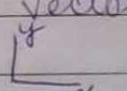
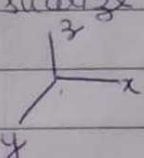
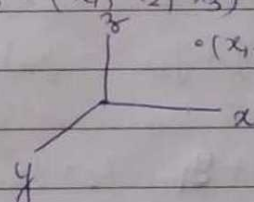


Module-2Vector spaces:-

The Euclidean space $\mathbb{R}^n$ and vector space, subspace, linear combination, span, linearly dependence, independence, bases, dimensions, finite dimensional vector space.

Def:- A vector is a quantity that has two independent properties magnitude and direction.

The geometric pictures of vectors can be easily visualize in $\mathbb{R} \xrightarrow{\quad} x$, $\mathbb{R}^2$  and in $\mathbb{R}^3$ 

for eg:- in $\mathbb{R}^3$, vector $x = (x_1, x_2, x_3)$ can be easily visualized as 

but for $n > 3$, its not easy to visualize

For all $x \in \mathbb{R}^n$, the two algebraic operations on vectors are

- (1) addition of two vectors
- (2) scalar multiplication of a vector by a scalar.

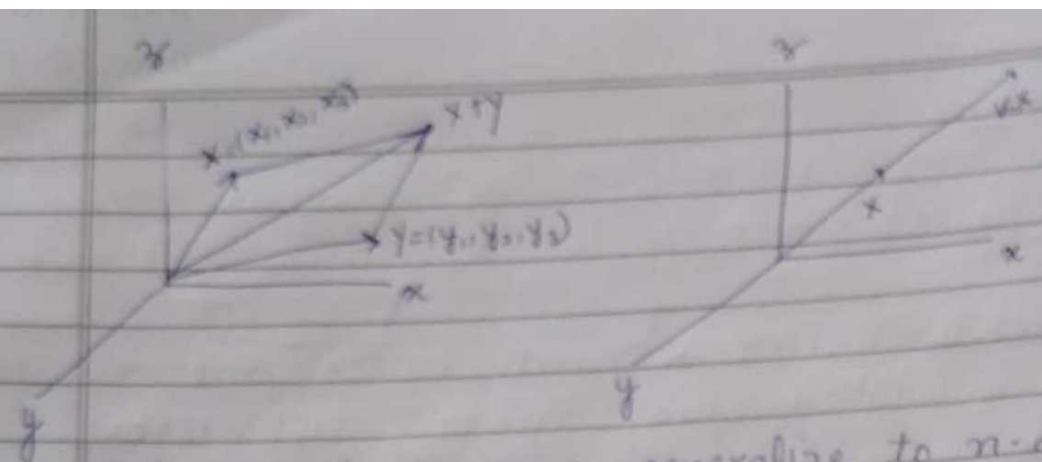
Let $n=3$

$$\text{i.e. } x = (x_1, x_2, x_3) \in \mathbb{R}^3 \quad y = (y_1, y_2, y_3) \in \mathbb{R}^3$$

then

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

$$kx = k(x_1, x_2, x_3) = (kx_1, kx_2, kx_3)$$



this concept or may generalize to n -dim space $\mathbb{R}^n$. An ordered n -tuple is defined as $x = (x_1, x_2, \dots, x_n)$
Euclidean space $\mathbb{R}^n$ is defined as collection of all ordered n -tuples.

i.e.

$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R} \}$$
 i.e. n dim. space.

For any
 $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$
 $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$

$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \rightarrow$ vector addition
 $Rx = (Rx_1, Rx_2, \dots, Rx_n) \rightarrow$ scalar multiplication
 $R \in \mathbb{K}$, $\mathbb{K}$ is set, called field of scalar.

1:- For any scalars R and l , and vectors

$x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n), z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n$
 the following rules hold

$x + y = y + x$

$x + (y + z) = (x + y) + z$

$x + 0 = 0 + x = x$

$0 = (0, 0, \dots, 0)$ is 0 vector.

$x + (-x) = 0$

- (5) $R(x+y) = Rx + Ry$, $R \in K$, $x, y \in \mathbb{R}^n$
 (6) $(R+I)x = Rx + Ix$, $R \in K$, $x, y \in \mathbb{R}^n$
 (7) $R(Ix) = (RI)x$, $R, I \in K$
 (8) $1 \cdot x = x$, 1 is unit scalar

where $0 = (0, 0, 0, \dots, 0)$ is zero vector.

Vector space:-

Let V be nonempty set of element called vectors and K is the field of scalar.

