Module-1: Higher order Differential Equations:

Second order Linear Differential Equations with constant Co-efficients:

An equation of the form

$$\frac{d^2y}{dx^2} + K_1\frac{dy}{dx} + K_2 y = Q(x), \longrightarrow (1)$$

where K_1 and K_2 are constants, is called a second order linear differential equation in y with constant co-efficients.

The operator form of eqn (1) is

$$D^2y + K_1Dy + K_2y = Q(x), \text{ where } D \equiv \frac{d}{dx}$$

or, $(D^2 + K_1 D + K_2)y = Q(x) \longrightarrow (2)$

Let $f(D) = D^2 + K_1 D + K_2$, then eqn (2) becomes

$$f(D)\,y = Q(x) \longrightarrow (3)$$

and $f(m) = 0$ is the Auxiliary Equation (A.E) of $f(D)\,y = 0$.

The general solution of eqn (3) is

$$y = y_c + y_p$$

$y_c \rightarrow$ Complementary Function (C.F)

$y_p \rightarrow$ Particular Integral (P.I)

Problems

1. Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

Sol: The operator form of the given differential equation is

$$(D^2 + 5D + 6)y = 0 \longrightarrow (1)$$

Let $f(D) = D^2 + 5D + 6$

Then the A.E. is $f(m) = 0$

i.e., $m^2 + 5m + 6 = 0$

$\Rightarrow m = -2, -3$

$\therefore y_c = c_1 e^{-2x} + c_2 e^{-3x}$

Hence, the general solution of (1) is

$y = y_c$

i.e., $y = c_1 e^{-2x} + c_2 e^{-3x}$

2. Solve $(D^2 + 4D + 4)y = 0$.

Sol: Given Diff. equation is

$$(D^2 + 4D + 4)y = 0 \longrightarrow (1)$$

Note:

(1) If $Q(x) = 0$, then $f(D)y = 0$ is called homogeneous equation and the general solution of $f(D)y=0$ is $y = y_c$

(2) If $Q(x) \neq 0$, then $f(D)y = Q(x)$ is called non-homogeneous.

(3) If the given differential equation is of order n, then its general solution contains n arbitrary constants and all these constants are in y_c only. So, y_p does not contains any arbitrary constant.

Finding C.F (y_c) for $f(D)y = Q(x)$

Let $f(D)y = Q(x)$ be the given second order equation.

(1) If m_1, m_2 are two distinct real roots of $f(m) = 0$, then $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

(2) If $m_1 = m_2 = m$ is a real root (repeated) of $f(m) = 0$, then $y_c = (c_1 + c_2 x) e^{mx}$

(3) If $m = \alpha \pm i\beta$, then $y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

(4) If $m = \alpha \pm \beta$, then $y_c = e^{\alpha x}(c_1 \cosh\beta x + c_2 \sinh\beta x)$

Let $f(D) = D^2+4D+4$.

Then the A.E. is $f(m)=0$

i.e., $m^2+4m+4=0$

$\Rightarrow m = -2, -2$

$\therefore y_c = (c_1+c_2x)e^{-2x}$

Hence, the general solution of ① is

$y = y_c$

i.e., $y=(c_1+c_2x)\,e^{-x}$.

3. Solve $(D^2+D+1)\,y = 0$

Sol: Given D.E. is

$(D^2+D+1)\,y=0 \longrightarrow ①$

Let $f(D) = D^2+D+1$.

Then the A.E. is $f(m)=0$

i.e., $m^2+m+1=0$

$\Rightarrow m = \dfrac{-1\pm\sqrt{-3}}{2} \Rightarrow m = \dfrac{-1}{2} \pm i\dfrac{\sqrt{3}}{2}$

$\therefore y_c = e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$

Hence, the general solution of eqn ① is

$y = y_c$ i.e., $y = e^{-\frac{x}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$.

(4) solve $(D^2 - D + 1)y = 0$, given that $y = 1$, $y' = 0$ when $x = 0$.

