NPTEL MOOC,JAN-FEB 2015
Week 2, Module 5

DESIGN AND ANALYSIS OF ALGORITHMS

Merge Sort

MADHAVAN MUKUND, CHENNAI MATHEMATICAL INSTITUTE
http://www.cmi.ac.in/~madhavan

$O(n^2)$ sorting algorithms

- Selection sort and insertion sort are both $O(n^2)$
- $O(n^2)$ sorting is infeasible for n over 100000

A different strategy?

- Divide array in two equal parts
- Separately sort left and right half
- Combine the two sorted halves to get the full array sorted

Combining sorted lists

- Given two sorted lists A and B, combine into a sorted list C
  - Compare first element of A and B
  - Move it into C
  - Repeat until all elements in A and B are over
- Merging A and B

Merging two sorted lists

32 74 89

21 55 64

Merging two sorted lists

32 74 89

21 55 64

21

Merging two sorted lists

~~32~~	74	89
~~21~~	55	64
21	32	

Merging two sorted lists

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Merging two sorted lists

21 32 55 64 74 89

Merge Sort

- Sort A[0] to A[n/2-1]
- Sort A[n/2] to A[n-1]
- Merge sorted halves into B[0..n-1]
- How do we sort the halves?
  - Recursively, using the same strategy!

Merge Sort

43	32	22	78	63	57	91	13

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43	32	22	78	63	57	91	13

43	32	22	78

63	57	91	13

43	32

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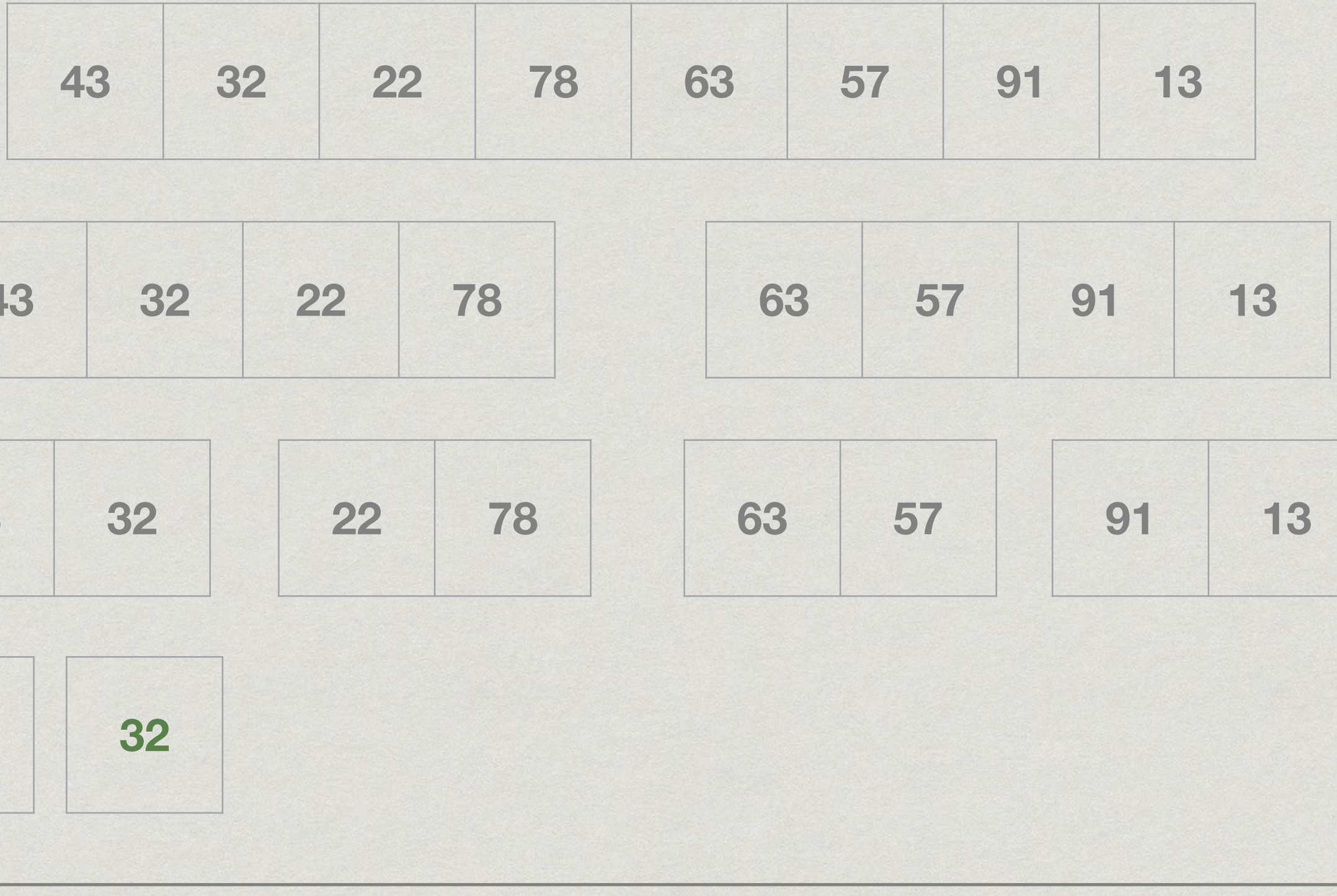
Merge Sort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Merge Sort


