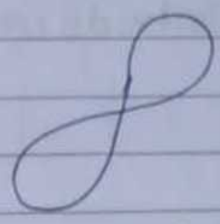
Simple closed curve:-

A closed curve, which does not intersect itself is called simple closed curve. (except initial & finial pt)

simple closed curve

Simple closed curve

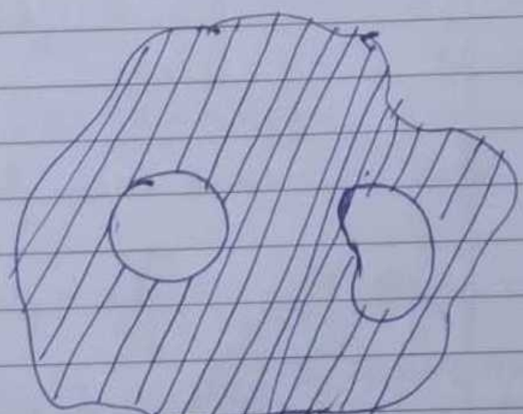


not Simple closed curve

Simply connected and Multiply connected Regions:-

Simply connected Region:- A region R is called simple connected region, if any simple closed curve which lies in R, can be shrunk to a point without leaving R. A region which is not simply connected is called **multiply connected.**

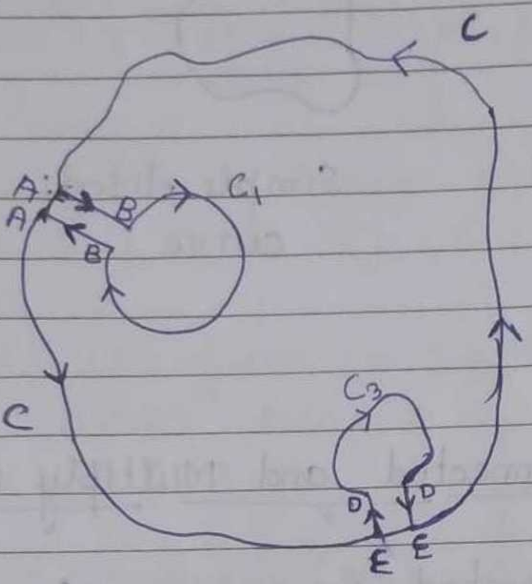
Simply connected region



multiply connected region

Obviously, a simple connected region, does not have any holes in it.

A multiply connected region can be converted in to simply connected regions by introducing cross cuts.



multiply connected ~~cor~~ region converted into simple connected region, by gives cross cuts.

Cauchy Integral theorem or Cauchy Fundamental theorem

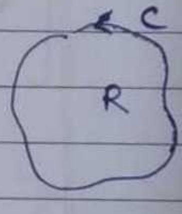
If $f(z)$ is analytic and $f'(z)$ is continuous at all points on and inside a simple closed curve C then $\boxed{\oint_C f(z)\,dz = 0}$

French mathematician E. Goursat shows that above result can be proved without assuming $f'(z)$ is continuous. i.e. Modified form of above theorem is

Cauchy Goursat theorem:-

If $f(z)$ is analytic inside and on a simple closed curve c then

$$\boxed{\oint_C f(z)\,dz = 0}$$



Proof:- Let $f(z) = u + iv$

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

as u_x, v_x, u_y, v_y are continuous in R.
by Green's theorem

$$\oint f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \text{—— (1)}$$

as $f(z)$ is analytic, $u_x = v_y$, $u_y = -v_x$

from (1)

$$\boxed{\oint_C f(z)\,dz = 0}$$

by Green's theo
$$\int M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Theorem :-

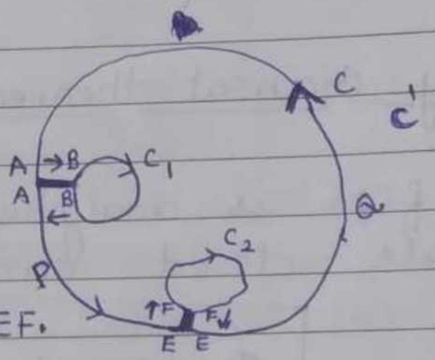
Extension of Cauchy's Integral ~~Formula~~

If $f(z)$ is analytic inside and on a multiply connected region, whose outer boundary in c and inner boundarys are $c_1, c_2, c_3, \ldots c_n$, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

where all the integrals are taken in the same sense (anticlock wise).

