

UNIT STEP FUNCTION :-

(or) (Heaviside's function)

The unit step function is denoted by $u(t-a)$ (or, $H(t-a)$) and defined as $u(t-a) = \begin{cases} 1 & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$ where $a > 0$.

This is called unit step function at $t=a$.

1. Express the following functions in terms of unit step functions and hence find their Laplace transforms.

$$(i) \quad f(t) = \begin{cases} e^t & \text{if } 0 < t < 1 \\ 1 & \text{if } t > 1 \end{cases}$$

$$(ii) \quad f(t) = \begin{cases} \cos t & \text{if } 0 < t < \pi/2 \\ 0 & \text{if } \pi/2 < t < \pi \\ 1 & \text{if } t > \pi \end{cases}$$

Sol. (i) We can express the given function as in terms of unit step function

$$f(t) = e^t (u(t-0) - u(t-1)) + 1 (u(t-1))$$

Now,

$$\begin{aligned}L\{f(t)\} &= \int_0^{\infty} e^{-st} \left\{ e^t(u(t-0) - u(t-1)) + 1(u(t-1)) \right\} dt \\&= \int_0^1 e^{-st} e^t dt + \int_1^{\infty} e^{-st} \cdot 1 dt \\&= \int_0^1 e^{-(s-1)t} dt + \left(\frac{e^{-st}}{-s} \right)_1^{\infty} \\&= \left(\frac{e^{-(s-1)t}}{-(s-1)} \right)_0^1 + \left(\frac{e^{-st}}{-s} \right)_1^{\infty} \quad (s > 1) \\&= \frac{-1}{(s-1)} \left(e^{-(s-1)} - e^0 \right) - \frac{1}{s} \left(e^{-\infty} - e^{-s} \right) \\&= \frac{-1}{(s-1)} \left(e^{-(s-1)} - 1 \right) + \frac{e^{-s}}{s}\end{aligned}$$

Second Shifting theorem :

$$\text{If } L\{f(t)\} = F(s) \text{ and } g(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases} \quad (a > 0)$$

$$\text{then } L\{g(t)\} = e^{-as} \cdot F(s)$$

Note: we can write $g(t)$ as $g(t) = f(t-a)u(t-a) + 0 \cdot u(t-a)$

(or) $\boxed{g(t) = f(t-a)u(t-a)}$

Examples

① Evaluate $L\{g(t)\}$, where

$$g(t) = \begin{cases} e^{-(t-2)} & \text{if } t > 2 \\ 0 & \text{if } t < 2 \end{cases}$$

Sol. Let $f(t) = e^{-t}$
Then $L\{f(t)\} = L\{e^{-t}\} = \frac{1}{s+1} = F(s)$

Here $a = 2$

By the second shifting theorem, we have

$$L\{g(t)\} = e^{-as} F(s)$$

Therefore, $L\{g(t)\} = e^{-2s} \frac{1}{s+1}$

② Evaluate $L\{(t-1)^2 u(t-1)\}$

Sol. Let $f(t) = t^2$

Then $L\{f(t)\} = L\{t^2\} = \frac{2}{s^3} = F(s)$

Here $a = 1$

By the second shifting theorem, we have

$$L\{f(t-a)u(t-a)\} = e^{-as} F(s)$$

Therefore, $L\{(t-1)^2 u(t-1)\} = e^{-s} \cdot \frac{2}{s^3}$

③ Evaluate $L \{ e^{2t} u(t-2) \}$

Sol. Let $f(t) = e^{2t}$

$$\text{Then, } L \{ f(t) \} = L \{ e^{2t} \} = \frac{1}{s-2} = F(s)$$

Here $a = 2$

By the second shifting theorem, we have

$$L \{ f(t-2) u(t-2) \} = e^{-2s} \cdot \frac{1}{s-2}$$

$$\begin{aligned} \text{Therefore, } L \{ f(t-2) u(t-2) \} &= L \{ e^{2(t-2)} u(t-2) \} \\ &= e^{-4} L \{ e^{2t} u(t-2) \} \end{aligned}$$

$$\begin{aligned} \text{Hence, } L \{ e^{2t} u(t-2) \} &= e^{+4} L \{ f(t-2) u(t-2) \} \\ &= e^{+4} \cdot e^{-2s} \frac{1}{s-2} \end{aligned}$$

$$\begin{aligned} \text{or, } L \{ e^{2t} \cdot u(t-2) \} &= e^4 L \{ e^{2(t-2)} \cdot u(t-2) \} \\ &= e^4 L \{ f(t-2) u(t-2) \} \\ &= e^4 e^{-2s} \cdot \frac{1}{s-2} \end{aligned}$$

Exercise: Evaluate the following

(i) $L \{ e^{-t} u(t-1) \}$ (ii) $L \{ g(t) \}$, where •

$$g(t) = \begin{cases} \cos(t - \pi/3) & ; t > \pi/3 \\ 0 & ; t < \pi/3 \end{cases}$$

Unit Impulse function (or, Dirac delta function)