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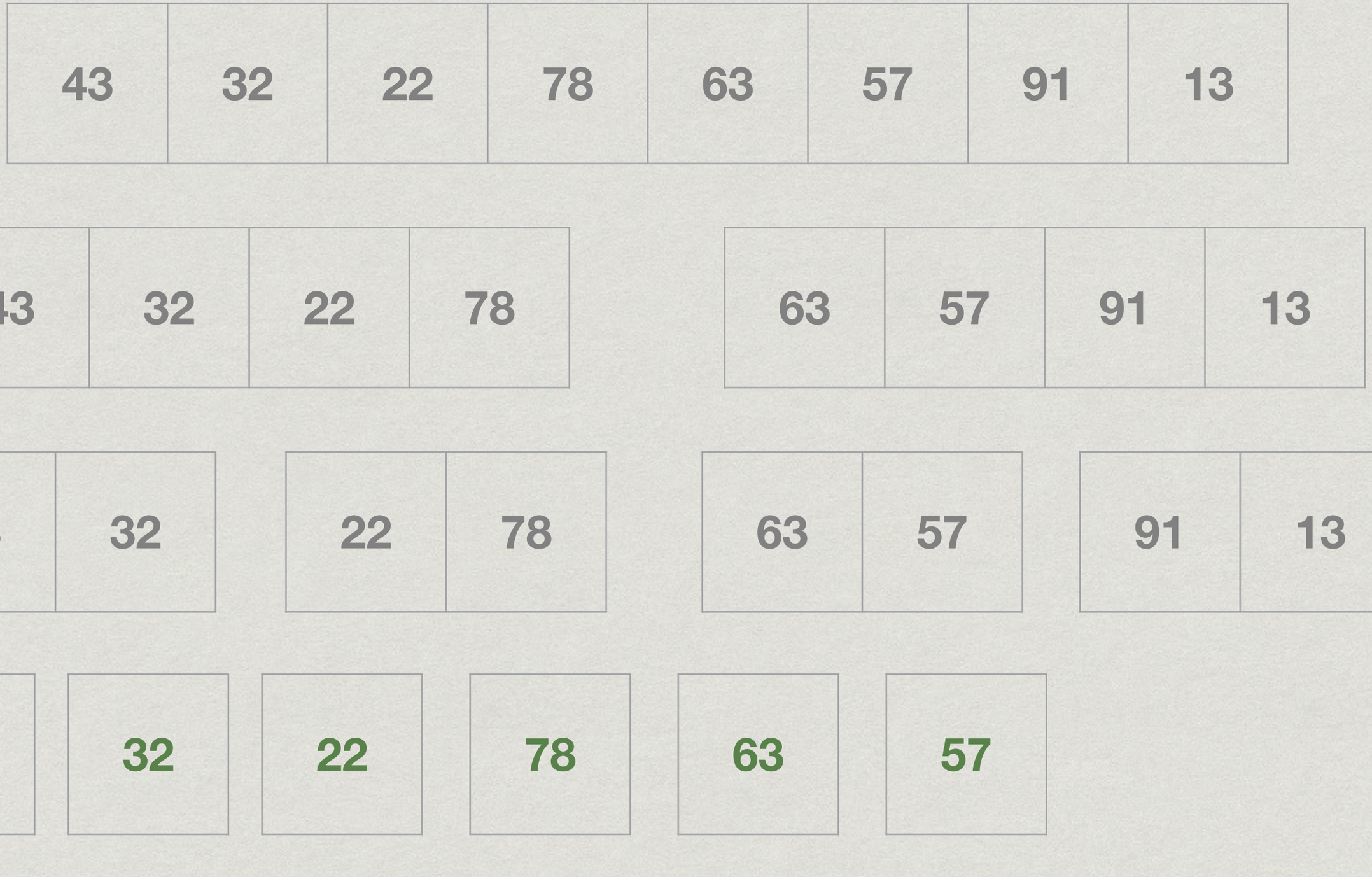
Merge Sort