Module 6.5:

Convolution Theorem: Convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

If $F\{f(x)\} = F(p)$ and $F\{g(x)\} = G(p)$, then the convolution theorem for Fourier transforms states that

$$F\{f(x) * g(x)\} = F\{f(x)\} \cdot F\{g(x)\} = F(p) \cdot G(p)$$

Problems:

Note: ① The Fourier transformation of the function $f(x) = e^{-ax} H(x)$ $(a>0)$, where $H(x)$ represents unit step function, is

$$F\{f(x)\} = F(p) = \frac{1}{a-ip}$$

i.e., $F\{e^{-ax} H(x)\} = \frac{1}{a-ip}$

② The Fourier transform of $f(x) = e^{-a|x|}$ $(a>0)$

is $\frac{2a}{a^2+p^2}$. i.e., $F\{e^{-a|x|}\} = F(p) = \frac{2a}{a^2+p^2}$

① If $f(x) = e^{-2|x|}$ and $g(x) = e^{-3|x|}$, then find the Fourier transform of the convolution of $f(x)$ and $g(x)$.

Sol: Suppose $f(x) = e^{-2|x|}$ and $g(x) = e^{-3|x|}$.

Then $F\{f(x)\} = F(p) = \frac{4}{4+p^2}$

and $F\{g(x)\} = G(p) = \frac{6}{9+p^2}$.

Therefore, $F\{f(x) * g(x)\} = F(p) \cdot G(p)$

$$= \frac{24}{(4+p^2)(9+p^2)}$$

② Using convolution theorem, find $F^{-1}\left[\frac{1}{(4-ip)(3-ip)}\right]$

Sol: Let $F(p) = \frac{1}{4-ip}$ and $G(p) = \frac{1}{3-ip}$.

Then $F^{-1}\{\frac{1}{4-ip}\} = e^{-4x} H(x) = f(x)$

and $F^{-1}[\frac{1}{3-ip}] = e^{-3x} \cdot H(x) = g(x)$ say.

By the convolution theorem, we have

$$F^{-1}\left[\frac{1}{(4-ip)(3-ip)}\right] = f(x) * g(x)$$

$$= \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

$$= \int_{-\infty}^{\infty} e^{-4u} H(u)\, e^{-3(x-u)} H(x-u)\, du$$

$$H(u)\, H(x-u) = \begin{cases} 1 & \text{if } u>0 \text{ and } x-u>0 \\ 0 & \text{if } u<0 \text{ or } x-u<0 \end{cases}$$

i.e., $H(u)\, H(x-u) = \begin{cases} 1 & \text{if } 0<u<x \\ 0 & \text{if } u<0 \text{ or } u>x \end{cases}$ $(x \geq 0)$

$$= e^{-3x}\int_{0}^{x} e^{-u} \cdot 1\, du$$

$$= e^{-3x}\left[-e^{-u}\right]_0^x = e^{-3x} - e^{-4x} \quad (x \geq 0)$$

Fourier Transforms of Derivatives

Let $u(x,t)$ be a function of two independent variables x and t. consider $u(x,t)$, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}, \ldots \to 0$ as $x \to \pm\infty$.

Then 1. $F\{\frac{\partial u}{\partial x}\} = -ip\, F\{u(x,t)\}$

$$F\{\frac{\partial^2 u}{\partial x^2}\} = (-ip)^2 F\{u(x,t)\}$$

$$\vdots$$

$$F\{\frac{\partial^n u}{\partial x^n}\} = (-ip)^n F\{u(x,t)\}$$

2. $F_s\{\frac{\partial^2 u}{\partial x^2}\} = p\, u(0,t) - p^2 F_s\{u(x,t)\}$

3. $F_c\{\frac{\partial^2 u}{\partial x^2}\} = -p^2 F_c\{u(x,t)\} - \left(\frac{\partial u}{\partial x}\right)_{x=0}$

Also, ① $F\{\frac{\partial u}{\partial t}\} = \frac{d}{dt} F\{u(x,t)\}$

② $F_s\{\frac{\partial u}{\partial t}\} = \frac{d}{dt} F_s\{u(x,t)\}$

③ $F_c\{\frac{\partial u}{\partial t}\} = \frac{d}{dt} F_c\{u(x,t)\}$.

Applications

1. The temperature $u(x,t)$ at any point of an infinite bar satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$ and the initial temperature along the length of the bar is given by

$$u(x,t) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1. \end{cases}$$

Determine the expression for $u(x,t)$.

Sol: Given $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $(-\infty < x < \infty, t > 0)$

Applying Fourier transform on both sides, we get

$$F\left\{\frac{\partial u}{\partial t}\right\} = F\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \frac{d}{dt} F\{u(x,t)\} = (-ip)^2 F\{u(x,t)\}$$

$$\Rightarrow \frac{d}{dt} \bar{u}(p,t) = -p^2 \bar{u}(p,t)$$

$$(F\{u(x,t)\} = \bar{u}(p,t))$$

$$\Rightarrow \frac{d\bar{u}}{\bar{u}} = -p^2 dt$$

Integrating, we get

$$\log \bar{u} = -p^2 t + \log A$$

$$\Rightarrow \bar{u}(p,t) = A e^{-p^2 t} \longrightarrow (1)$$

Taking $t=0$ on both sides, we get

$$\bar{u}(p,0) = A \longrightarrow (2)$$

Now, given that
$$u(x,0) = \begin{cases} 1 & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$$

Taking Fourier transform on both sides,

we get

$$F\{u(x,0)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)\, e^{ipx}\, dx$$

$$\Rightarrow \bar{u}(p,0) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ipx}\, dx$$

$$\Rightarrow \bar{u}(p,0) = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipx}}{ip}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ip} - e^{-ip}}{ip}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2i\sin p}{ip}\right]$$

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{\sin p}{p}\right) \longrightarrow (3)$$

$\therefore$ From ②, $A = \frac{2}{\sqrt{2\pi}}\left(\frac{\sin p}{p}\right)$

and hence, from ①, we have

$$F\{u(x,t)\} = \bar{u}(p,t) = \frac{2}{\sqrt{2\pi}}\left(\frac{\sin p}{p}\right) e^{-p^2 t}$$

By the inverse Fourier transform, we have

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{u}(p,t) \cdot e^{-ipx}\, dp$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin p}{p}\right) e^{-p^2 t} e^{-ipx}\, dp$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin p}{p}\right) e^{-p^2 t} (\cos px - i \sin px)\, dp$$

$$= \frac{2}{\pi} \int_{0}^{\infty} e^{-p^2 t} \left(\frac{\sin p \cos px}{p}\right) dp.$$