The Unit impulse function is considered as the limiting form of the following function

$$f_{\epsilon}(t-a) = \begin{cases} \frac{1}{\epsilon} & \text{if } a \leq t \leq a+\epsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Now, } \delta(t-a) = \lim_{\epsilon \rightarrow 0} f_{\epsilon}(t-a)$$

$$L\{f_{\epsilon}(t-a)\} = \int_0^{\infty} e^{-st} f_{\epsilon}(t-a) dt$$

$$= \int_a^{a+\epsilon} e^{-st} \cdot \frac{1}{\epsilon} dt$$

$$= \frac{1}{\epsilon} \left(\frac{e^{-st}}{-s} \right)_a^{a+\epsilon}$$

$$= \frac{-1}{s\epsilon} \left[e^{-s(a+\epsilon)} - e^{-as} \right]$$

$$= \frac{e^{-as}(1 - e^{-s\epsilon})}{s\epsilon}$$

$$\text{So } L\{\delta(t-a)\} = L\left\{ \lim_{\epsilon \rightarrow 0} f_{\epsilon}(t-a) \right\}$$

$$= \lim_{\epsilon \rightarrow 0} L\{f_{\epsilon}(t-a)\}$$

$$= \lim_{\epsilon \rightarrow 0} \frac{e^{-as}(1 - e^{-s\epsilon})}{s\epsilon}$$

Evaluation of Integrals :

① Evaluate $\int_0^{\infty} e^{-t} \left(\frac{1-\cos t}{t} \right) dt$

Sol. we have $\int_0^{\infty} e^{-st} f(t) dt = L\{f(t)\}$

So, $\int_0^{\infty} e^{-st} \left(\frac{1-\cos t}{t} \right) dt = L\left\{ \frac{1-\cos t}{t} \right\} \quad *$

We know that $L\left\{ \frac{1-\cos t}{t} \right\} = \log\left(\frac{\sqrt{s^2+1}}{s}\right)$

Therefore, $\int_0^{\infty} e^{-st} \left(\frac{1-\cos t}{t} \right) dt = \log\left(\frac{\sqrt{s^2+1}}{s}\right)$

Taking $s=1$, we get

$$\int_0^{\infty} e^{-t} \left(\frac{1-\cos t}{t} \right) dt = \log \sqrt{2}.$$

② Evaluate $\int_0^{\infty} \left(\frac{e^{-t} - e^{-2t}}{t} \right) dt$

Sol. we have $\int_0^{\infty} e^{-st} f(t) dt = L\{f(t)\}$

So $\int_0^{\infty} e^{-st} \left(\frac{e^{-t} - e^{-2t}}{t} \right) dt = L\left\{ \frac{e^{-t} - e^{-2t}}{t} \right\} \quad *$

We know that $L\left\{ \frac{e^{-t} - e^{-2t}}{t} \right\} = \log\left(\frac{s+2}{s+1}\right)$

Therefore, $\int_0^{\infty} e^{-st} \left(\frac{e^{-t} - e^{-2t}}{t} \right) dt = \log\left(\frac{s+2}{s+1}\right)$

Taking $s=0$, we get $\int_0^{\infty} \left(\frac{e^{-t} - e^{-2t}}{t} \right) dt = \log 2.$

$$= e^{-as} \lim_{\epsilon \rightarrow 0} \frac{\epsilon e^{-s\epsilon}}{\epsilon} \quad (\text{L'Hospital's rule})$$

$$= e^{-as} \cdot 1.$$

Therefore $L\{\delta(t-a)\} = e^{-as}$.

We define $\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases}$

Such that $\int_0^{\infty} \delta(t-a) dt = 1 \quad (a > 0)$.

Note: we have $\int_0^{\infty} e^{-st} \delta(t-a) dt = L\{\delta(t-a)\}$
 $= e^{-as}$

Taking $s=0$, we get

$$\boxed{\int_0^{\infty} \delta(t-a) dt = 1}$$

③ Evaluate $\int_0^{\infty} e^{-2t} t \sin t \, dt$

Sol. We have $\int_0^{\infty} e^{-st} f(t) \, dt = L\{f(t)\}$

So, $\int_0^{\infty} e^{-st} t \sin t \, dt = L\{t \sin t\}$ ——— *

We know that $L\{t \sin t\} = \frac{2s}{(s^2+1)^2}$

Therefore, from (*), we have

$$\int_0^{\infty} e^{-st} t \sin t \, dt = \frac{2s}{(s^2+1)^2}$$

Taking $s=2$, we get

$$\int_0^{\infty} e^{-2t} t \sin t \, dt = \frac{4}{25}$$

Exercise : Evaluate the following

(i) $\int_0^{\infty} e^t + \cos t \, dt$

(ii) $\int_0^{\infty} e^{-2t} u(t-3) \, dt$

(iii) $\int_0^{\infty} e^{-t} t \sin t \cos t \, dt$

(iv) $\int_0^{\infty} e^{-3t} t^2 \delta(t-1) \, dt$

(v) $\int_0^{\infty} e^{-t} \left(\frac{e^t - \cos t}{t} \right) dt$