Type I:

Solve completely the following :—

(1) $\sqrt{p} + \sqrt{q} = 1$

Solution. Let $p = a$, arbitrary constant. Then the equation becomes

$$\sqrt{a} + \sqrt{q} = 1$$

or $$q = (1 - \sqrt{a})^2.$$

Substituting $p = a$ and $q = (1-\sqrt{a})^2$ in the total differential $dz = p\,dx + q\,dy$, we get

$$dz = a\,dx + (1-\sqrt{a})^2\,dy$$

$$\Rightarrow \quad \cancel{dz} \quad z = ax + (1-\sqrt{a})^2 y + b$$

will be the complete integral.

(2) $pq + p + q = 0$. $\longrightarrow$ (i)

Solution. Let $p = a$ in (i) then

$$aq + a + q = 0 \quad \text{or} \quad q = -\frac{a}{a+1}$$

Hence
$$dz = p\,dx + q\,dy$$

gives
$$dz = a\,dx - \frac{a}{1+a}\,dy$$

$$\Rightarrow \quad z = ax - \frac{a}{a+1}\,y + b$$

is the complete integral of (i).

(3) $p^3 - q^3 = 0$

Ans. C.I is $z = ax + by + c$ where

$$b = a \text{ or } \frac{-a \pm \sqrt{a^2 - 4a^2}}{2}$$

(4) $p^2 + q^2 = 4$

Ans. C.I. is $z = ax + \sqrt{4 - a^2}\, y + b$

(5) $p^2 + p = q^2$

Ans. C.I. is $z = ax + \sqrt{a(a+1)}\, y + b$

(6) $2p + 3q = 1$

Ans. CI is $z = ax + \left(\frac{1-2a}{3}\right) y + c$

(7) $p = e^q$

Ans. CI is $z = ax + (\log_e a) y + c$

or $z = e^a x + ay + c$.

(8) $p^2 + q^2 = npq$.

Solution :- This is of the form $f(p,q) = 0$.

$\therefore$ Let $p = a$ in this eqn, we get

$$a^2 + q^2 = naq$$

$$\Rightarrow \quad q^2 - (na) q + a^2 = 0$$

$$\Rightarrow \quad q = \frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2}$$

$$= \frac{a\left[n \pm \sqrt{n^2 - 4}\right]}{2}$$

$\therefore$ The total differential

$$dz = p\,dx + q\,dy \Rightarrow$$

$$dz = a\,dx + \frac{a\left[n \pm \sqrt{n^2-4}\right]}{2} dy$$

$$\Rightarrow \quad z = ax + \frac{a\left[n \pm \sqrt{n^2-4}\right]}{2} y + b \rightarrow \text{C.I.}$$

(9) $p = q^2$

Ans. CI is $z = ax + \sqrt{a}\, y + c$

(10) $q^2 - 3q + p = 2$

Ans. $z = ax + by + c$ where $b = \frac{3 \pm \sqrt{17 - 4a}}{2}$

Problems

Solve the following completely. Type II

(1) $q = xp + p^2$.

Solution. The eqn is of the form $f(x,p,q) = 0$.

$\therefore$ Let $q = a^2$ in this, we get

$$a^2 = px + p^2 \quad \text{or} \quad p^2 + px - a^2 = 0$$

$$\Rightarrow \quad p = \frac{-x \pm \sqrt{x^2 + 4a^2}}{2}$$

$$\therefore \quad dz = p\,dx + q\,dy$$

$$\Rightarrow \quad dz = \left\{ \frac{-x \pm \sqrt{x^2 - 4a^2}}{2} \right\} dx + a^2\,dy.$$

Integrating,

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left\{ \frac{x}{2}\sqrt{x^2 + 4a^2} + \frac{4a^2}{2}\sinh^{-1}\left(\frac{x}{2a}\right) \right\} + a^2 y + c \longrightarrow \text{The C.I.}$$

(2) $\sqrt{p} + \sqrt{q} = x$.

Solution:– It is of the form $f(x,p,q) = 0$.