The set V with two algebraic operations (vector addition and scalar multiplication), over the field K , is called vector space if following axiom holds:-
 (1-10)

rules for addition:-

- (1) $\forall u, v \in V$, $u+v \in V$ (closure prop)
 (2) $u+v = v+u$ (commutative prop)
 (3) $(u+v)+w = u+(v+w)$ (associative law)
 (4) ~~There exists~~ $\exists$ an element $0 \in V$ (called zero vector) such that
 $u+0 = 0+u = u$ (0 is additive identity)

- (5) $\forall u \in V$, $\exists$ a vector $-u \in V$ (called negative of u)
 s.t. $u+(-u) = 0$ [e]

rules of scalar multiplication

- (6) $\forall u \in V$ and $R \in K$, $Ru \in V$
 (7) $R(u+v) = Ru + Rv$, where $R \in K$ (distributive law over vector addition)
 (8) $(R+I)u = Ru + Iu$, where $R, I \in K$ (" " " scalar multiplication)
 (9) $R(Iu) = (RI)u$, where $R, I \in K$
 (10) $1 \cdot u = u$, for unit ~~vector~~ scalar $1 \in K$.

Notes: real vector space has real no's as scalars,
 i.e. field K is the set of real no's ($K = \mathbb{R}$)
 For complex vector space, field K is the
 set of complex no's. (i.e. $K = \mathbb{C}$)
 for vector space

ote: 0 , the zero vector is unique, and for any vector u ,
 negative $-u$, is unique.

Example of vector space:

Ex:1 $V = \mathbb{R}^n$ is always a vector space with usual
 addition and scalar multiplication

$$u + v = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$\alpha u = (\alpha u_1, \alpha u_2, \dots, \alpha u_n)$$

$$\text{where } u = (u_1, u_2, \dots, u_n), v = (v_1, v_2, \dots, v_n) \in V$$

$$\text{and } \alpha \in \mathbb{R} \text{ (set of real no's)}$$

three most important cases of $V = \mathbb{R}^n$ are

- (i) $V = \mathbb{R}$ (set of real no) over field $K = \mathbb{R}$
- (ii) $V = \mathbb{R}^2$ (set of all ordered pair) " " "
- (iii) $V = \mathbb{R}^3$ (set of all ordered triple) over $K = \mathbb{R}$

Ex2 vector space of polynomials:-

$V = P_n$ set of all polynomials of degree n or
 less than n with scalars $\alpha \in \mathbb{R}$
 (i.e. degree $\leq n$)

i.e. $V = \{ a_0 + a_1x + \dots + a_nx^n \mid a_i \in \mathbb{R}, \text{ degree} \leq n \}$ over field $K = \mathbb{R}$ is
 vector space with usual addition and scalar
 multiplication

$$u + v = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n)x^n, \text{ vector addition}$$

$$\alpha u = \alpha a_0 + \alpha a_1x + \dots + \alpha a_nx^n, \text{ scalar mult}$$

Ex 3 Vector space of matrices:-

$M_{m \times n}$ set of all matrices of order $m \times n$, with their element as real no's is always a vector space over field $\mathbb{R}$, with usual matrix addition and multiplication of matrix by scalar.

ie $M_{m \times n} = \{ [a_{ij}]_{m \times n} \mid a_{ij} \in \mathbb{R} \}$ over $\mathbb{R}$ is vector space.

Ex 4 $\mathbb{C}^n$, set of all n -tuples of complex no is a vector space over $\mathbb{C}$.

ie $\mathbb{C}^n = \{ u \mid u = (u_1, u_2, \dots, u_n), u_i \in \mathbb{C} \}$ is vector space over $\mathbb{C}$.

Ex 4 The set of all polynomials of degree equal to n is not a vector space as closure prop are not ~~not~~ satisfied.

for example sum of two polynomials of degree n , need not to have degree n .

e.g:- Let $f(x) = 1 + 2x + 2x^2$ $g(x) = 4 + 2x - 2x^2$

$$f(x) + g(x) = 1 + 2x + 2x^2 + 4 + 2x - 2x^2 = 5 + 4x$$

$$V = \{ a_0 + a_1x + a_2x^2, a_2 \neq 0, a_0, a_1, a_2 \in \mathbb{R} \}$$

$f+g$ is a poly of degree 1 $\notin V$

ie V is not a vector space as V is not closed under vector addition.