Proof: (doubly) Given region is multiple connected, ~~by~~ which we convert into simple connected region by introducing the cross cut AB and EF.



by Cauchy's Integral ~~formula~~ Theorem :-

$$\int_{C'} f(z)\,dz = 0 \quad , \text{ where } c' \text{ includes}$$

c described in anticlockwise
c_1 described in clockwise
and c_2 described in clockwise

$$\int_{C'} f(z)\,dz = 0$$

$$\Rightarrow \int_{C} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{EF} f(z)\,dz$$

$$+\int_{C_2 \downarrow} f(z)\,dz + \int_{FE} f(z)\,dz = 0 \quad \text{--- (1)}$$

Here $\int_{BA} f(z)\,dz = -\int_{AB} f(z)\,dz$

$$\int_{FE} f(z)\,dz = -\int_{EF} f(z)\,dz$$

i.e from (1)

$$\int_{C \uparrow} f(z)\,dz + \int_{C_1 \downarrow} f(z)\,dz + \int_{C_2 \downarrow} f(z)\,dz = 0$$

$$\int_{C \uparrow} f(z)\,dz = -\int_{C_1 \downarrow} f(z)\,dz - \int_{C_2 \downarrow} f(z)\,dz$$

$$\left[\int_{C \uparrow} f(z)\,dz = \int_{C_1 \uparrow} f(z)\,dz + \int_{C_2 \uparrow} f(z)\,dz\right]$$

or

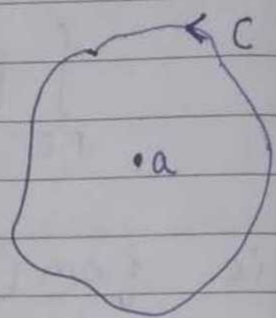
$$\int_{EQA} f(z)dz + \int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{BA} f(z)dz + \int_{APE} f(z)dz + \int_{EF} f(z)dz + \int_{C_2} f(z)dz + \int_{FE} f(z)dz = 0$$

$$\Rightarrow \int_{C \uparrow} f(z)\,dz + \int_{C_1 \downarrow} f(z)\,dz + \int_{C_2 \downarrow} f(z)\,dz = 0$$

Cauchy's Integral Formula:-

If $f(z)$ is analytic inside and on a simple closed curve c and a is any point inside c then

$$\boxed{f(a) = \frac{1}{2\pi i} \oint_{c} \frac{f(z)}{z-a}\, dz} \quad \text{—(1)}$$

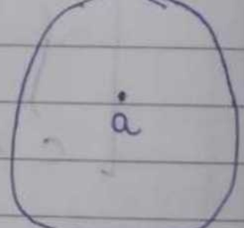


or $$\oint \frac{f(z)}{z-a}\, dz = 2\pi i \times f(a)$$

Cauchy Integral formula for derivative of analytic function:-

If $f(z)$ is analytic inside and on a simple closed curve c, and a is any point inside c then its derivative at $z = a$ can be obtain by differentiating ① w. r. to a

$$f'(a) = \frac{1}{2\pi i} \oint_{c} \frac{\partial}{\partial a}\left(\frac{f(z)}{z-a}\right) dz$$



$$\boxed{f'(a) = \frac{1}{2\pi i} \oint_{c} \frac{f(z)}{(z-a)^2}\, dz} \quad \text{—②}$$

or $$\oint_{c} \frac{f(z)}{(z-a)^2}\, dz = 2\pi i \times f'(a)$$

Similarly

$$\boxed{f''(a) = \frac{2!}{2\pi i} \oint_c \frac{f(z)}{(z-a)^3}\, dz}$$

⋮

$$\boxed{f^{(n)}(a) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}}\, dz} \quad \text{--- (3)}$$

from (3)

i.e. $$\left[\oint \frac{f(z)}{(z-a)^{n+1}}\, dz = 2\pi i \times \frac{f^{(n)}(a)}{n!} \right]$$

