Example Problems:

1. Solve $(D^2+1)y = e^x$, where $D \equiv \frac{d}{dx}$, using the method of Undetermined Co-efficients.

Sol: Given differential equation is

$$(D^2+1)y = e^x. \quad \left(D \equiv \frac{d}{dx}\right) \longrightarrow (1)$$

Let $f(D) \equiv D^2+1$. Then the A.E. is $f(m)=0$.

i.e., $m^2+1=0 \Rightarrow m = 0 \pm i$

Therefore, $y_c = e^{0x}(c_1 \cos x + c_2 \sin x)$

or, $y_c = c_1 \cos x + c_2 \sin x$

Now, to find particular Integral (P.I).

Let $y^* = Ae^x$ be the trail solution of y_p for eq^n (1).

Then, from (1), we have

$$D^2 y^* + y^* = e^x$$

$$\Rightarrow D^2(Ae^x) + Ae^x = e^x$$

$$\Rightarrow Ae^x + Ae^x = e^x \Rightarrow 2Ae^x = e^x \Rightarrow A = \frac{1}{2}$$

$\therefore$ $y_p = \frac{1}{2}e^x$ and hence the general solution of (1) is $y = y_c + y_p$

i.e., $y = c_1 \cos x + c_2 \sin x + \frac{e^x}{2}$

2) Solve $(D-2)y = e^{2x}$ $\left(D \equiv \frac{d}{dx}\right)$ using the method of undetermined coefficients.

Sol: Given differential equation is

$$(D-2)y = e^{2x} \longrightarrow (1)$$

clearly $y_c = c_1 e^{2x}$.

Let $y^* = Ae^{2x}$ be the trail solution of y_p for (1). Then, from (1), we have

$$Dy^* - 2y^* = e^{2x}$$

$$\Rightarrow D(Ae^{2x}) - 2Ae^{2x} = e^{2x}$$

$$\Rightarrow 2Ae^{2x} - 2Ae^{2x} = e^{2x}$$

$$\Rightarrow \boxed{0 = e^{2x}} \times \text{ (not possible)}$$

Let $y^* = Axe^{2x}$ be the trail solution of y_p for (1). Then, from (1), we get

$$D(Axe^{2x}) - 2Axe^{2x} = e^{2x}$$

$$\Rightarrow A[2xe^{2x} + e^{2x}] - 2Axe^{2x} = e^{2x}$$

$\Rightarrow Ae^{2x} = e^{2x} \Rightarrow A = 1$

$\therefore y_p = xe^{2x}$ and hence the general solution of ① is $y = y_c + y_p$

i.e., $y = c_1 e^{2x} + xe^{2x}$.

3) Solve $(D^2+D+1)y = x\cos x$ $\left(D \equiv \frac{d}{dx}\right)$ Hint: $y^* = (A+Bx)(C\cos x + D\sin x)$

4) Solve $(D^2+4D+3)y = x^2$ $\left(D \equiv \frac{d}{dx}\right)$

5) Solve $(D^2+1)y = \sin x$ $\left(D \equiv \frac{d}{dx}\right)$

Sol: Given differential equation is

$$(D^2+1)y = \sin x \longrightarrow ①$$

Clearly $y_c = c_1\cos x + c_2 \sin x$.

Let $y^* = A\sin x + B\cos x$ be the trail solution of y_p for ①. Then, from ①, we have

$$D^2 y^* + y^* = \sin x$$

$$\Rightarrow D^2(A\sin x + B\cos x) + (A\sin x + B\cos x) = \sin x$$

$$\Rightarrow -A\sin x - B\cos x + A\sin x + B\cos x = \sin x$$

$$\Rightarrow 0 = \sin x \quad \times$$

Let $y^* = x(A\sin x + B\cos x)$ be the trail solution of y_p for ①. Then, from ①, we have

$$D^2[x(A\sin x + B\cos x)] + x(A\sin x + B\cos x) = \sin x$$

$$\Rightarrow D[A(x\cos x + \sin x) + B(-x\sin x + \cos x)] + x(A\sin x + B\cos x) = \sin x$$

$$\Rightarrow A(-\cancel{x\sin x} + \cos x + \cos x) + B(-\cancel{x\cos x} - \sin x - \sin x) + x(A\cancel{\sin x} + B\cancel{\cos x}) = \sin x$$

$$\Rightarrow \quad 2A\cos x - 2B\sin x = 1\cdot\sin x + 0\cdot\cos x$$

$$\Rightarrow \quad 2A = 0 \quad \text{and} \quad -2B = 1$$

$$\Rightarrow \quad A = 0 \quad \text{and} \quad B = -\frac{1}{2}$$

$\therefore y^* = -\frac{x}{2}\cos x$ and hence the general solution of ① is

$y = y_c + y_p$ i.e., $y = c_1\cos x + c_2\sin x - \frac{x}{2}\cos x$.

6) Solve $(D^2 + D + 2)y = x^2 e^x$ $\left(D \equiv \frac{d}{dx}\right)$ using the method of undetermined coefficients.

[Hint : $y^* = e^x(A + Bx + Cx^2)$]

(7) Solve $(D^2+1)y = e^x$, where $D \equiv \frac{d}{dx}$, using the method of Variation of parameters.

Sol: Given differential equation is

$$(D^2+1)\,y = e^x \longrightarrow (1)$$

Clearly $y_c = c_1 \cos x + c_2 \sin x$.

Let $y_p = A(x)\,u(x) + B(x)\,V(x)$, where $u(x) = \cos x$,

$V(x) = \sin x$. Here $R(x) = e^x$ and $uv' - vu' = 1 \neq 0$

Now, $A(x) = -\int \frac{R(x)\,V(x)}{uv' - vu'}\,dx$

$$= -\int \frac{e^x \cdot \sin x}{1}\,dx$$

$\left[\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)\right]$

$$= -\frac{e^x}{1^2+1^2}(1 \cdot \sin x - 1 \cdot \cos x)$$

$$= -\frac{e^x}{2}[\sin x - \cos x]$$

and $B(x) = \int \frac{R(x)\,u(x)}{uv' - vu'}\,dx$

$$= \int \frac{e^x \cos x}{1}\,dx$$

$\left[\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)\right]$

$$= \frac{e^x}{2}[\cos x + \sin x]$$

$$\therefore y_p = \frac{e^x}{2}(\cos x - \sin x)\cos x + \frac{e^x}{2}(\cos x + \sin x)\sin x.$$

$$= \frac{e^x}{2}\left[\cos^2 x + \sin^2 x\right]$$

$$= \frac{e^x}{2}.$$

Hence the general solution of the given differential equation is

$$y = y_c + y_p$$

i.e., $y = C_1 \cos x + C_2 \sin x + \frac{e^x}{2}$

(8) Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = e^x$

(9) Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = x^2$

(10) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^2+1}$.

[Hint: $u(x) = e^x$, $v(x) = xe^x$

and $uv' - vu' = e^{2x}$.

$A(x) = -\frac{1}{2}\log_e(x^2+1)$, $B(x) = \tan^{-1}(x)$]