Ex 5 The set of all functions define on a given interval, is a vector space over field $\mathbb{R}$ under usual addition of functions and scalar multiplication with a function

Q - show that the set $V = \mathbb{R}^3$ with standard vector addition and scalar multiplication defined as $\alpha(u_1, u_2, u_3) = (0, 0, \alpha u_3)$ is not a vector space.

Soln $V = \{u \mid u = (u_1, u_2, u_3) \in \mathbb{R}^3\}$

with standard vector addition, in $V = \mathbb{R}^3$ all the properties w.r.t. addition will be satisfied as we know $V = \mathbb{R}^3$ is a vector space. But for this scalar multiplication

$$\begin{aligned} 1 \cdot u &= 1 \cdot (u_1, u_2, u_3) \\ &= (0, 0, u_3) \\ &\neq (u_1, u_2, u_3) \\ &\neq u \end{aligned}$$

i.e. the property $1 \cdot u = u$ is not satisfied. i.e. $V = \mathbb{R}^3$ is not a vector space w.r.t. given scalar multiplication.

check whether $V = \mathbb{R}^2$ is a vector space with ~~$\mathbb{R}^2$~~ standard addition and scalar multiplication define.

check whether $V = \mathbb{R}^2$ is a vector space with the following operations

$$\begin{aligned} u + v &= (u_1 + 2v_1, u_2 + v_2) \\ (u_1, u_2) \cdot \alpha u &= \alpha(u_1, u_2) \\ &= (\alpha u_1, \alpha u_2) \end{aligned}$$

$$\begin{aligned} \text{where } u &= (u_1, u_2) \\ v &= (v_1, v_2) \end{aligned}$$

Solⁿ check for commutativity

$$u+v = (u_1, u_2) + (v_1, v_2) \\ = (u_1 + 2v_1, u_2 + v_2)$$

$$v+u = (v_1, v_2) + (u_1, u_2) \\ = (v_1 + 2u_1, v_2 + u_2)$$

Since

$u+v \neq v+u$ i.e. V is not a vector space.

Q² check whether $V = \mathbb{R}^2$ is a vector space w.r.to the following operations

$$u+v = (u_1 + v_1 - 2, u_2 + v_2 - 3)$$

$$\alpha u = (\alpha u_1 + 2\alpha - 2, \alpha u_2 - 3\alpha + 3), \alpha \in \mathbb{R}$$

$$u = (u_1, u_2), v = (v_1, v_2)$$

Solⁿ:- Let $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$ are scalar

(1) $u+v = (u_1 + v_1 - 2, u_2 + v_2 - 3)$, which also in $\mathbb{R}^2$.
i.e. $\forall u, v \in \mathbb{R}^2, u+v \in \mathbb{R}^2$, closure prop satisfied

(2) $u+v = (u_1 + v_1 - 2, u_2 + v_2 - 3)$

$$v+u = (v_1 + u_1 - 2, v_2 + u_2 - 3)$$

$$u+v = v+u$$

commutative prop.

$$\begin{aligned} (3) \quad u+(v+w) &= (u_1, u_2) + \{(v_1, v_2) + (w_1, w_2)\} \\ &= (u_1, u_2) + (v_1 + w_1 - 2, v_2 + w_2 - 3) \\ &= (u_1 + v_1 + w_1 - 2, u_2 + v_2 + w_2 - 3) \\ &= (u_1 + v_1 - 2, u_2 + v_2 - 3) + (w_1, w_2) \\ &= (u+v) + w \end{aligned}$$

$$u+(v+w) = (u+v) + w \quad \text{associativity}$$

(4) Let $(a, b) \in \mathbb{R}^2$ such that

$$u + (a, b) = u$$

$$(u_1, u_2) + (a, b) = (u_1, u_2)$$

$$(u_1 + a - 2, u_2 + b - 3) = (u_1, u_2)$$

$$a - 2 = 0, \quad b - 3 = 0$$

$$a = 2, \quad b = 3$$

ie $(2, 3)$ is the zero vector in V

$$\text{s.t. that } u + (2, 3) = u$$

$$\bar{0} = (2, 3) \in V$$

(5) Let $u + \bar{u} = (2, 3)$, $\bar{u}$ be inverse of u .