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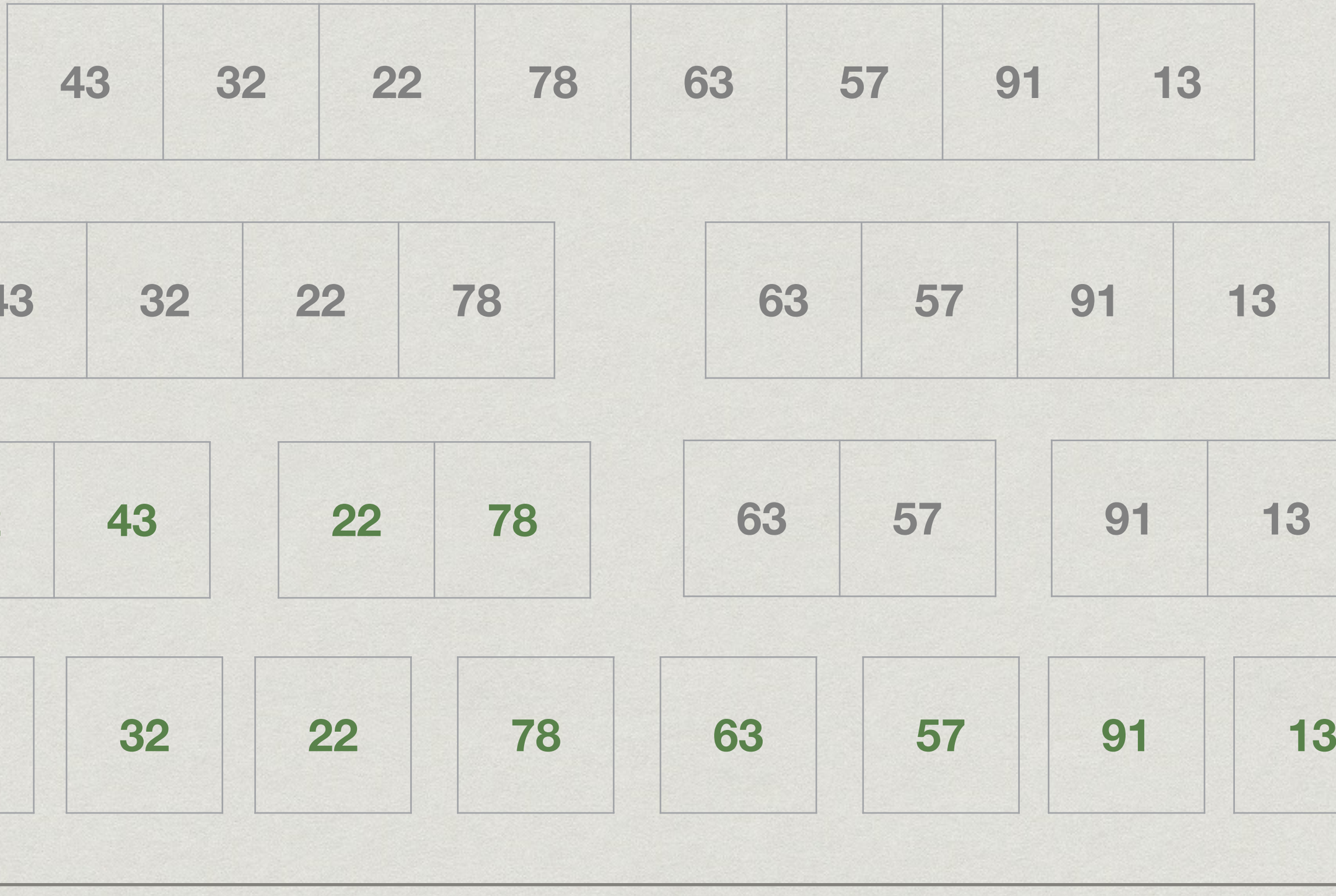
Merge Sort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