$n = 1, 2, 3, \ldots$

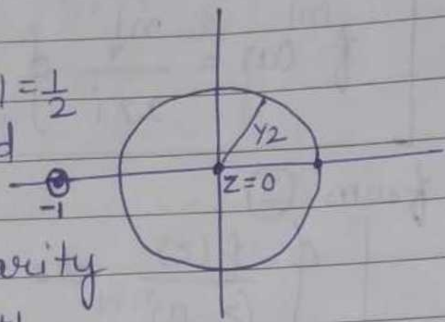
Q: Find the integral $\oint_C \frac{3z^2+7z+1}{z+1} dz$ where

c is circle $|z| = 1/2$.

Soln Poles of integrand are

$$z = -1$$

The given circle $|z| = \frac{1}{2}$ with center $z=0$ and radius $\frac{1}{2}$, does not enclose any singularity of $f(z) = \frac{3z^2+7z+1}{z+1}$



therefor by cauchy Goursat theorem

$$\oint_C f(z)\, dz = 0$$

$$\boxed{\oint_C \frac{3z^2+7z+1}{z+1} dz = 0}$$

(2) $\oint_C (x^2-y^2+2ixy)\, dz$ where $c: |z|=1$

Soln $f(z) = x^2-y^2+2ixy$ is analytic everywhere within the circle $|z|=1$ therefore by cauchy integral theorem

$$\oint_C f(z)\, dz = 0$$

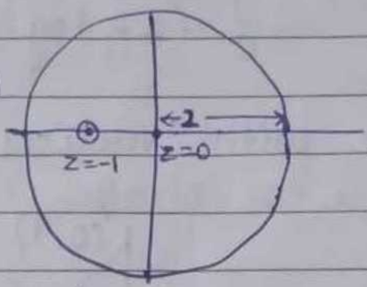
$$\oint_C (x^2-y^2+2ixy)\, dz = 0$$

(3) Evaluate $\oint_C \frac{e^{-z}}{z+1}\,dz$, where c is circle

(a) $|z| = 2$ (b) $|z| = \frac{1}{2}$

(a) $|z|=2$

circle with center 0 and radius 2



$f(z) = e^{-z}$ is analytic function & $a=-1$ point lies inside the circle $|z|=2$

By cauchy's Integral formula

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$

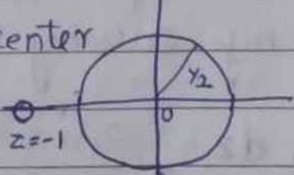
$a=-1$

$$\oint \frac{f(z)}{z-(-1)}\,dz = 2\pi i \times f(-1)$$

$$\oint \frac{f(z)}{z+1}\,dz = 2\pi i\,(e^{-z})_{z=-1}$$

$$\left[\oint \frac{e^{-z}}{z+1}\,dz = 2\pi i e\right]$$ Ans

(b) $|z|=\frac{1}{2}$, circle with center 0 and radius $\frac{1}{2}$



Now $f(z) = \frac{e^{-z}}{z+1}$ is analytic inside and on circle $|z| = 1/2$

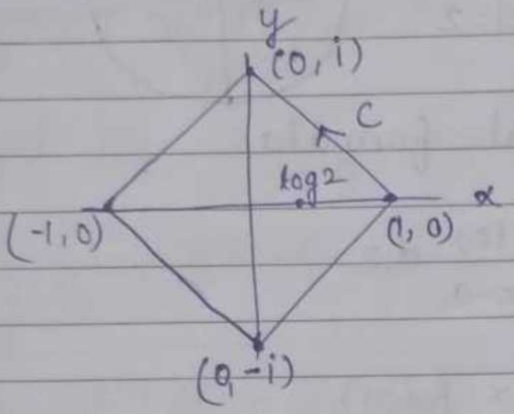
therefor by cauchy's Integral theorem

$$\boxed{\oint_C \frac{e^{-z}}{z+1}\,dz = 0}$$

$C : |z| = \frac{1}{2}$

Q:- Evaluate $\int_C \frac{e^{3z}}{(z-\log 2)^4}\,dz$, where C is the square with vertices at $\pm 1, \pm i$