$\therefore$ Let $q = a^2$ in the eqn. Then

$$\sqrt{p} + a = x \quad \text{or} \quad p = (x-a)^2$$

$$\therefore \quad dz = p\,dx + q\,dy \Rightarrow dz = (x-a)^2 dx + a^2\,dy.$$

Integrating, $z = \dfrac{(x-a)^3}{3} + a^2 y + c \longrightarrow$ C.I.

(3) $pq = x$.

Ans. $z = \dfrac{x^2}{2a} + ay + b$.

(4) $q^2 = yp^4$.

Solution. This is of the form $f(y,p,q) = 0$.

$\therefore$ Let $p = a$ in it. Then

$$q^2 = ya^4 \quad \Rightarrow \quad q = \pm a^2\sqrt{y}.$$

$$\therefore \quad dz = p\,dx + q\,dy \Rightarrow dz = a\,dx \pm a^2\sqrt{y}\,dy.$$

Integrating, $z = ax \pm \dfrac{2a^2 y^{3/2}}{3} + c \longrightarrow$ C.I.

(5) $p^2 + pq = z^2$.

Sol: This is of the form $f(z,p,q) = 0$.

$\therefore$ Let $q = pa$, a being arbitrary constant.

Then we have $p^2 + p(pa) = z^2$

$$\Rightarrow p^2(1+a) = z^2$$

$$\Rightarrow p = \frac{z}{\sqrt{1+a}}.$$

$\therefore$ $dz = p\,dx + q\,dy$ implies that

$$dz = p\,dx + ap\,dy$$

$$\Rightarrow dz = p(dx + a\,dy)$$

$$\Rightarrow dz = \frac{z}{\sqrt{1+a}}(dx + a\,dy)$$

$$\Rightarrow \sqrt{1+a}\cdot\frac{dz}{z} = dx + a\,dy.$$

Integrating, $\sqrt{1+a}\cdot\log z = x + ay + c \rightarrow$ The

(6) $p(1+q) = qz$.

Sol. It is of the form $f(z,p,q) = 0$.

$\therefore$ Let $q = pa$ in this. Then we get

$$p(1+ap) = apz$$

$$\Rightarrow p = \frac{az-1}{a}.$$

$$\therefore dz = p\,dx + q\,dy = p(dx + a\,dy)$$

$$\Rightarrow dz = \cancel{p}\,\frac{az-1}{a}(dx + a\,dy)$$

$$\Rightarrow \frac{dz}{\left(z - \frac{1}{a}\right)} = dx + a\,dy.$$

Integrating, $\log(z - 1/a) = x + ay + c \rightarrow$ C.I

(7) $p(1+q^2) = q(z-1)$.

Sol. This is of the form: $f(p,q,z) = 0$.

$\therefore$ Let $q = ap$ in this, we get

$$p(1 + a^2p^2) = ap(z-1)$$

$$\Rightarrow a^2p^2 = a(z-1) - 1$$

$$\Rightarrow p = \frac{\sqrt{a(z-1)-1}}{a}$$

(8) $\sqrt{p}+\sqrt{q}=\sqrt{y}$.

Ans. $z = ax + \frac{y^2}{2} + ay - \frac{4}{3}\sqrt{a}\cdot y^{3/2} + b$.

(9) $z = p^2+q^2$.

Ans. $4(a^2+1)z = (x+ay+c)^2$

(10) $p+q = z$

Ans. $x+ay = (1+a)\log z + c$

(11) $p^3 = qz$.

Ans. $2\sqrt{z} = \sqrt{a}\,(x+ay+b)$.

(12) $p^2 = qz$.

Ans. $a(x+y)+b-\log z = 0$.

(13) $z^2 = p^2+q^2+1$.

Sol. It is of the form $f(p,q,z)=0$.

$\therefore$ Let $q = ap$ in this,

$$\Rightarrow z^2 = p^2 + a^2p^2 + 1$$

$$\Rightarrow p^2 = \frac{z^2-1}{a^2} \Rightarrow p = \frac{\sqrt{z^2-1}}{\sqrt{a^2+1}}.$$

$$\therefore dz = p\,dx + q\,dy \Rightarrow dz = p(dx+a\,dy)$$

$$\Rightarrow dz = \frac{\sqrt{z^2-1}}{\sqrt{1+a^2}}(dx+dy)$$

$$\Rightarrow \frac{\sqrt{(1+a^2)}\,dz}{\sqrt{z^2-1}} = dx + a\,dy. \text{ Integrating,}$$

$$\sqrt{1+a^2}\cdot\cosh^{-1}z = (x+ay)+c \longrightarrow \text{C.I.}$$

(14) $z = p^2q^2$.