$$(u_1, u_2) + (\bar{u}_1, \bar{u}_2) = (2, 3)$$

$$(u_1 + \bar{u}_1 - 2, u_2 + \bar{u}_2 - 3) = (2, 3)$$

$$u_1 + \bar{u}_1 - 2 = 2, \quad u_2 + \bar{u}_2 - 3 = 3$$

$$\bar{u}_1 = 4 - u_1, \quad \bar{u}_2 = 6 - u_2$$

ie $\forall u \in V, \exists \bar{u} = (4 - u_1, 6 - u_2) \in V$

$$\text{s.t. } \boxed{u + \bar{u} = (2, 3)}$$

(6) Let $\alpha \in \mathbb{R}$

$$\alpha(u, u_2) = (\alpha u_1 + 2\alpha - 2, \alpha u_2 - 3\alpha + 3) \in V$$

$$\begin{aligned} \alpha(u + v) &= \alpha(u_1 + v_1 - 2, u_2 + v_2 - 3) \\ &= (\alpha(u_1 + v_1 - 2) + 2\alpha - 3, \alpha(u_2 + v_2 - 3) - 3\alpha + 3) \\ &= (\alpha u_1 + \alpha v_1 - 2\alpha + 2\alpha - 3, \alpha u_2 + \alpha v_2 - 3\alpha - 3\alpha + 3) \\ &= (\alpha u_1 + \alpha v_1 - 3, \alpha u_2 + \alpha v_2 - 3) \end{aligned}$$

$$\alpha u + \alpha v$$

$$= \alpha(u_1, u_2) + \alpha(v_1, v_2)$$

$$= (\alpha u_1 + 2\alpha - 2, \alpha u_2 - 3\alpha + 3) + (\alpha v_1 + 2\alpha - 2, \alpha v_2 - 3\alpha + 3)$$

$$= (\alpha u_1 + 2\alpha - 2 + \alpha v_1 + 2\alpha - 2, \alpha u_2 - 3\alpha + 3 + \alpha v_2 - 3\alpha + 3)$$

$$= (\alpha u_1 + \alpha v_1 + 4\alpha - 4, \alpha u_2 + \alpha v_2 - 6\alpha + 6)$$

$$\text{ie } \alpha(u + v) \neq \alpha u + \alpha v$$

Hence $\mathbb{R}^2$ is not a vector space, with the following given operations.

Q check whether the set $V = \{(x_1, x_2, x_3) \mid x_2 = 2x_1, x_3 = x_1 + 1\}$ is a vector space or not.

Solⁿ:- Let $u = (u_1, 2u_1, u_1 + 1)$, $v = (v_1, 2v_1, v_1 + 1) \in V$
 then $u + v = (u_1, 2u_1, u_1 + 1) + (v_1, 2v_1, v_1 + 1)$
 $= (u_1 + v_1, 2(u_1 + v_1), u_1 + v_1 + 2)$
 $\neq v$

ie

$\forall u, v \in V$, $u + v \notin V$, closure prop is not satisfied w.r.t vector addition.

ie V is not a vector space.

Q $V = \mathbb{Z}$, set of integer is a vector space or not ?

Ans (1) let $a, b \in V$

then $a + b \in V$ as $a + b$ is also an integer
 ie closure prop w.r.t vector addition is satisfied

(2) Let $\alpha \in \mathbb{R}$, $a \in V$

then $\alpha a \notin V$

ex say $\alpha = \frac{1}{3}$, $a = 4$

then $\alpha \cdot a = \frac{4}{3} \notin V$ as $\frac{4}{3}$ is not integer

ie closure prop w.r.t scalar multiplication is not satisfied

ie the set of integer is not a vector space

Q: check whether the following sets form a vector space or not

(i) $V = \mathbb{R}^2$, with usual addition and scalar multiplication define as $a(x, y) = (ax, 0)$, $\forall (x, y) \in V$

(ii) $M = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$

(iii) $M = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

(NO)

(iv) $V = \{(x, y); x \geq 0\}$

(NO)

(v) $\{(x, x); x \in \mathbb{R}\}$

(vi) $V = \{A; A \text{ is } 2 \times 2 \text{ order matrix and } A \text{ is singular}\}$

(vii) set of 2×2 diagonal matrix over $\mathbb{R}$.

(viii) set of non singular matrix of order 2×2

ie $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{matrix} ad - bc \neq 0 \\ a, b, c, d \in \mathbb{R} \end{matrix} \right\}$

(NO)

Q:- show that set of all solutions of the homogenous system $AX = 0$, forms a vector space

ie $V = \{x \in \mathbb{R}^n \mid Ax = 0\}$ is a vector space.