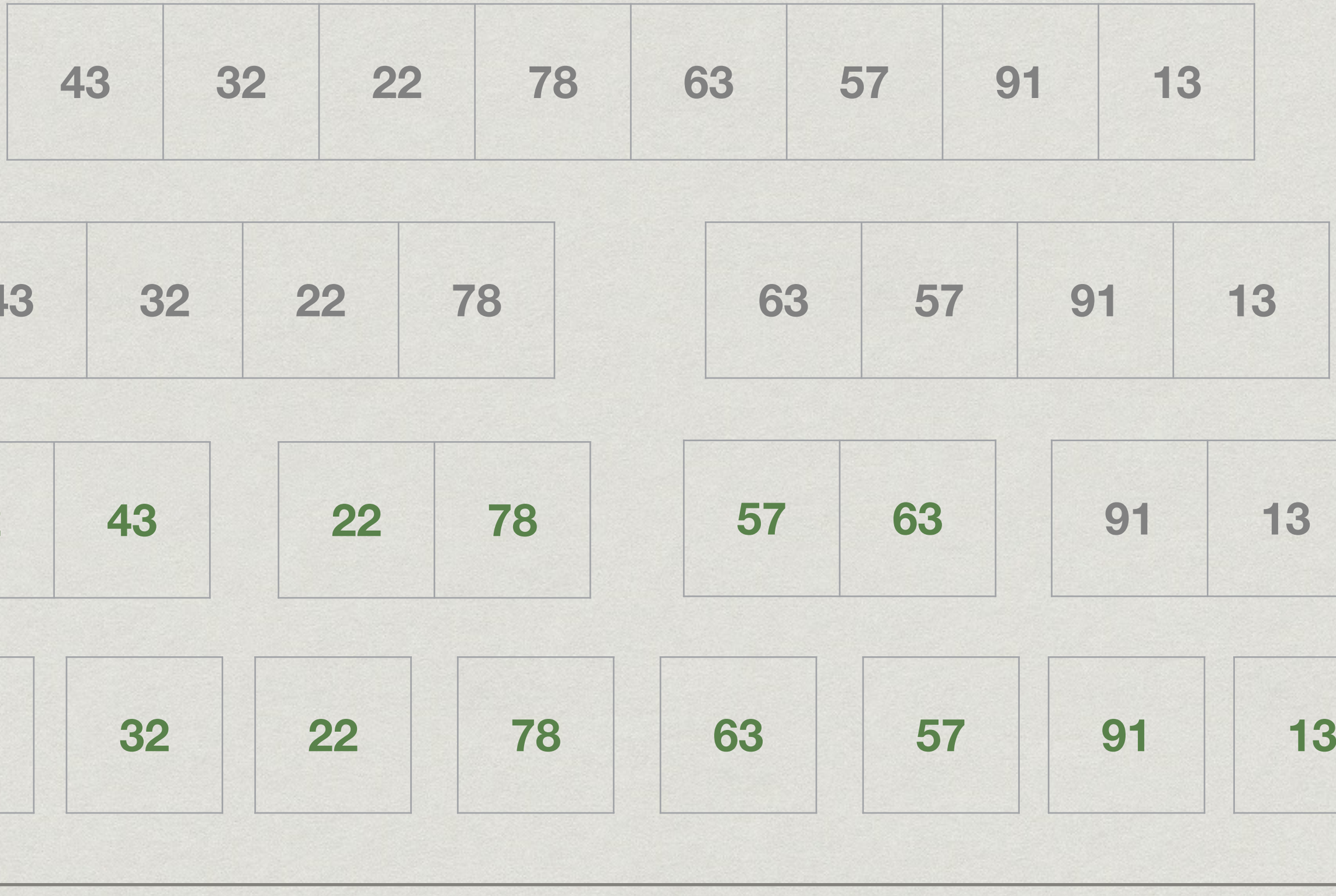
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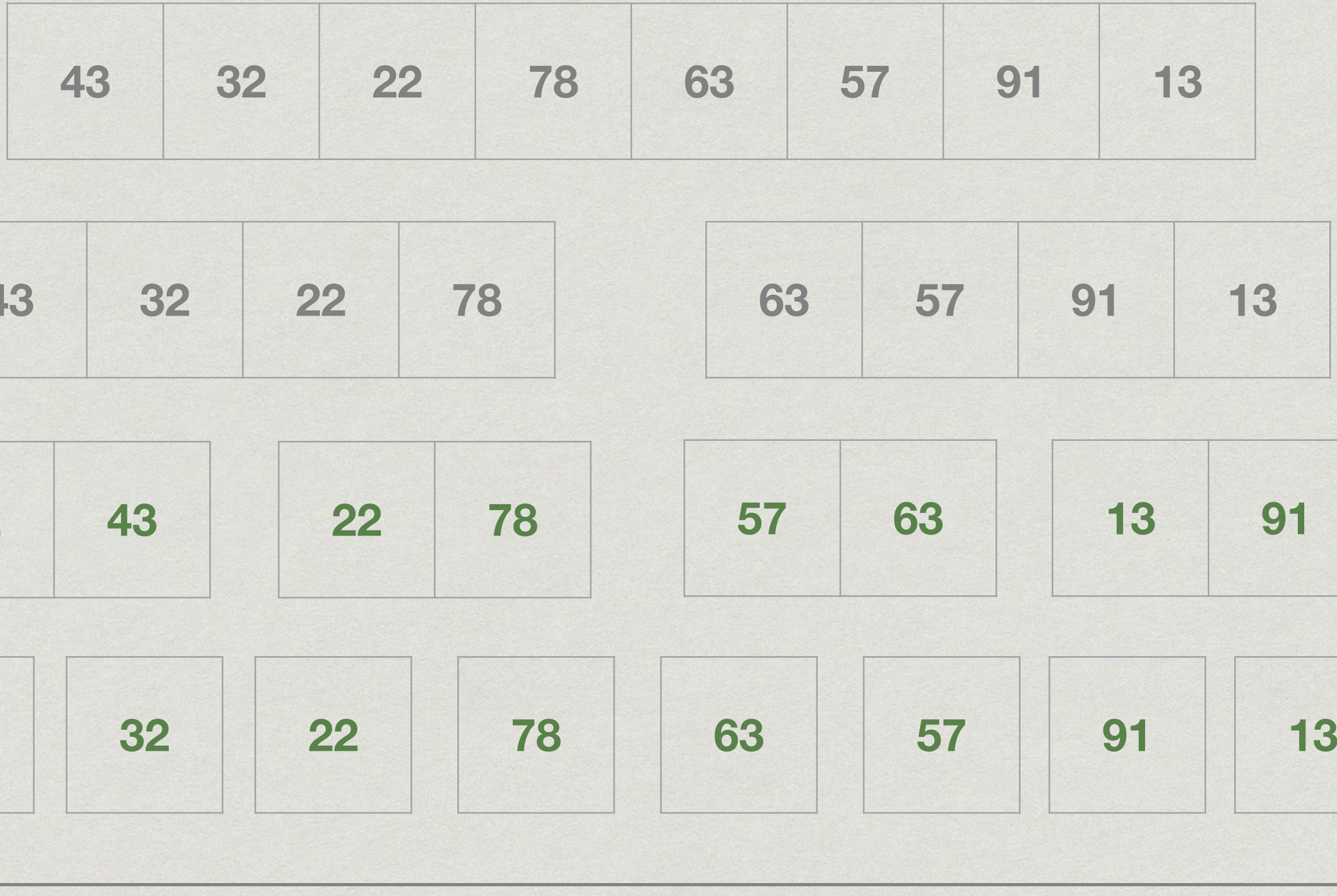