Solⁿ



$\log 2 = \cdot 301$

poles of integrand are

$$(z - \log 2)^4 = 0$$

$z = \log 2$ is a pole of order 4.

Let $f(z) = e^{3z}$, which is analytic inside and on the ~~con~~ simple closed contoure C & $z = \log 2$ lies inside C. therefore ~~it~~ by cauchy Integral formula

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i \times f^{(n)}(a)}{n!}$$

put $n=3$, $a=\log 2$

$$\oint \frac{e^{3z}}{(z-\log 2)^4}\,dz = \frac{2\pi i\, f^{(3)}(a)}{3!}$$

$$= \frac{2\pi i}{6}\left(\frac{d^3}{dz^3}\cdot e^{3z}\right)_{z=a}$$

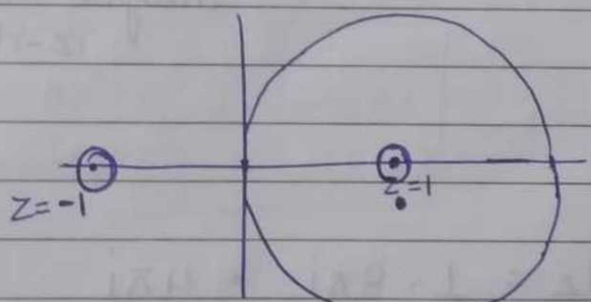
$$= \frac{2\pi i}{6}\left(27\, e^{3z}\right)_{z=\log 2}$$

$$= 9\pi i\; e^{3\log 2} = 9\pi i\,(2)^3 = 72\pi i$$

$$\boxed{\oint \frac{e^{3z}}{(z-\log 2)^4} = 72\pi i}$$

Q:- Evaluate $\oint_c \frac{3z^2+z}{z^2-1}\,dz$, where c is the circle $|z-1|=1$.

Soln



integrand is not analytic at

$$z^2-1=0 \Rightarrow z=\pm 1$$

ie $z=1$ and $z=-1$ are poles of order 1.

Method I

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)}$$

$$\frac{1}{z^2+1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$$\oint_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\oint_C \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\oint_C \frac{3z^2+z}{z+1}\,dz \quad \text{—} ①$$

$z=1$ lies inside the $|z-1|=1$ therefor by cauchy's Integral formula

$$\oint_C \frac{3z^2+z}{z-1}\,dz = 2\pi i\, f(1), \text{ where } f(z)=3z^2+z$$

$$= 2\pi i\,(3+1) = 8\pi i$$

Now by cauchy's Integral Theorem

$$\oint_C \frac{3z^2+z}{z+1}\,dz = 0,$$ as integrand is analytic inside $|z-1|=1$

from ①

$$\oint_C \frac{3z^2+z}{(z^2-1)}\,dz = \frac{1}{2}\times 8\pi i = 4\pi i$$

$$\boxed{\oint_C \frac{3z^2+z}{z^2-1}\,dz = 4\pi i}$$

Method 2

$$\oint_C \frac{3z^2+z}{z^2-1}\,dz = \oint \frac{\left(\frac{3z^2+z}{z+1}\right)}{z-1}\,dz$$