Sol: Given diff. equation is

$$(D^2 - D + 1)y = 0 \rightarrow (1)$$

let $f(D) = D^2 - D + 1$.

Then the A. E. is $f(m) = 0$

i.e., $m^2 - m + 1 = 0$

$$\Rightarrow m = \frac{1 \pm i\sqrt{3}}{2}$$

$$\therefore y_c = e^{\frac{x}{2}}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right)$$

Hence the general solution of (1) is

$$y = y_c$$

i.e., $y = e^{\frac{x}{2}}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right). \rightarrow (2)$

Taking $y = 1$ when $x = 0$ in (2), we get

$$c_1 = 1$$

From (2), we have

$$y' = \frac{dy}{dx} = e^{\frac{x}{2}}\left(-\frac{\sqrt{3}}{2} c_1 \sin \frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{2} c_2 \cos \frac{\sqrt{3}}{2}x\right) + \frac{1}{2} \cdot e^{\frac{x}{2}}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right) \rightarrow (3)$$

Taking $y' = 0$ when $x = 0$ in (3), we get

$$\frac{\sqrt{3}}{2} c_2 + \frac{1}{2} c_1 = 0$$

$$\Rightarrow \frac{\sqrt{3}}{2} c_2 = -\frac{1}{2} \quad (\text{since } c_1 = 1)$$

$$\Rightarrow c_2 = -\frac{1}{\sqrt{3}}$$

Therefore, the particular solution of ① is

$$y = e^{\frac{x}{2}}\left(\cos\frac{\sqrt{3}}{2}x - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right).$$

I. Finding P.I. for $f(D)y = Q(x)$, when $Q(x) = e^{ax}$ (a is a constant)

$\boxed{\text{General Formula}}$

$$\frac{1}{D-\alpha} Q(x) = e^{\alpha x} \int e^{-\alpha x} \cdot Q(x)\, dx$$

$$* \ \text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}, \quad \text{if } f(a) \neq 0$$

Problems:

① Solve $(D^2+7D+6)y = e^{2x}$

Sol: Given D.E is

$$(D^2+7D+6)y = e^{2x} \longrightarrow ①$$

Clearly $y_c = c_1 e^{-x} + c_2 e^{-6x}$

Now,

$$y_p = P.I. = \frac{1}{D^2+7D+6} e^{2x}$$

$$= \frac{e^{2x}}{4+14+6} = \frac{e^{2x}}{24}$$

Therefore, the general solution of eqn (1) is

$$y = y_c + y_p$$

i.e., $$y = c_1 e^{-x} + c_2 e^{-6x} + \frac{e^{2x}}{24}.$$

(2) Solve $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = e^{-t}$

Sol: The operator form of the given differential equation is

$$(D^2+4D+3)\,x = e^{-t} \longrightarrow (1)$$

Clearly, $x_c = c_1 e^{-t} + c_2 e^{-3t}$

Now,

$$x_p = PI = \frac{1}{D^2+4D+3} e^{-t}$$

$$\boxed{X = \frac{e^{-t}}{1-4+3} = \frac{e^{-t}}{0} \quad X}$$

$$= \frac{t}{2D+4} e^{-t}$$

(Multiply numerator with 't' and differentiate denominator w.r.t. D)

$$= \frac{te^{-t}}{-2+4} = \frac{te^{-t}}{2}$$

Hence, the general solution of eqⁿ ① is

$$x = x_c + x_p$$

i.e., $x = c_1 e^{-t} + c_2 e^{-3t} + \frac{te^{-t}}{2}$

II. Finding P.I. for $f(D)\,y = Q(x)$,

When $Q(x) = \sin ax$ (or, $\cos ax$) where a is a constant.