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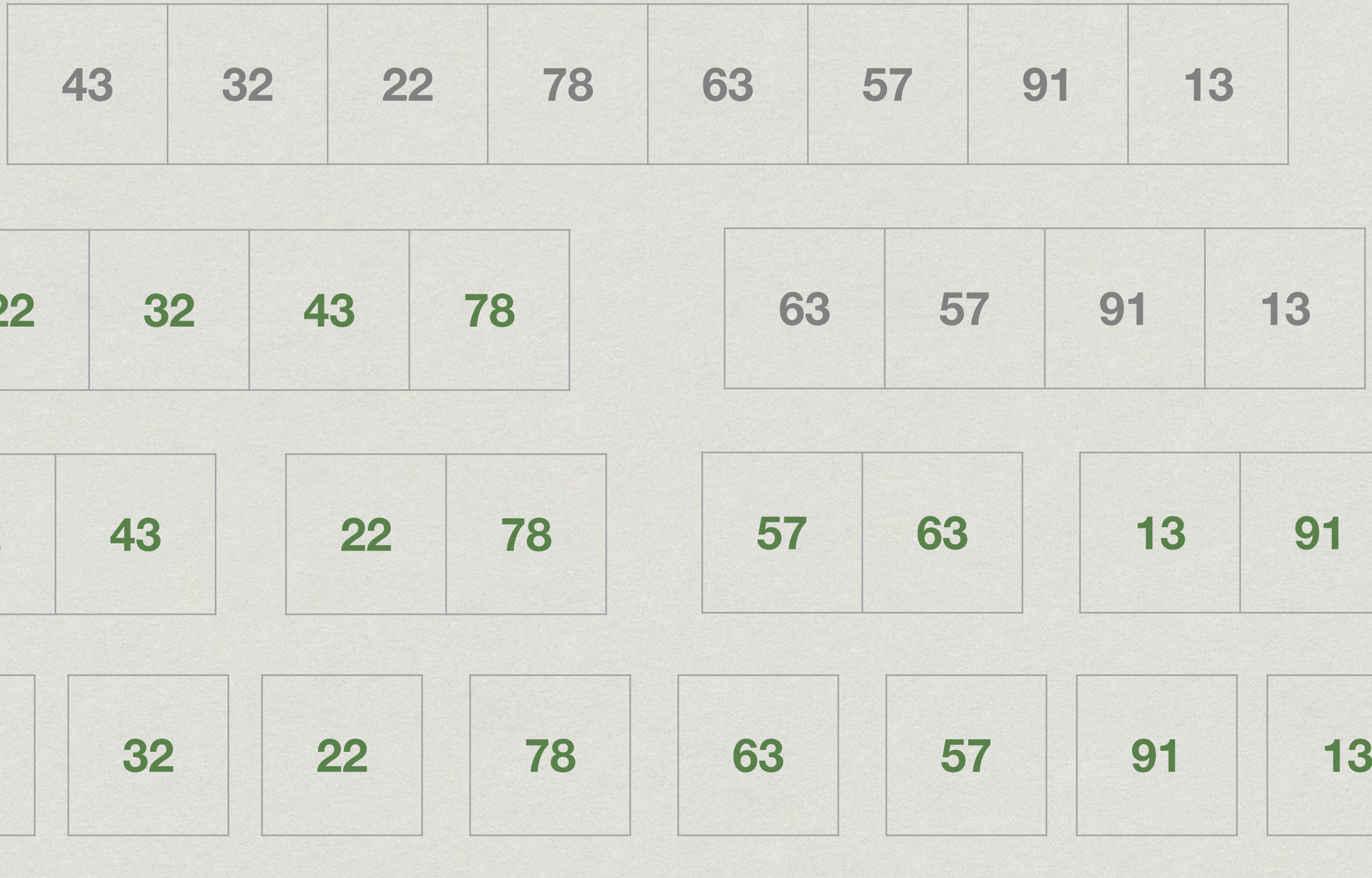