$f(z) = \frac{3z^2+z}{z+1}$ is analytic inside and on

$|z-1|=1$ and $z=1$ lies inside c

there for by cauchy's integral formula

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(z)\Big|_{z=a}$$

here $a=1$

$$\oint_C \frac{\frac{3z^2+z}{z+1}}{z-1}\,dz = 2\pi i \left(\frac{3z^2+z}{z+1}\right)_{z=1}$$

$$= 2\pi i \left(\frac{4}{2}\right) = 4\pi i$$

$$\boxed{\oint_C \frac{3z^2+z}{(z-1)^2}\,dz = 4\pi i}$$

Q Evaluate $\oint_C \frac{z+1}{z^2+2z+4} dz$, $c: |z+1+i| = 2$

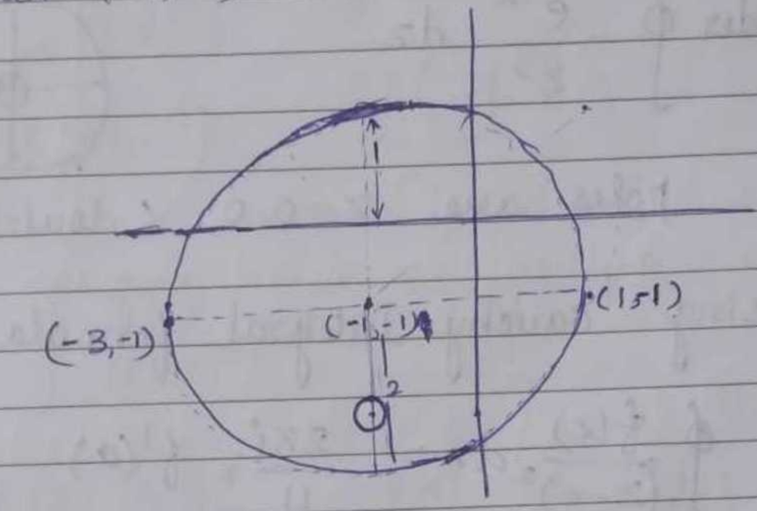
Sol^n integrand has pole at

$$z^2 + 2z + 4 = 0$$

$$z = -1 \pm \sqrt{3}i$$

$z = -1+\sqrt{3}i$, $z = -1-\sqrt{3}i$ are poles of order 1 (simple pole).

$c: |z+1+i| = 2$ is a circle with radius 2 and center $(-1,-1)$



clearly $-1+\sqrt{3}i$ lies outside the circle therefore

$$\oint_C \frac{z+1}{z^2+2z+4} dz = \oint \frac{\frac{z+1}{z-(-1+\sqrt{3}i)}}{z-(-1-\sqrt{3}i)} dz$$

Using cauchy's Integral formula $a = -1-\sqrt{3}i$

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i \, f(a)$$

here $f(z) = \frac{z+1}{z-(-1+\sqrt{3}i)}$

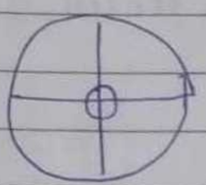
ie $\oint \frac{z+1}{z^2+2z+4} dz = 2\pi i \times \left(\frac{z+1}{z-(-1+\sqrt{3}i)}\right)_{(-1-\sqrt{3}i)}$

$$= 2\pi i \left(\frac{-i\sqrt{3}}{-2i\sqrt{3}} \right)$$

$$\boxed{\oint \frac{z+1}{z^2+2z+4}\, dz = \pi i}$$

Q) Evaluate $\oint \frac{dz}{z^2 e^z} dz \qquad c: |z|=1$

Sol^n consider $\oint \frac{e^{-z}}{z^2}\, dz$



poles are $z = 0, 0$ (double pole)

Using cauchy Integral formula

$$\oint \frac{f(z)}{(z-a)^2}\, dz = \frac{2\pi i}{1!} \times f'(a)$$

here $f(z) = e^{-z}$, $a = 0$

$f'(z) = -e^{-z}$

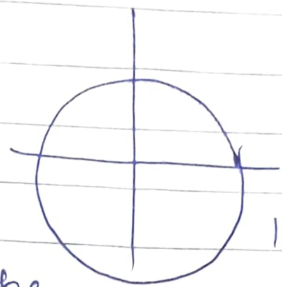
$f'(a) = -1$

$$\oint \frac{e^{-z}}{z^2}\, dz = 2\pi i \times -1$$

$$= -2\pi i \quad \text{Ans}$$

Q:- Evaluate $\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz \qquad C: |z|=1$

