Experiment 3–A

Lagrange's multiplier Optimization method

Aim:

Finding the Maxima/Minima of a function of several variables under a given constraint by using Lagrange's method of multipliers.

Introduction:

In many practical and theoretical applications, it is required to find the maximum and minimum of a function of several variables, where the variables are connected by some relation or condition known as constraint.

If $f(x, y, z)$ is a function of three independent variables where x, y, z are related by a known constraint $g(x, y, z) = k$, then the problem is to find extreme values of $f(x, y, z)$ subject to $g(x, y, z) = k.$

Lagrange's method of multipliers:

Using the Lagrange's method of multipliers, we can obtain the stationary points of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. An additional unknown constant λ known as Lagrange multiplier is introduced to find the stationary points.

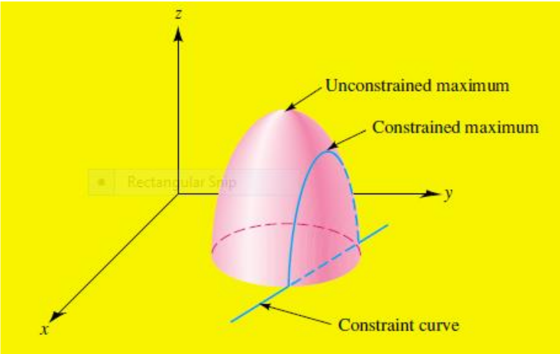
This method involves in the following steps:

If $f(x, y, z)$ is a function subject to the constraint $g(x, y, z) = k$:

Step1: Form the auxiliary equation $F(x, y, z) = f(x, y, z) + \lambda\, g(x, y, z)$.

Step2: Partially differentiate F with respect to x, y, z respectively.

Step3: Solve the four equations $F_x = 0$, $F_y = 0$, $F_z = 0$ and $g(x, y, z) = k$ for the Lagrange's Multiplier λ and the stationary values x, y, z.



Example 1. Find the minimum of $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 10$.

Given $f(x, y) = x^2 + y^2$ subject to the constraint $x + y = 10$.

Let $g(x, y) = x + y - 10$.

The auxiliary equation is $F(x, y) = f(x, y) + \lambda\, g(x, y)$.

$$= (x^2 + y^2) + \lambda\,(x + y - 10)$$

$F_x = 2x + \lambda$, $F_y = 2y + \lambda$.

Solving $F_x = 0$, $F_y = 0$, we get $x = -\frac{\lambda}{2}$, $y = -\frac{\lambda}{2}$, $\lambda = -10$.

Hence $\min\{f(x, y)\} = f(5, 5) = 5^2 + 5^2 = 50$.

MATLAB commands required:

Following are the new MatLab commands one needs to know for the present experiment.

`fimplicit(f)`	plots the implicit function defined by f(x,y)=0 over the default interval [-5,5] for x and y.
`v=get(h,propertyName)`	returns the value for the specific property, propertyName. Use single quotes around the property name, for example, get(h,'Color').

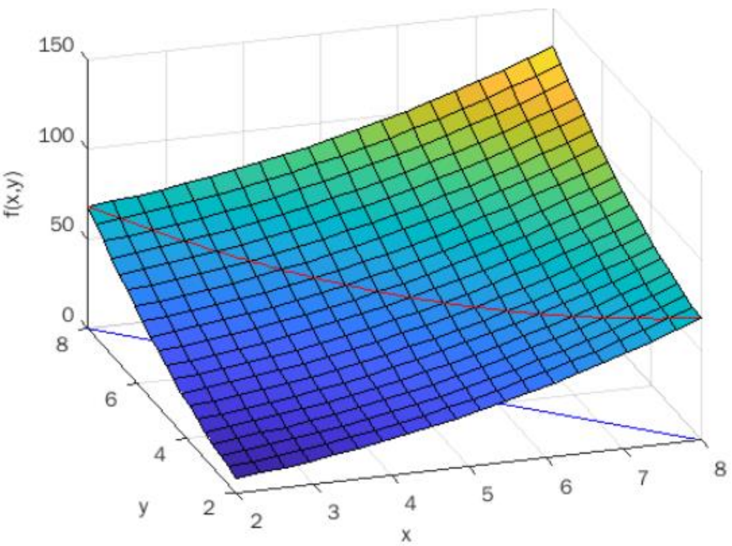
MATLAB code for Lagrange's multiplier method for two variables:

```
clear
clc
syms x y z L
f(x,y)=x^2+y^2; % Function
g(x,y)=x+y-10; % Constraint
F=f+L*g;
Fx=diff(F,x);Fy=diff(F,y);
S=solve(g,Fx,Fy,'Real',true);  %Solving only for Real x and y
St_pts=[S.x,S.y];        % Stationary points
St_pts=double(St_pts)
Fun_Val=f(S.x,S.y);      % Function values at Stationary points
Fun_Val=double(Fun_Val)
% Visualization of Function and constraint
X=linspace(min(S.x)-3,max(S.x)+3,20);X=double(X); % X data
Y=linspace(min(S.y)-3,max(S.y)+3,20);Y=double(Y); % Y data
[X,Y]=meshgrid(X,Y);
Z=f(X,Y);Z=double(Z);
surf(X,Y,Z);hold on; % Function surface
xlabel('x');ylabel('y');zlabel('f(x,y)');
gv=fimplicit(g,'b'); % Constraint curve
hold on;
xv=get(gv,'XData');
yv=get(gv,'YData');
fv=f(xv,yv);fv=double(fv);
plot3(xv,yv,fv,'-r'); %Constrained extrema on the Function surface.
```