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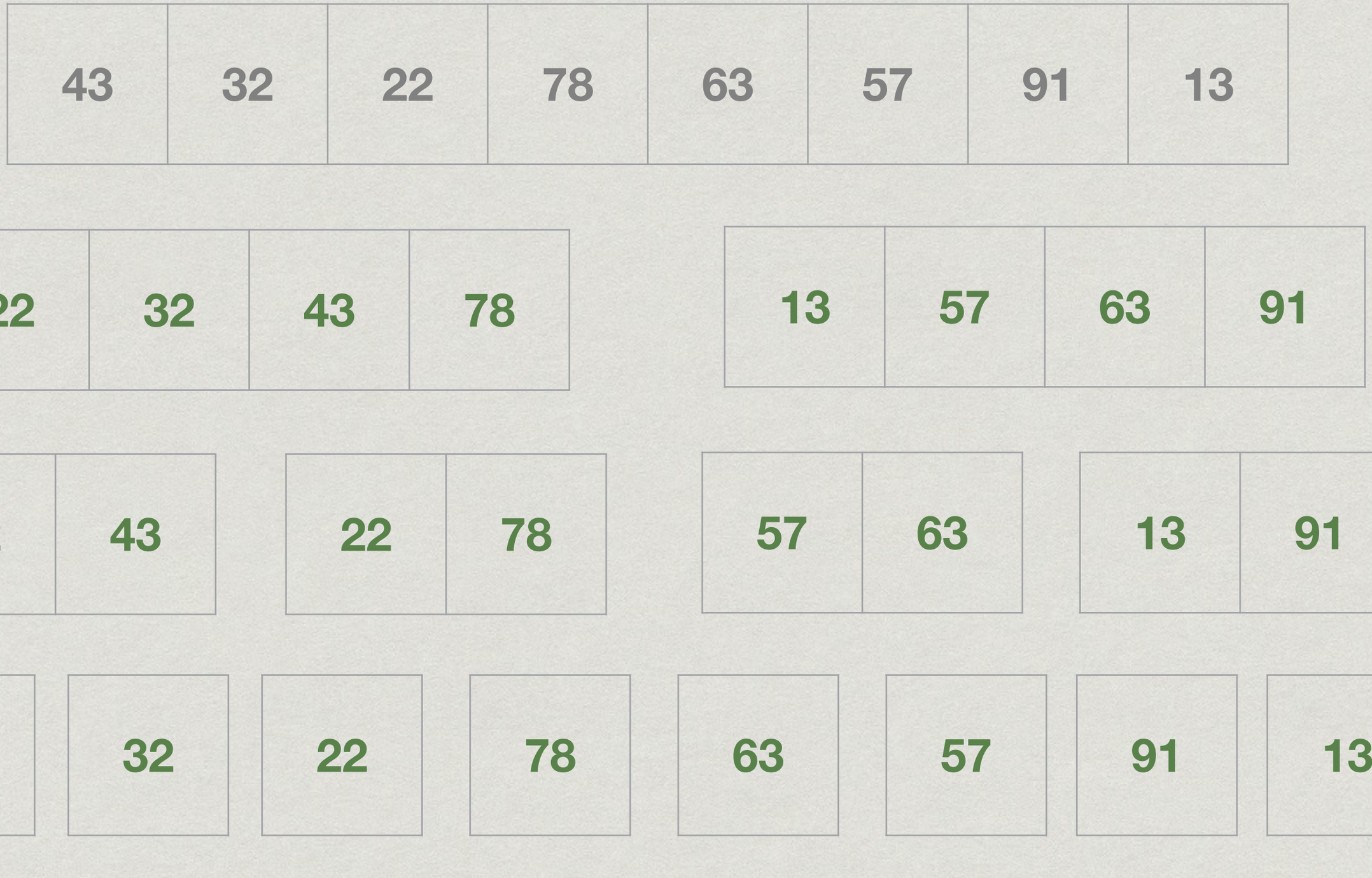
Merge Sort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