Solⁿ:- integrand has pole of order 3 at $z=\pi/6$



$|z|=1$

$\pi/6 = .52$

clearly $\pi/6$ lies inside the circle

using cauchy's Integral formula for derivative

$$\oint \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$n=2$

$$\oint \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a)$$

i.e

$$\oint \frac{f(z)}{(z-\pi/6)^3} dz = \frac{2\pi i}{2!} f''(\pi/6) \quad \text{—— (1)}$$

$f(z) = \sin^6 z, \quad f'(z) = 6\sin^5 z \cos z$

$f''(z) = 6(-\sin^6 z + \cos^2 z \cdot 5\sin^4 z)$

$$f''(\pi/6) = 6\left[-\left(\frac{1}{2}\right)^6 + \left(\frac{\sqrt{3}}{2}\right)^2 \cdot 5\left(\frac{1}{2}\right)^4\right].$$

$$= 6\left[-\frac{1}{64} + \frac{15}{64}\right] = 6 \times \frac{14}{64} = \frac{21}{16}$$

from (1)

$$\oint \frac{\sin^6 z}{(z-\pi/6)^3} dz = \pi i \times \frac{21}{16} = \frac{21\pi i}{16}$$

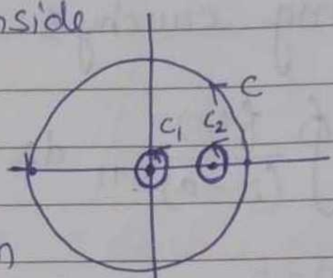
Q:- Evaluate $\oint \frac{4-3z}{z(z-1)(z-2)}\, dz$, using cauchy's Integral formula. $c : |z| = 3/2$

Sol^n method 3:-

Poles are
$z = 0, 1, 2$, all are simple pole.

$c : |z| = 3/2$

$z=0$ and $z=1$ lies inside c.



Using cauchy integral theorem for multiply connected domain

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_{C_1} \frac{4-3z}{z(z-1)(z-2)}\,dz + \int_{C_2} \frac{4-3z}{z(z-1)(z-2)}\,dz$$

$$= \int_{C_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{z}\,dz + \int_{C_2} \frac{\frac{4-3z}{z(z-2)}}{z-1}\,dz$$

using cauchy's Integral formula

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i \times \left(\frac{4-3z}{(z-1)(z-2)}\right)_{z=0}$$

$$+\,2\pi i\left(\frac{4-3z}{z(z-2)}\right)_{z=1}$$

$$= 2\pi i \times 2 + 2\pi i(-1) = 2\pi i$$

$$\boxed{\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i}$$

Q $\displaystyle\oint_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz, \quad C: |z|=10$

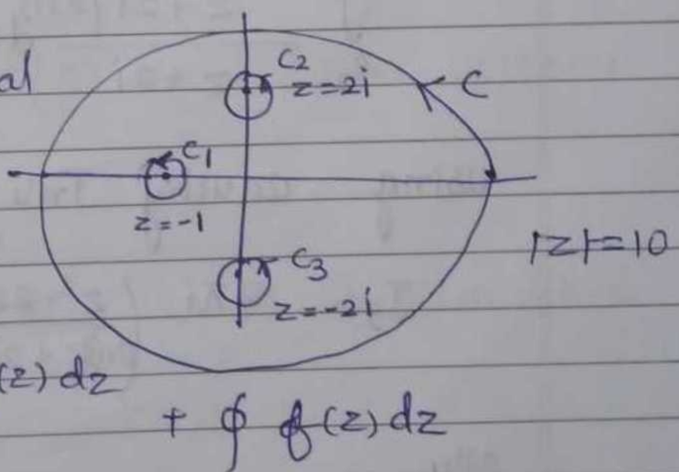
Hint:- poles $z=-1$ (order 2)