Ans. $\frac{4a}{3}z^{3/4} = x+ay+c$.

(15) $pq = x$ Ans. $z = \frac{a}{2}\cdot x^2 + \frac{1}{a}\cdot y + b$

(16) $z^2(p^2+q^2+1) = 1$. Ans. $(1+a^2)(1-z^2) = (x+ay+c)^2$

(17) $4(1+z^3) = 9z^4pq$.

Ans. $a^2(1+z^3) = (x+a^2y+b)^2$

(18) $q^2 = z^2p^2(1-p^2)$

Ans. $z^2 = a^2+(x+ay+c)^2$.

19) $p^2z^2+q^2=1$

Sol: it is of the form $f(p,q,z)=0$

$\therefore$ Write $q=ap$. Then $dz=p(dx+ady)$

But $q=ap$ in the given eqt results

$$p^2(z^2+a^2)=1 \Rightarrow p=\frac{1}{\sqrt{z^2+a^2}}$$

$$\therefore \quad dz=\frac{1}{\sqrt{z^2+a^2}}(dx+ady)$$

$\Rightarrow \sqrt{z^2+a^2}\,dz = dx+ady$, on integration

$$\frac{z}{2}\sqrt{z^2+a^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{z}{a}\right)=(x+ay)+c$$

$\longrightarrow$ C.I

20) $9(p^2z+q^2)=4$

Ans: $(z+a^2)^{3/2}=x+ay+c$.

Type 3: Equations of form

$$f(x,y)=g(y,q) \longrightarrow (1)$$

Where, the independent variable z is not involved

Assume that $f(x,p)=g(y,q)=a$ (const)

Then $f(x,p)=a$ or $p=\phi_1(x,a)$

$g(y,q)=a$ or $q=\phi_2(y,a)$

with these substitutions,

$$dz=pdx+qdy$$

$$dz=\phi_1(x,a)dx+\phi_2(y,a)dy$$

$z=\int\phi_1(x,a)\,dx+\int\phi_2(y,a)\,dy+c$, is the C.I

No Singular solution.

Solve completely the following

$$p^2-q^2 = x-y$$

Sol: let $p^2-x = q^2-y = a$ say

then $p^2 = x = a,\ q^2-y=a$

$\Rightarrow$ $p = \frac{\partial z}{\partial x} = \sqrt{x+a}\ ,\ q = \frac{\partial z}{\partial y} = \sqrt{y+a}$

But, $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

$= \sqrt{x+k}\ dx + \sqrt{y+k}\ dy$

$\Rightarrow$ $z = \frac{2}{3}(x+k)^{3/2} + \frac{2}{3}(y+k)^{3/2} + \frac{1}{3}c$

$\Rightarrow$ $3z = 2(x+k)^{3/2} + 2(y+k)^{3/2} + c.$ Ans.

2) $$\sqrt{p}+\sqrt{q} = x+y$$

Sol: let $\sqrt{p}-x = -\sqrt{q}+y = a$

$\sqrt{p} = x+a\ ,\ \sqrt{q} = y-a$

$p = (x+a)^2,\ q = (y-a)^2$

$dz = p\,dx + q\,dy$

$\Rightarrow$ $dz = (x+a)^2dx + (y-a)^2dy$

$\Rightarrow$ $3dz = 3(x+a)^2dx + (y-a)^2dy$

$\Rightarrow$ $C.I = 3z = (x+a)^3 + (y-a)^3 + b.$ Ans.