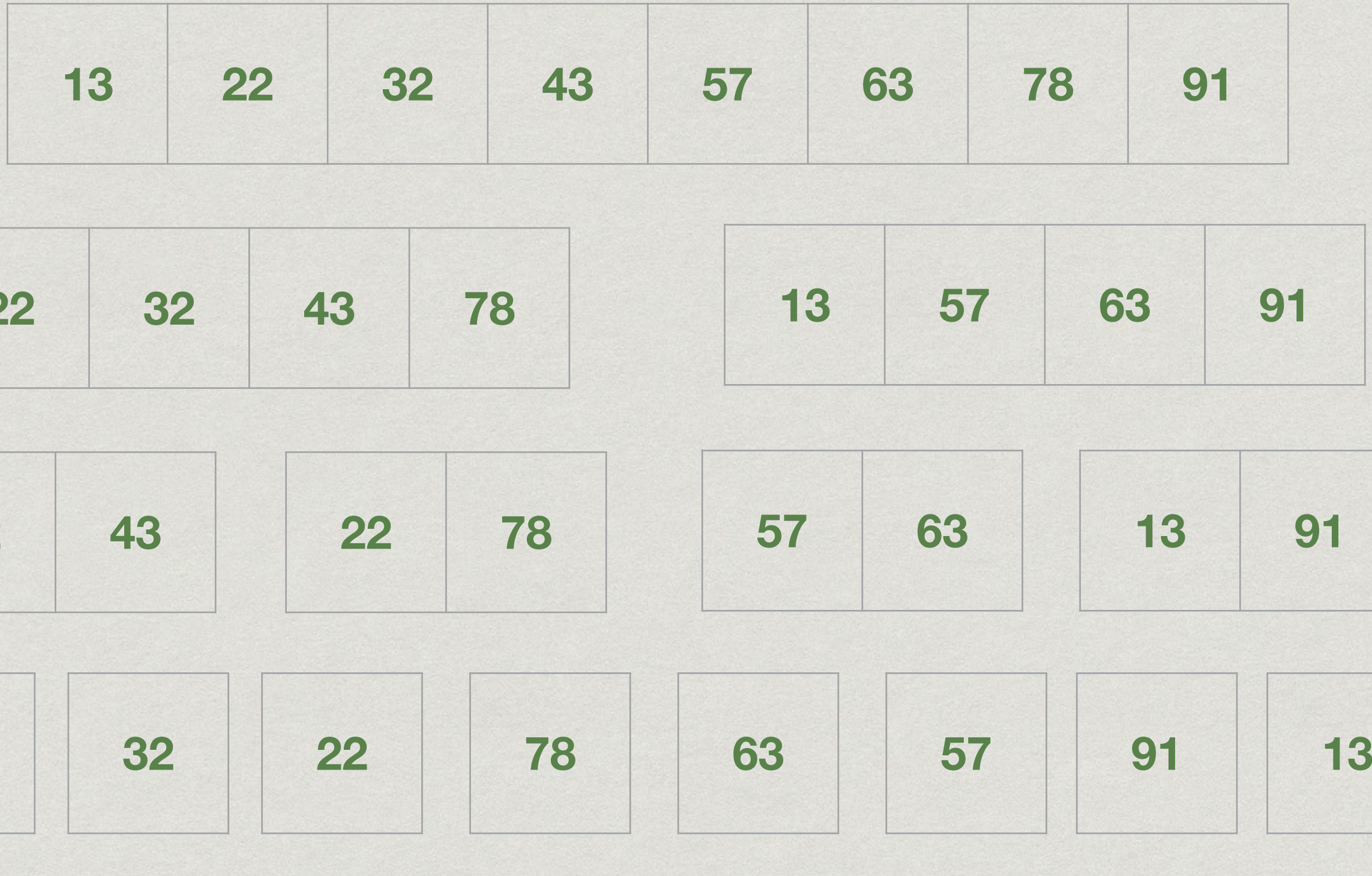
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Divide and conquer

