_1 + b_3)$$

$$\Rightarrow \left.\begin{aligned} b_1 + b_2 &= 1 \\ b_2 + b_3 &= 2 \\ b_1 + b_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} b_1 &= -\tfrac{1}{2} \\ b_2 &= \tfrac{3}{2} \\ b_3 &= \tfrac{1}{2}. \end{aligned}$$

$$v_3 = (2,0,3) = c_1 w_1 + c_2 w_2 + c_3 w_3$$
$$= c_1(1,0,1) + c_2(1,1,0) + c_3(0,1,1)$$
$$= (c_1, 0, c_1) + (c_2, c_2, 0) + (0, c_3, c_3)$$
$$= (c_1 + c_2,\ c_2 + c_3,\ c_1 + c_3)$$

$$\left.\begin{aligned} c_1 + c_2 &= 2 \\ c_2 + c_3 &= 0 \\ c_1 + c_3 &= 3 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 5/2 \\ c_2 &= -1/2 \\ c_3 &= 1/2. \end{aligned}$$

$$[id]_{\alpha}^{\beta} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & 5/2 \\ 2 & 3/2 & -1/2 \\ 1 & 1/2 & 1/2 \end{bmatrix}$$

2) Find the transition matrix from a basis α to another basis β for the 3-space R^3, where,
$\alpha = \{(1,0,1), (1,1,0), (0,1,1)\}$, $\beta = \{(2,3,1), (1,2,0), (2,0,3)\}$.

Solution:

$$\alpha = \{\overset{v_1}{(1,0,1)}, \overset{v_2}{(1,1,0)}, \overset{v_3}{(0,1,1)}\}$$
$$\beta = \{\underset{w_1}{(2,3,1)}, \underset{w_2}{(1,2,0)}, \underset{w_3}{(2,0,3)}\}$$

$$v_1 = a_1 w_1 + a_2 w_2 + a_3 w_3 = a_1(2,3,1) + a_2(1,2,0) + a_3(2,0,3)$$

$$\Rightarrow \left.\begin{aligned} 2a_1 + a_2 + 2a_3 &= 1 \\ 3a_1 + 2a_2 &= 0 \\ a_1 + 3a_3 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} a_1 &= -2 \\ a_2 &= 3 \\ a_3 &= 1 \end{aligned}$$

$$v_2 = (1,1,0) = b_1 w_1 + b_2 w_2 + b_3 w_3 = b_1(2,3,1) + b_2(1,2,0) + b_3(2,0,3)$$

$$\Rightarrow \left.\begin{aligned} 2b_1 + b_2 + 2b_3 &= 1 \\ 3b_1 + 2b_2 &= 1 \\ b_1 + 3b_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} b_1 &= -3 \\ b_2 &= 5 \\ b_3 &= 1 \end{aligned}$$

$v_3 = (0,1,1) = c_1 w_1 + c_2 w_2 + c_3 w_3 = c_1(2,3,1) + c_2(1,2,0) + c_3(2,0,3)$

$$\Rightarrow \left.\begin{aligned} 2c_1 + c_2 + 2c_3 &= 0 \\ 3c_1 + 2c_2 &= 1 \\ c_1 + 3c_3 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 7 \\ c_2 &= -10 \\ c_3 &= -2 \end{aligned}$$

$$[Id]_\alpha^\beta = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 7 \\ 3 & 5 & -10 \\ 1 & 1 & -2 \end{bmatrix}.$$

Similarity:-

For any square matrix A and B, A is said to be similar to B if there exists a non-singular matrix Q such that $B = Q^{-1} A Q$.

Remark:- $[T]_\alpha = [Id]_\beta^\alpha [T]_\beta [Id]_\alpha^\beta$

$[T]_\beta = [Id]_\alpha^\beta [T]_\alpha [Id]_\beta^\alpha$

$= P^{-1} [T]_\alpha P$, where, $P^{-1} = [Id]_\alpha^\beta$, $P = [Id]_\beta^\alpha$.

1) If $T: R^3 \to R^3$ be the linear transformation defined by $T(x_1, x_2, x_3) = (2x_1 + x_2,\ x_1 + x_2 + 3x_3,\ -x_2)$ and $\alpha = (e_1, e_2, e_3)$ is the standard basis and $\beta = \{(-1,0,0), (2,1,0), (1,1,1)\}$ is any ordered basis. Find, $[T]_\alpha$ and hence $[T]_\beta = P^{-1}[T]_\alpha P$.

Solution:-

Given, $T(x_1, x_2, x_3) = (2x_1 + x_2,\ x_1 + x_2 + 3x_3,\ -x_2)$

$T(e_1) = T(1,0,0) = (2,1,0) = 2e_1 + e_2 + 0\cdot e_3$

$T(e_2) = T(0,1,0) = (1,1,-1) = e_1 + e_2 - e_3$

$T(e_3) = T(0,0,1) = (0,3,0) = 3e_2$

$$[T]_\alpha = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{bmatrix}.$$

$[T]_\beta = P^{-1}[T]_\alpha P$, where, $P = [Id]_\beta^\alpha$.

$v_1 = (-1, 0, 0) = -e_1 + 0 \cdot e_2 + 0 \cdot e_3$

$v_2 = (2, 1, 0) = 2e_1 + e_2 + 0 \cdot e_3$

$v_3 = (1, 1, 1) = e_1 + e_2 + e_3$

$$P = [Id]_\beta^\alpha = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$ [$\because$ using any method, for example Gauss-Jordan (or) $A^{-1} = \frac{1}{|A|} \operatorname{adj} A$.

$$[T]_\beta = P^{-1}[T]_\alpha P = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{bmatrix}.$$

2) Let $T: R^3 \to R^3$, $[T]_\beta = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}$, where, $\beta = \{(1,1,0), (1,0,1), (0,1,1)\}$.