1. Solve $(D^2 + a^2)\,y = \sin ax$

Sol: Given Diff. equation is

$$(D^2 + a^2)\,y = \sin ax \longrightarrow ①$$

Clearly,

$$y_c = c_1 \cos ax + c_2 \sin ax$$

Now,

$$y_p = P.I = \frac{1}{D^2 + a^2} \cdot \sin ax$$

$$\boxed{\cancel{X} = \frac{\sin ax}{-a^2 + a^2} = \frac{\sin ax}{0} \times}$$

$$* \boxed{\frac{1}{D^2 + a^2} \sin ax = \frac{-x}{2a} \cos ax}$$

$$= \frac{x}{2D} \sin ax$$

$$= -\frac{x}{2a} \cos ax$$

Hence, the general solution of 1 is

$$y = y_c + y_p$$

i.e., $$y = c_1 \cos ax + c_2 \sin ax - \frac{x}{2a} \cos ax.$$

2 Solve $(D^2+a^2)y = \cos ax$

Sol: Given Diff. equation is

$$(D^2+a^2)\, y = \cos ax \longrightarrow (1)$$

Clearly, $$y_c = c_1 \cos ax + c_2 \sin ax$$

Now,

$$y_p = PI = \frac{1}{D^2+a^2} \cos ax$$

$$\boxed{= \frac{\cos ax}{-a^2+a^2} = \frac{\cos ax}{0}}$$

$$* \boxed{\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax}$$

$$= \frac{x}{2D} \cos ax$$

$$= \frac{x}{2a} \sin ax$$

$$\therefore \quad y = y_c + y_p$$

i.e., $$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{2a} \sin ax.$$

(3) solve $(D^2-2D-15)y = \sin 3x$

Sol: Given D.E. is

$$(D^2-2D-15)\,y = \sin 3x \longrightarrow (1)$$

clearly, $y_c = c_1 e^{-3x} + c_2 e^{5x}$

Now,
$$y_p = \frac{1}{D^2-2D-15}\sin 3x$$

$$= \frac{1}{-9-2D-15}\sin 3x$$

$$= \frac{1}{-(2D+24)}\sin 3x$$

$$= -\frac{1}{2}\left[\frac{1}{D+12}\sin 3x\right]$$

$$= -\frac{1}{2}\cdot\left[\frac{D-12}{D^2-144}\sin 3x\right]$$

$$= -\frac{1}{2}\left(\frac{3\cos 3x - 12\sin 3x}{-9-144}\right)$$

$$= \frac{1}{256}\left(3\cos 3x - 12\sin 3x\right)$$

$\therefore\ y = y_c + y_p$

i.e., $y = c_1 e^{-3x} + c_2 e^{5x} + \frac{3}{256}\left(\cos 3x - 4\sin 3x\right)$

III Finding P.I. for $f(D)y = Q(x)$, when $Q(x) = x^k$ $(k \in \mathbb{Z}^+)$

1. Solve $(3D^2 + 3D - 18)y = x^2$.

Sol: Given differential equation can be written as

$$(D^2 + D - 6)y = \frac{1}{3} \cdot x^2 \longrightarrow (1)$$

Clearly, $y_c = c_1 e^{-3x} + c_2 e^{2x}$

Now,

$$y_p = P.I = \frac{1}{3} \frac{1}{D^2 + D - 6} x^2$$

$$= \frac{1}{3} \cdot \frac{-1}{6} \frac{1}{\left[1 - \left(\frac{D^2 + D}{6}\right)\right]} x^2$$

$$= \frac{-1}{18} \left[1 - \left(\frac{D + D^2}{6}\right)\right]^{-1} x^2$$

$$= \frac{-1}{18} \left[1 + \left(\frac{D + D^2}{6}\right) + \left(\frac{D + D^2}{6}\right)^2\right] x^2$$

$$= \frac{-1}{18} \left[1 + \left(\frac{D + D^2}{6}\right) + \frac{D^2}{36}\right] x^2$$

$$= \frac{-1}{18} \left[x^2 + \frac{2x}{6} + \frac{7(2)}{36}\right]$$

$$= \frac{-1}{18} \left[x^2 + \frac{x}{3} + \frac{7}{18}\right]$$

Therefore, the general solution of (1) is

$$y = y_c + y_p$$

i.e., $$y = c_1 e^{-3x} + c_2 e^{2x} - \frac{1}{18}\left[x^2 + \frac{x}{3} + \frac{7}{18}\right].$$