A is $m \times n$
 $x \rightarrow n \times 1$

(i) Let $x_1, x_2 \in V$ i.e. $Ax_1 = 0$, $Ax_2 = 0$

$$\text{Now } A(x_1 + x_2) = Ax_1 + Ax_2 = 0$$

$$\Rightarrow x_1 + x_2 \in V$$

(ii) $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$ $\forall x_1, x_2, x_3 \in V$
easy to check as $x_1, x_2, x_3 \in \mathbb{R}^n$

$$(iii) x_1 + x_2 = x_2 + x_1$$

(iv) Let e be zero vector

then for $x \in V$

$$x + e = x \Rightarrow e = 0 \in V$$

as $A0 = 0$, zero solⁿ of $Ax = 0$.

(v) Let $x \in V$ i.e. $Ax = 0$.

Let $\bar{x}$ be inverse of x

$$x + \bar{x} = 0$$

$$\Rightarrow \bar{x} = -x \quad \text{where } A(-x) = -Ax = 0$$

$$\text{i.e. } \exists -x \in V \text{ s.t. } x + (-x) = 0 = (-x) + x$$

(vi) Let $\alpha \in \mathbb{R}$, $x \in V$ i.e. $Ax = 0$

$$\text{Consider } A(\alpha x) = \alpha(Ax) = 0$$

$$\text{i.e. } \alpha x \in V$$

$$(vii) \alpha(x_1 + x_2) = \alpha x_1 + \alpha x_2$$

as $x_1, x_2 \in \mathbb{R}^n$ so prop is obvious

$$(viii) (\alpha + \beta)x = \alpha x + \beta x, \text{ true for } x \in \mathbb{R}^n$$

$$(ix) \alpha(\beta x) = (\alpha\beta)x \quad "$$

$$(x) 1 \cdot x = x \quad "$$

Hence V is a vector space.

check whether $V = \{(x_1, x_2, x_3) \mid x_1 \text{ is even integer, } x_2, x_3 \text{ are odd}\}$

is vector space or not.

~~not a vector space~~

Subspace:-

Let V be a vector space and W be non-empty subset of V . Then W is a subspace of V if W is itself a vector space under the operations of vector addition and scalar multiplication defined in V .

ex:- The set $A = \{(x, 0) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ and A itself a vector space under the same operation, i.e. A is a subspace of $\mathbb{R}^2$.

Test for a subspace:-

Interesting, we don't need to check all the properties of a vector space to be a subspace but only two properties are sufficient:-

o:- If W is a nonempty subset of a vector space V , then W is a subspace of V iff the two closure prop listed below hold:-

1) for all $u, v \in W \Rightarrow u + v \in W$
i.e. W should be closed under vector addition.

$\forall u \in W$ and $\alpha \in \mathbb{R}$ (scalar field)
 $\Rightarrow \alpha u \in W$

i.e. W should be closed under scalar multiplication.

ie:- If W is a subspace of V , then the identity element (zero vector) of V is in W i.e. $0 \in W$. i.e. V and W has same identity element.

(Hint)

Proof

Only these properties guarantee that W is itself a vector space, under the same operation as in V .

as properties 2, 3, 7-10 are automatically true in W because they are already satisfied by all element of V .

for prop 5:- ~~let~~

let $u \in W$ and $-1 \in \mathbb{R}$
~~then~~ $\Rightarrow -1 \cdot u \in W$ (as W is closed under scalar multi)
 $\Rightarrow -u \in W$

prop 4:-

$u \in W, -u \in W$
 $u + (-u) \in W$
 i.e. $0 \in W$ (W has same identity as V)

Note:-

let V be any vector space then V automatically contains two subspaces

(1) the set $\{0\}$, consisting of the ^(identity element) zero vector alone

(2) the whole space V itself.

these two are called trivial subspaces of V .

Ex:-1 check whether

(1) $W = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$ is a subspace of $\mathbb{R}^2$.

(2) $W = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0, a, b, c \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

2. Soln

W is set of all points (x, y) in $\mathbb{R}^2$, which lies in the first quadrant

Prove

Let $u = (x_1, y_1)$ $v = (x_2, y_2) \in \mathbb{R}^2$, $x_1, x_2, y_1, y_2 \geq 0$
 $u + v = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$ as $x_1 + x_2 \geq 0$,
 $y_1 + y_2 \geq 0$

ie W is closed under vector addition

Let $u = (x, y) \in W$
 $\alpha \in \mathbb{R}$

then αu may not belong to W
for example

$$\alpha = -1, u = (1, 1)$$

then $\alpha u = (-1, -1) \notin W$

ie W is not closed under scalar multiplication.

$$W = \{ (x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = 0, a, b, c \in \mathbb{R} \}$$

Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2) \in W$
ie $ax_1 + by_1 + cz_1 = 0$
 $ax_2 + by_2 + cz_2 = 0$

$u + v$

$$= (x_1, y_1, z_1) + (x_2, y_2, z_2)$$
$$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

Now

$$a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2)$$
$$= (ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2)$$
$$= 0$$

ie $u + v \in W$

W is closed under vector addition.