Output

```
St_pts =
5     5

Fun_Val =
    50
```

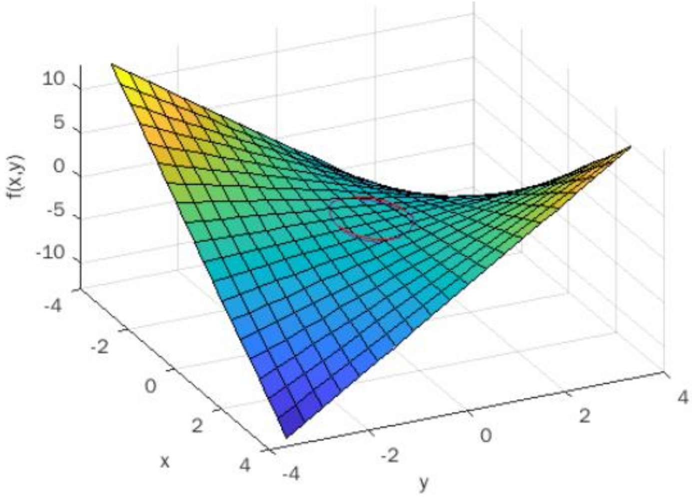


Example 2. Find the minimum of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 - 1$.

Output

```
St_pts =
   -0.7071   -0.5000
    0.7071   -0.5000
   -0.7071    0.5000
    0.7071    0.5000

Fun_Val =
    0.3536
   -0.3536
   -0.3536
    0.3536
```



MATLAB code for Lagrange's multiplier method for three variables:

```
clear
clc
syms x y z L
f(x,y,z)=x^2+y^2+z^2; % Function
g(x,y,z)=3*x^2+4*x*y+6*y^2-140; % Constraint
F=f+L*g;
Fx=diff(F,x);Fy=diff(F,y); Fz=diff(F,z);
S=solve(g,Fx,Fy,Fz,'Real',true);  %Solving only for Real x,y and z
St_pts=[S.x,S.y, S.z];        % Stationary points
St_pts=double(St_pts)
Fun_Val=f(S.x,S.y,S.z);  % Function values at Stationary points
Fun_Val=double(Fun_Val)
```

Example 3. Find the maximum and minimum distances from the origin to the curve

$$3x^2 + 4xy + 6y^2 = 140.$$

Given $f(x, y, z) = d^2 = x^2 + y^2 + z^2$ subject to the constraint $g(x, y) = 3x^2 + 4xy + 6y^2 - 140.$

Output

```
St_pts =
   -2.0000   -4.0000         0
    2.0000    4.0000         0
   -7.4833    3.7417         0
    7.4833   -3.7417         0

Fun_Val =
    20
    20
```

```
70
70
```

Exercise:

1. The highway department is planning to build a picnic area for motorists along a major highway. It is to be rectangular with an area of 5,000 square yards and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing that will be needed to complete the job?

2. Let $f(x, y) = x^2y^2$ represents the utility function or customer satisfaction derived by a consumer from the consumption of a certain amount of product x and certain amount of product y. Maximize the utility function subject to the constraint $2x + 4y = 40$.

3. Find the dimension of rectangular box with the largest possible volume with an open top and one portion to be constructed from 162 sq. inches of cardboard. (Note: The amount of the material used in construction of box is $xy + 2xz + 2yz = 162$).