IV Finding P.I. for $f(D)\,y = Q(x)$, when $Q(x) = e^{ax}.\sin bx$ (or, $\cos bx$)

where a, b are constants.

$$\left[\frac{1}{f(D)} \cdot e^{ax}\sin bx = e^{ax} \cdot \frac{1}{f(D+a)} \sin bx\right]$$

1. Find the P.I. of $(D^2+4D+3)y = e^{2x}.\sin x + 5$

Sol: $$P.I = \frac{1}{D^2+4D+3}\left(e^{2x}\sin x + 5\right)$$

$$= \frac{1}{D^2+4D+3} e^{2x}.\sin x + \frac{1}{D^2+4D+3}.5e^{0x}$$

$$= e^{2x} \frac{1}{(D+2)^2+4(D+2)+3}.\sin x + \frac{5}{3}$$

$$= e^{2x} \frac{1}{D^2+8D+15} \sin x + \frac{5}{3}$$

$$= e^{2x} \frac{1}{-1+8D+15} \sin x + \frac{5}{3}$$

$$= e^{2x} \frac{1}{8D+14} \sin x + \frac{5}{3}$$

$$= \frac{e^{2x}}{2} \cdot \frac{1}{4D+7} \sin x + \frac{5}{3}$$

$$= \frac{e^{2x}}{2} \frac{(4D-7)}{16D^2-49} \sin x + \frac{5}{3}$$

$$= \frac{e^{2x}}{2} \frac{(4\cos x - 7\sin x)}{-16-49} + \frac{5}{3}$$

$$= -\frac{e^{2x}}{130} (4\cos x - 7\sin x) + \frac{5}{3}$$

V Finding P.I. for $f(D)y = Q(x)$, when $Q(x) = e^{ax} \cdot x^k$ $(k \in Z^+)$

where a is a constant.

1. Find the P.I of $(D^2+D+2)y = e^{-x} \cdot x$

Sol: $$P.I = \frac{1}{D^2+D+2} e^{-x} \cdot x$$

$$= e^{-x} \frac{1}{(D-1)^2+(D-1)+2} \cdot x$$

$$= e^{-x} \frac{1}{D^2-D+2} x$$

$$= \frac{e^{-x}}{2} \cdot \frac{1}{\left[1 + \left(\frac{D^2 - D}{2}\right)\right]} x$$

$$= \frac{e^{-x}}{2} \left[1 + \frac{D^2 - D}{2}\right]^{-1} x$$

$$= \frac{e^{-x}}{2} \left[1 - \left(\frac{D^2 - D}{2}\right) + \cdots\right] x$$

$$= \frac{e^{-x}}{2} \left[x + \frac{1}{2}\right]$$

$$= \frac{e^{-x}}{4} [2x + 1]$$

VI Finding P.I. for $f(D)y = x^k \cdot \sin ax$ (or, $\cos ax$) ($k \in \mathbb{Z}^+$) where a is a constant.

1. Find the P.I. of $(D^2 + D + 1)y = x \cdot \sin x$

Sol: $$P.I = \frac{1}{D^2 + D + 1} x \sin x$$

$$\left[e^{ix} = \underbrace{\cos x}_{\substack{\downarrow \\ R.P}} + i \underbrace{\sin x}_{\substack{\downarrow \\ I.P}} \right]$$

$$= \text{I.P. of } \frac{1}{D^2 + D + 1} e^{ix} \cdot x$$

$$= \text{I.P. of } e^{ix} \frac{1}{(D+i)^2 + (D+i) + 1} x$$

$$= \text{I.P. of } e^{ix} \frac{1}{D^2+2Di+D+i} \cdot x$$

$$= \text{I.P. of } \frac{e^{ix}}{i} \frac{1}{\left[1+\frac{(D^2+2Di+D)}{i}\right]} x$$