3) $$p+q = \sin x + \sin y$$

Sol: let $p - \sin x = \sin y - q = a$

$dz = \sin x + a\ ,\ q = \sin y - a$

$dz = p\,dx + q\,dy \Rightarrow dz = (\sin x + a)dx + (\sin y - a)dy$

$$z = -\cos x + ax - \cos y - ay + b$$

$$z + \cos x + \cos y = a(x-y) + b. \quad \text{Ans.}$$

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$$px + qx + pq = 0$$

Sol:

$$py + qx = -pq \quad (or) \quad \frac{y}{q} - 1 = -\frac{x}{p}$$

Let $\frac{y}{q} - 1 = -\frac{x}{p} = a$

then $\frac{y}{q} - 1 = a$ and $-\frac{x}{p} = a$

$$dz = p\,dx + q\,dy$$

$$= -\frac{x}{a}dx + \frac{y}{a+1}dy$$

$$2dz\,(a)(a+1) = -(a+1)\,2x\,dx + a \cdot 2y\,dy$$

$$2a(a+1)z = -(a+1)x^2 + ay^2 + b \quad \text{Ans.}$$

(on integration)

;)

$$q = xyp^2$$

Sol:

$$\frac{q}{y} = p^2 x = a$$

$$\Rightarrow \frac{q}{y} = a^2, \quad p^2 x = a^2$$

$$q = a^2 y, \quad p = \frac{a^2}{\sqrt{x}}$$

$$dz = p\,dx + q\,dy$$

$$\Rightarrow a^2 (x)^{-1/2} dx + a^2 y\,dy$$

$$\Rightarrow z = 2\sqrt{x}\,a^2 + \frac{a^2 y^2}{2} + \frac{b}{2}$$

$$(or) \quad 2z = (4\sqrt{x} + y^2)a^2 + b.$$

$$x^2 p = y^2 q$$

Sol: Let $x^2 p = y^2 q = a$

$$p = \frac{a}{x^2}, \quad q = \frac{a}{y^2}$$

$$\therefore \; dz = \frac{a}{x^2} dx + \frac{a}{y^2} dy$$

$$= \frac{-a}{x} - \frac{a}{y} + b$$

$$\Rightarrow \quad xyz + a(x+y) = bxy$$

(or) $xy(z-b) + a(x+y) = 0$ Ans.

1)

$$pq + qx = y$$

$$p + x = y/q = a.$$

$$p = a - x, \quad q = y/a$$

$$z = \int (a-x)\, dx + \frac{1}{a} \int y\, dy$$

$$z = (ax - x^2) + \frac{y^2}{2a} + c$$

(or) $2az = 2a^2 x - 2ax^2 + y^2 + c_1$

$$yp - x^2 q^2 = x^2 y$$

$$\frac{p^2}{x^2} - \frac{q^2}{y}, = 1$$

$$\frac{p}{x^2} = 1 + \frac{q^2}{y} = a^2 \qquad \therefore dz = x^2 a^2 + \sqrt{a^2-1}\sqrt{y}\, dy$$

$$p = x^2 a^2, \quad \frac{q^2}{y} = a^2 - 1$$

(or) $q = \sqrt{a^2-1}\sqrt{y}$

$$z = \frac{x^3 a^2}{3} + \sqrt{a^2-1} \cdot 2\sqrt{y} + b$$

(9) $\left(\frac{p}{2}+x\right)^2+\left(\frac{q}{2}+y\right)^2=1$.

Ans. $z=-x^2+2(\sqrt{a+1})x-y^2+2\sqrt{a}\,y+c$.

(10) $xp-y^2q^2=1$.

Ans. $z=(a^2+1)\log x+a\log y+c$.

(11) $p^2+q^2=x+y$

Ans. $z=\frac{2}{3}\left[(a+x)^{3/2}+(y-a)^{3/2}\right]+c$.

(12) $py+qx=pq$.

Ans. $z=\frac{x^2}{2a}+\frac{y^2}{2(1-a)}\quad b$

(13) $pq=xy$

Ans $z=\frac{ax^2}{2}+\frac{y^2}{2a}+c$.

(14) $p^2+q^2=x^2+y^2$.

Solution.

let $p^2+q^2=x^2+\ ^2 \Rightarrow p^2-x^2=y^2-q^2=a^2$, say.

