Vector and scalar functions

Vector functions: Functions whose values are vectors depending on the points P in space,

$$\mathbf{v} = \mathbf{v}(P) = v_1(P)\mathbf{i} + v_2(P)\mathbf{j} + v_3(P)\mathbf{k}\ .$$

Definition: A scalar valued function is a function that takes one or more values but produces a single value.

The $f(x,y,z) = x^2+2yz^5$ is an example of a scalar valued function.

A n-variable scalar valued function acts as a map from the space $\mathbf{R}^n$ to the real number line. That is, f: $\mathbf{R}^n$ --------> $\mathbf{R}$.

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f.

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Divergence *The divergence of a continuously differentiable vector point function* **F** *is denoted by div* **F** *and is defined by the equation*

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \mathbf{I} \cdot \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \cdot \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \cdot \frac{\partial \mathbf{F}}{\partial z}$$

If $\mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$

then

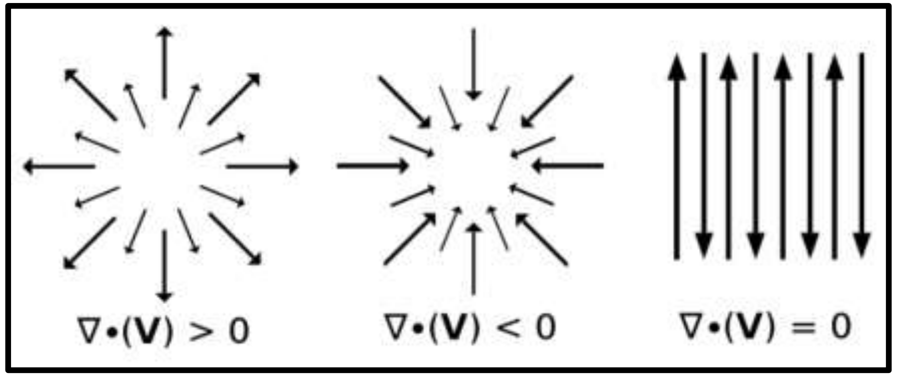
$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right) \cdot (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}) = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$$

Curl *The curl of a continuously differentiable vector point function* **F** *is defined by the equation*

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \mathbf{I} \times \frac{\partial \mathbf{F}}{\partial x} + \mathbf{J} \times \frac{\partial \mathbf{F}}{\partial y} + \mathbf{K} \times \frac{\partial \mathbf{F}}{\partial z}$$

If $\mathbf{F} = f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K}$ then $\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \left(\mathbf{I}\frac{\partial}{\partial x} + \mathbf{J}\frac{\partial}{\partial y} + \mathbf{K}\frac{\partial}{\partial z}\right) \times (f\mathbf{I} + \phi\mathbf{J} + \psi\mathbf{K})$

$$= \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = \mathbf{I}\left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial z}\right) + \mathbf{J}\left(\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\right) + \mathbf{K}\left(\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial y}\right).$$



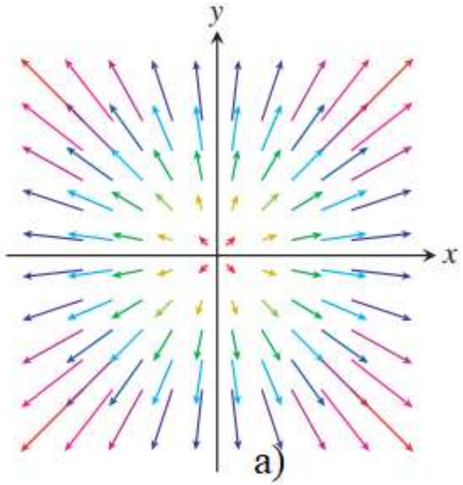
- ✓ A field which has zero divergence everywhere is called solenoidal.
- ✓ The divergence of a vector field is a scalar function.
- ✓ The curl of a vector field is a vector field.
- ✓ Curl gives the measure of angular velocity of an object.
- ✓ If Curl is zero, it means the object is not rotating (irrotational).
- ✓ If Curl is not zero, its magnitude represents the speed of the object and its direction denotes the axis of rotation.