Find, $[T]_\alpha$ if α is standard basis.

Solution:

We know that, $[T]_\beta = P^{-1}[T]_\alpha P$ and

where, $P = [id]_\beta^\alpha$ and $P^{-1} = [id]_\alpha^\beta$.

$$[T]_\alpha = P[T]_\beta P^{-1}.$$

$v_1 = (1, 1, 0) = e_1 + e_2$

$v_2 = (1, 0, 1) = e_1 + e_3$

$v_2 = (0, 1, 1) = e_2 + e_3$

$$\Rightarrow P = [Id]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and $[Id]_{\alpha}^{\beta} = P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$\therefore [T]_{\alpha} = [Id]_{\beta}^{\alpha} [T]_{\beta} [Id]_{\alpha}^{\beta}$.

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 1 & \cancel{1}1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 & 2 & 2 \\ 3 & -1 & 1 \\ -1 & 1 & 7 \end{bmatrix}.$$

3) Let D be the differential operator on the vector space $P_2(\mathbb{R})$. Given, two ordered bases $\alpha = \{1, x, x^2\}$ and $\beta = \{1, 2x, 4x^2-2\}$ for $P_2(\mathbb{R})$. To find, $[D]_{\alpha}$ and $[D]_{\beta}$.

Solution:

Given, $\alpha = \{1, x, x^2\}$, $\beta = \{1, 2x, 4x^2-2\}$

$D(1) = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 = 0$

$D(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$$\therefore [D]_{\alpha} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly, $D(1) = 0 = 0 \cdot 1 + 0 \cdot 2x + 0 \cdot (4x^2-2)$

$D(2x) = 2 = 2 \cdot 1 + 0 \cdot 2x + 0 \cdot (4x^2-2)$

$D(4x^2-2) = 8x = 0 \cdot 1 + 4 \cdot 2x + 0 \cdot (4x^2-2)$

$$[D]_{\beta} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, the transition matrix Q from $\beta = \{1, 2x, 4x^2-2\}$ to $\alpha = \{1, x, x^2\}$ calculated as,

$1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$$4x^2-2 = (-2)\cdot 1 + 0\cdot x + 4\cdot x^2$$

$$P = [Id]_\beta^\alpha = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Then, $P^{-1} = [Id]_\alpha^\beta = \frac{1}{4}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore [D]_\beta = P^{-1}[D]_\alpha P$$

$$= \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

SIMILARITY OF MATRICES

If A and B are two square matrices then B is said to be similar to A, if there exists a non-singular matrix P such that $B = P^{-1}AP$

Properties of Similar Matrices

(i) Similar matrices have the same determinant.
(ii) Similar matrices have the same rank.
(iii) Similar matrices have the same nullity.
(iv) Similar matrices have the same trace.
(v) Similar matrices have the same characteristic polynomial.
(vi) Similar matrices have the same eigenvalues.
(vii) If λ is an eignvalue of two similar matrices, the eigenspace of both the similar matrices corresponding to λ have the same dimension.

Two matrices representing the same linear operator $T : V \to V$ with respect to different bases are similar. If S_1 and S_2 are two different bases for a vector space V then matrices $[T]_{S_1}$ and $[T]_{S_2}$ are similar.

Hence, $$\det\left([T]_{S_1}\right) = \det\left([T]_{S_2}\right)$$

Example 1: Show that the matrices $\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ are similar but that $\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$ and $\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$ are not.

Solution: Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, C = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$

$$\det(A) = \begin{vmatrix} 1 & 1 \\ -1 & 4 \end{vmatrix} = 5$$

$$\det(B) = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$$

$$\det(C) = \begin{vmatrix} 3 & 1 \\ -6 & -2 \end{vmatrix} = 0$$

$$\det(D) = \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

Since $\det(A) = \det(B)$, matrices A and B are similar.
Since $\det(C) \neq \det(D)$, matrices C and D are not similar.

Example 2: Let $T: R^3 \to R^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ -x_2 \\ x_1 + 7x_3 \end{bmatrix}$$

S_1 is the standard basis for R^3 and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Verify that
$$\det(T) = \det\left([T]_{S_1}\right) = \det\left([T]_{S_2}\right).$$

Solution: The standard matrix of T is

$$[T] = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{bmatrix}$$

Since S_1 is the standard basis for R^3,

$$[T]_{S_1} = [T] = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{bmatrix}$$

$$\det(T) = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{vmatrix} = -8$$

Since S_1 is the standard basis for R^3,

$$[\mathbf{w}_1]_{S_1} = [\mathbf{w}_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$[\mathbf{w}_2]_{S_1} = [\mathbf{w}_2] = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$[\mathbf{w}_3]_{S_1} = [\mathbf{w}_3] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Transition matrix from S_2 to S_1 is

$$P = \left[[\mathbf{w}_1]_{S_1} \,\middle|\, [\mathbf{w}_2]_{S_1} \,\middle|\, [\mathbf{w}_3]_{S_1} \right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus
$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix of T w.r.t the basis S_2 is

$$[T]_{S_2} = P^{-1}[T]_{S_1}P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{bmatrix}$$

$$\det([T]_{S_2}) = \begin{vmatrix} 1 & 4 & 3 \\ -1 & -2 & -9 \\ 1 & 1 & 8 \end{vmatrix} = -8$$

Hence, $\det(T) = \det\left([T]_{S_1}\right) = \det\left([T]_{S_2}\right)$