$\Rightarrow p^2-x^2=a^2,\ y^2-q^2=a^2$

$\Rightarrow p=\sqrt{x^2+a^2},\quad q=\sqrt{y^2-a^2}$

$\therefore dz=p\,dx+q\,dy$

$\Rightarrow dz=\sqrt{x^2+a^2}\,dx+\sqrt{y^2-a^2}\,dy$. Integrating,

$$z=\frac{x}{2}\sqrt{x^2+a^2}\quad\frac{a}{2}\sinh^{-1}\left(\frac{x}{a}\right)$$
$$+\sqrt{y^2-a^2}-\frac{a}{2}\cosh^{-1}\left(\frac{y}{a}\right)+c\rightarrow \text{C.I.}$$

(15) $yp=2yx+\log q$.

Ans. $z=x^2+ax+\frac{1}{a}e^{ay}+b$.

(16) $pe^y=qe^x$.

Ans. $z=ae^x+ae^y+b$

(17) $p+q=\sin x+\sin y$.

Ans. $z=a(x-y)-(\cos x+\cos y)+c$.

If the Lagrangian method of linear equations is used the answer will be

general integral : $f(z-\cos x-\cos y,\ x-y)=0$.

—o—

Type 4: clairaut's form

$$z = px + qy + f(p,q)$$

its C.I is $z = ax + by + f(a,b) \rightarrow (1)$

where a & b are arbitrary constants

The Singular Solution is obtained from (1) and the relations, $\frac{\partial z}{\partial a} = 0$ and $\frac{\partial z}{\partial b} = 0$

Sometimes, the clairaut's form is given by

$$F(z - px - qy, p, q) = 0.$$

problems:

Solve the following completely, singular integrals also

$(1-x)p + (2-y)q = 3-z$

Sol:

$$z = px + qy + (3 - p - 2q) \rightarrow (1)$$

which is the clairaut's form

$$z = px + qy + f(p,q)$$

where, $f(p,q) = 3 - p - 2q$

The C.I is obtained by replacing p&q in a&b in (1) Then C.I is

$$z = xa + by + f(a,b)$$

$$z = ax + by + (3a - a - 2b)$$

$z = 3$ is the S.I.

$$z = px + qy + p^2q^2$$

Ans: C.I is $z = ax + by + a^2b^2$; S.I is $16z^3 + 27x^2y^2 = 0$

$$z = px + qy + \sqrt{1 + p^2 + q^2}$$

Ans: C.I is $z = ax + by + \sqrt{1 + a^2 + b^2} \rightarrow (1)$

(4) $z = px + qy - 2\sqrt{pq}$.

Ans. C.I. is $z = ax + by - 2\sqrt{ab}$

S.I. is $xy = 1$.

(5) $z = px + qy + pq$

Ans. C.I. is $z = ax + by + ab$.

S.I. is $z = xy$.

(6) $z = px + qy + \frac{p}{q} - p$.

Solution. C.I. is $z = ax + by + \frac{a}{b} - a$. $\rightarrow$ (i)

Diff. w.r.t. a and b partially and equating to zero.

$$x + \frac{1}{b} - 1 = 0 \quad \text{and} \quad y - \frac{a}{b^2} = 0 .$$

$$\Rightarrow \quad b = \frac{1}{1-x} \quad \text{and hence} \quad a = b^2 y$$

$$\text{or } a = \frac{y}{(1-x)^2} .$$

Hence the S.I. is

$$z = \frac{xy}{(1-x)^2} + \frac{y}{(1-x)} + \frac{y}{(1-x)^2}\cdot(1-x) - \frac{y}{(1-x)^2}$$

$$\Rightarrow \quad z = \frac{xy + 2y(1-x) - y}{(1-x)^2}$$

$$\Rightarrow \quad z = \frac{y(1-x)}{(1-x)^2} \quad \text{or} \quad (1-x)z = y .$$

(7) $z = px + qy + (p^2 - q^2)$.

Ans. CI is $z = ax + by + (a^2 - b^2)$

S.I. is $x^2 - y^2 + 4z = 0$.

(8) $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$.

Ans. CI. $z = ax + by + (ab)^{3/2}$.

(9) $pqz = p^2(x + p^2) + q^2(yp + q^2)$.

$$\Rightarrow \quad z = \frac{(p^2q)x + (q^2p)y + p^4 + q^4}{pq}$$

$$\Rightarrow \quad z = px + qy + \left(\frac{p^4 + q^4}{pq}\right) .$$