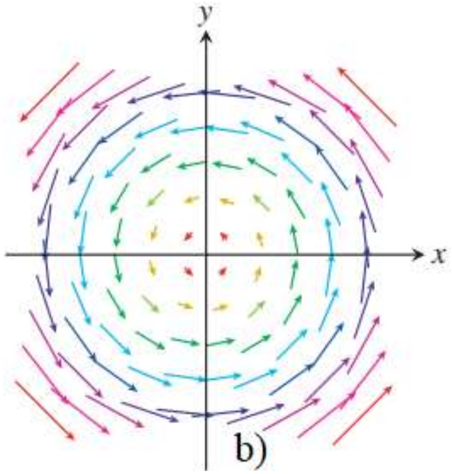
(a) *Uniform expansion or compression:* $\mathbf{F}(x, y) = cx\mathbf{i} + cy\mathbf{j}$

(b) *Uniform rotation:* $\mathbf{F}(x, y) = -cy\mathbf{i} + cx\mathbf{j}$

(c) *Shearing flow:* $\mathbf{F}(x, y) = y\mathbf{i}$

(d) *Whirlpool effect:* $\mathbf{F}(x, y) = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$





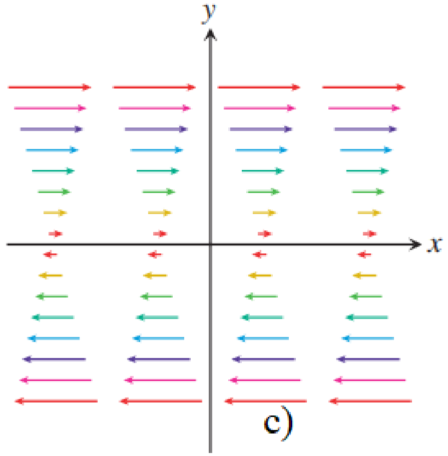
Solution

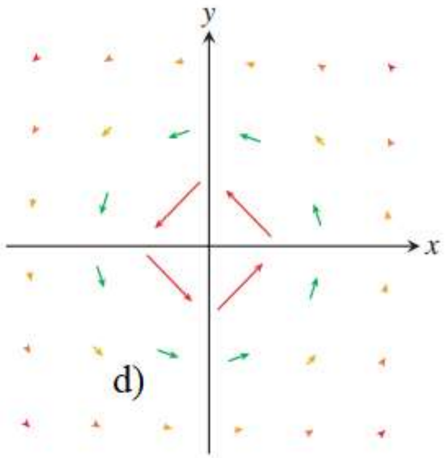
(a) $\text{div}\,\mathbf{F} = \dfrac{\partial}{\partial x}(cx) + \dfrac{\partial}{\partial y}(cy) = 2c$: If $c > 0$, the gas is undergoing uniform expansion; if $c < 0$, it is undergoing uniform compression.

(b) $\text{div}\,\mathbf{F} = \dfrac{\partial}{\partial x}(-cy) + \dfrac{\partial}{\partial y}(cx) = 0$: The gas is neither expanding nor compressing.

(c) $\text{div}\,\mathbf{F} = \dfrac{\partial}{\partial x}(y) = 0$: The gas is neither expanding nor compressing.

(d) $\text{div}\,\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{-y}{x^2 + y^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2 + y^2}\right) = \dfrac{2xy}{(x^2 + y^2)^2} - \dfrac{2xy}{(x^2 + y^2)^2} = 0$: Again, the divergence is zero at all points in the domain of the velocity field.





Example Compute $\operatorname{div}\vec{F}$ for $\vec{F} = x^2y\,\vec{i} + xyz\,\vec{j} - x^2y^2\,\vec{k}$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-x^2y^2) = 2xy + xz$$

Example Verify $\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = 0$

for the vector field $\vec{F} = yz^2\,\vec{i} + xy\,\vec{j} + yz\,\vec{k}$.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & xy & yz \end{vmatrix}$$

$$= z\,\vec{i} + 2yz\,\vec{j} + y\,\vec{k} - z^2\vec{k}$$

$$= z\vec{i} + 2yz\,\vec{j} + \left(y - z^2\right)\vec{k}$$

Now compute the divergence of this.

$$\operatorname{div}\left(\operatorname{curl}\vec{F}\right) = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}\left(y - z^2\right) = 2z - 2z = 0$$