$$= \text{I.P. of } \frac{e^{ix}}{i} \left[1+\frac{D^2+2Di+D}{i}\right]^{-1} x$$

$$= \text{I.P. of } \frac{e^{ix}}{i} \left[1-\frac{D^2+2Di+D}{i}\right] x$$

$$= \text{I.P. of } (-i)e^{ix}\left[x-(-i)(2i+1)\right]$$

$$\left[\because \frac{1}{i}=\frac{i}{i^2}=-i\right]$$

$$= \text{I.P. of } (-i)(\cos x + i\sin x)\left[(x-2)+i\right]$$

$$= \text{I.P. of } (-i\cos x + \sin x)\left[(x-2)+i\right]$$

$$= -(x-2)\cos x + \sin x$$

$$= (2-x)\cos x + \sin x$$

VII Finding P.I. for $f(D)\cdot y = e^{ax}\cdot x^K \cdot \cos bx$ (or, $\sin bx$)

$(K \in \mathbb{Z}^+)$ where a, b are constants.

1. Find the P.I. of $(D^2 - D)y = xe^x \cos x$

Sol:

$$P.I = \frac{1}{D^2 - D} xe^x \cos x$$

$$= e^x \frac{1}{(D+1)^2 - (D+1)} x.\cos x$$

$$= e^x \frac{1}{D^2 + D} x.\cos x$$

$$= e^x.\ R.P.\ of\ \frac{1}{D^2+D} e^{ix}.x$$

$$= e^x\ R.P.\ of\ e^{ix} \frac{1}{(D+i)^2 + (D+i)} x$$

$$= e^x\ R.P.\ of\ e^{ix} \frac{1}{D^2 + 2Di + D + i - 1}.x$$

$$= e^x\ R.P.\ of\ \frac{e^{ix}}{(i-1)} \frac{1}{\left[1 + \left(\frac{D^2+2Di+D}{i-1}\right)\right]} x$$

$$= e^x.\ R.P\ of\ \frac{(i+1)}{2} e^{ix}\left[1 + \frac{D^2+2Di+D}{i-1}\right]^{-1} x$$

$$= e^x.\ R.P.\ of\ \frac{(i+1)}{2} e^{ix}\left[1 - \left(\frac{i+1}{2}\right)(D^2+2Di+D)\right]x$$

$$= \frac{e^x}{2} \text{ R.P. of } (i+1)\, e^{ix}\left[x - \left(\frac{i+1}{2}\right)(2i+1)\right]$$

$$= \frac{e^x}{4} \text{ R.P. of } (i+1)(\cos x + i\sin x)\left[2x - (-2+3i+1)\right]$$

$$= \frac{e^x}{4} \text{ R.P. of } \left[(i\cos x + i\sin x) + (\cos x - \sin x)\right]\left[2x - (3i-1)\right]$$

$$= \frac{e^x}{4} \text{ R.P. of } \left[i(\cos x + \sin x) + (\cos x - \sin x)\right]\left[(2x+1) - 3i\right]$$

$$= \frac{e^x}{4}\left[(2x+1)(\cos x - \sin x) + 3(\cos x + \sin x)\right]$$

— x —

Exercise: Solve the following:

1. $(D^2 + D + 1)\, y = \sinh x$, given that $y = 0$, $\frac{dy}{dx} = 0$ when $x = 0$
2. $(D^2 + 3D + 2)\, y = \sin x \cdot \cos 2x$
3. $(D^2 - D + 1)\, y = e^{-x} \cdot \cos 3x$
4. $(D^2 + D + 2)\, y = x^2 \cdot \sin x$
5. $(D^2 + 2D + 1)\, y = e^{x} \cdot x^3$
6. $(D^2 - D + 2)\, y = x \cdot e^{2x} \cdot \sin x$