Now, let $\alpha \in \mathbb{R}$

then $\alpha \cdot u = \alpha(x_1, y_1, z_1) = (\alpha x_1, \alpha y_1, \alpha z_1)$

$$\begin{aligned}\text{Now } a \cdot x_1 + b \cdot x_2 + c \cdot x_3 \\ = \alpha(ax_1 + bx_2 + cx_3) \\ = 0\end{aligned}$$

$$\text{i.e. } \alpha \cdot u \in W$$

i.e. W is closed under scalar multiplication.

i.e. W is a subspace of $\mathbb{R}^3$. also it represents a plane passing through origin.

Note:- here zero vector is $(0,0,0) \in W$ i.e. W has the same additive identity as V has.

Q: Does $W = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$ a subspace of $\mathbb{R}^3$ with standard operation.

Hint:- No, as $(0,0,0) \notin W$
 W is not a subspace of $\mathbb{R}^3$.

Q:- Examine whether the following subset of $\mathbb{R}^3$ are its subspace or not

(a) $W = \{(a_1, a_2, a_3) \mid a_2, a_3 \text{ are arbitrary, } a_1 \geq 0\}$

(b) $W = \{(a_1, a_2, a_3) \mid a_1, a_2 \geq 0, a_3 \text{ arb}\}$

(c) $E = \{(a_1, a_2, a_3) \mid a_3 = 1 + a_1 - a_2, a_1, a_2 \text{ arb}\}$

(d) $W = \{(a_1, a_2) \mid (a_1, a_2) \in \mathbb{R}^2, a_1 = 1\}$

(e) $W = \{(x, y, z) \in \mathbb{R}^3 : xyz = 0\}$

(f) $W = \{(2t, 3t, 4t) \in \mathbb{R}^3 : t \in \mathbb{R}\}$

(h) $W = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 - z^2 = 0\}$

Intersection of two subspaces of a vector space:-

Theorem:- The intersection of two subspaces of a vector space V is a subspace of V .

Proof:- Let V be a vector space and, W_1 and W_2 be subspaces of V .

to show:- $W_1 \cap W_2$ is also a subspace of V .

Now, let $u, v \in W_1 \cap W_2$
 $\Rightarrow u, v \in W_1$ and $u, v \in W_2$

W_1 is a subspace of V
 i.e. for $u, v \in W_1$, $\alpha \in \mathbb{R}$
 $\Rightarrow u+v \in W_1$ and $\alpha u \in W_1$, ($\alpha \in \mathbb{R}$) — (1)

Similarly W_2 is a subspace of V
 i.e. for $u, v \in W_2$, $\alpha \in \mathbb{R}$
 $\Rightarrow u+v \in W_2$ and $\alpha u \in W_2$ — (2)

from (1) & (2)
 we have

$$u+v \in W_1 \cap W_2$$

$$\alpha u \in W_1 \cap W_2$$

i.e. $W_1 \cap W_2$ is also a subspace of V .

Union of Subspaces:-

Let W_1 and W_2 be subspaces of a vector space V
 then $W_1 \cup W_2$ need not to be a subspace of V

Pf:- counter example:-

$$\text{let } W_1 = \{(x, 0, 0) : x \in \mathbb{R}\}$$

$$W_2 = \{(0, y, 0) : y \in \mathbb{R}\}$$

W_1 and W_2 are subspaces of $V = \mathbb{R}^3$

Now

$$W_1 \cup W_2 = \{(x, 0, 0) \cup (0, y, 0) : x, y \in \mathbb{R}\}$$

$$\text{Let } u = (1, 0, 0) \in W_1 \cup W_2$$

$$v = (0, 1, 0) \in W_1 \cup W_2$$

but

$$u + v = (1, 0, 0) + (0, 1, 0)$$

$$= (1, 1, 0) \notin W_1 \cup W_2$$

ie $W_1 \cup W_2$ is not closed under vector addition

ie $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^3$.