- Break up problem into disjoint parts
- Solve each part separately
- Combine the solutions efficiently

Merge Sort

To sort A[0..n-1] into B[0..n-1]

- If n is 1, nothing to be done
- Otherwise
  - Sort A[0..n/2-1] into L (left)
  - Sort A[n/2..n-1] into R (right)
  - Merge L and R into B

Merge Sort

```
function MergeSort(A,left,right,B)
   // Sort the segment A[left..right-1] into B

   if (right - left == 1) // Base case
      B[0] = A[left]

   if (right - left > 1)  // Recursive call

      mid = (left+right)/2

      MergeSort(A,left,mid,L)
      MergeSort(A,mid,right,R)

      Merge(L,mid-left,R,right-mid,B)
```

Merging sorted lists

Combine two sorted lists A and B into C

- If A is empty, copy B into C
- If B is empty, copy A into C
- Otherwise, compare first element of A and B and move the smaller of the two into C
- Repeat until all elements in A and B have been moved

Merging

```
function Merge(A,m,B,n,C)
    // Merge A[0..m-1], B[0..n-1] into C[0..m+n-1]

    i = 0; j = 0; k = 0;
    // Current positions in A,B,C respectively

    while (k < m+n)
        // Case 1: Move head of A into C
        if (j==n or A[i] <= B[j])
            C[k] = A[i]; i++; k++

        // Case 2: Move head of B into C
        if (i==m or A[i] > B[j])
            C[k] = B[j]; j++; k++
```

Analysis of Merge

How much time does Merge take?

- Merge A of size m, B of size n into C
- In each iteration, we add one element to C
  - At most 7 basic operations per iteration
  - Size of C is m+n
  - $m+n \lesssim 2 \max(m,n)$
- Hence $O(\max(m,n)) = O(n)$ if $m \approx n$

Merge Sort

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  - Merge L and R into B

Analysis of Merge Sort ...

- t(n): time taken by Merge Sort on input of size n
  - Assume, for simplicity, that $n = 2^k$
  - $t(n) = 2t(n/2) + n$
    - Two subproblems of size n/2
    - Merging solutions requires time O(n/2+n/2) = O(n)
  - Solve the recurrence by unwinding

Analysis of Merge Sort ...

Analysis of Merge Sort ...

- t(1) = 1

Analysis of Merge Sort …

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- t(n) = 2t(n/2) + n

Analysis of Merge Sort …

- $t(1) = 1$
- $t(n) = 2t(n/2) + n$

$= 2 [2t(n/4) + n/2] + n = 2^2 t(n/2^2) + 2n$

Analysis of Merge Sort …

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…

Analysis of Merge Sort …

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…

$= 2^j t(n/2^j) + jn$

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- When $j = \log n$, $n/2^j = 1$, so $t(n/2^j) = 1$

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- When $j = \log n$, $n/2^j = 1$, so $t(n/2^j) = 1$
  - log n means $\log_2 n$ unless otherwise specified!
- $t(n) = 2^j t(n/2^j) + jn = 2^{\log n} + (\log n)\, n = n + n \log n = O(n \log n)$

O(n log n) sorting

- Recall that O(n log n) is much more efficient than $O(n^2)$
- Assuming 10^8 operations per second, feasible input size goes from 10,000 to 10,000,000 (10 million or 1 crore)

Merge Sort: Shortcomings

- Merging A and B creates a new array C
  - No obvious way to efficiently merge in place
- Extra storage can be costly
- Inherently recursive
  - Recursive call and return are expensive

Alternative approach

- Extra space is required to merge
- Merging happens because elements in left half must move right and vice versa
- Can we divide so that everything to the left is smaller than everything to the right?
  - No need to merge!