$z = 2i$ (simple pole)

$z = -2i$ (simple pole)

all lies inside the circle $|z|=10$

- Using cauchy's Integral theorem for multiply connected domain



$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz$$

$$\oint_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz = \oint_{C_1} \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz + \oint_{C_2} \frac{z^2-2z}{(z+1)^2(z+4)}\,dz + \oint_{C_3} \frac{z^2-2z}{(z+1)^2(z+4)}\,dz$$

$$I_1 + I_2 + I_3$$

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$$I_1 = \int_{C_1} \frac{z^2-2z}{(z+1)^2(z^2+4)} dz$$

$$= \oint \frac{\left(\frac{z^2-2z}{z^2+4}\right)}{(z+1)^2} dz$$

using cauchy's integral ~~the~~ formula for derivatives

$I_1 = \frac{2\pi i}{1!} f'(-1)$, where $f(z) = \frac{z^2-2z}{(z+1)^2}$

$$f'(z)\Big|_{z=-1} = -\frac{14}{25}$$

$$I_1 = 2\pi i\left(-\frac{14}{25}\right) = \frac{-28\pi i}{25}$$

$$I_2 = \oint_{C_2} \frac{\frac{z^2-2z}{z+2i\,(z+1)^2}}{z-2i} dz$$

using cauchy Integral formula

$$I_2 = 2\pi i \left(\frac{z^2-2z}{(z+1)^2(z+2i)}\right)_{z=2i} = \frac{2\pi i(1+i)}{4+3i}$$

slly

$$I_3 = 2\pi i\left(\frac{i-1}{3i-4}\right)$$

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$I = I_1 + I_2 + I_3$

$$I = -\frac{28\pi i}{25} + 2\pi i\left(\frac{1+i}{4+3i}\right) + 2\pi i\left(\frac{i-1}{3i-4}\right)$$

$$\boxed{I = 0}$$

Q Evaluate $\oint_c \frac{z-3}{z^2+2z+5}\,dz$, where c is circle

(i) $|z|=1$ (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$

Q Using cauchy's Integral formula $\frac{1}{2\pi i}\oint_c \frac{e^{az}}{z^2+1}\,dz$

$a>0$, $C: |z|=2$.

Ans :- $\sin \alpha$

Q Evaluate $\oint_c (z+1)\cot\left(\frac{z}{2}\right) dz$, where $c: |z|=1$

using residue theorem.

Q. Evaluate $\oint_c \frac{z \cdot \cosh \pi z}{z^4+5z^2+4}$, using residue theo.

where $c: |z|=4$.

Q. Evaluate $\oint \frac{\cot z}{z-\pi/2}\,dz$, ~~using~~ where

c is square joining point $(2,2)$, $(-2,2)$

$(-2,-2)$ & $(2,-2)$

Ans $-4i$

Q. Using cauchy Integral

Q:- Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, where c is circle $|z|=3$.

Ans $4\pi i$

- Evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, $c: |z|=3$

Ans:- $4\pi i$

- evaluate $\int_C \frac{z^3 - z}{(z-2)^3} dz$, $c: |z|=3$

Ans $12\pi i$

using
$$\left[\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a) \right]$$

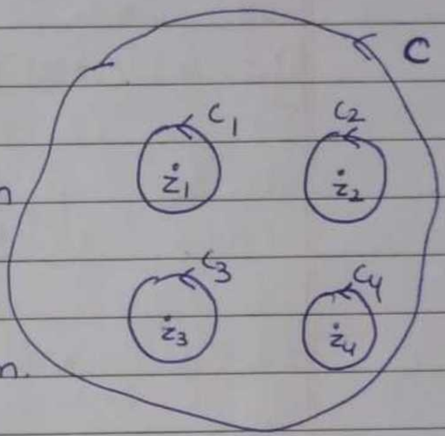
evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, $c: |z|=2$

Ans $\frac{8\pi i}{3} e^{-2}$

Cauchy's Residue theorem:-

If $f(z)$ is analytic inside and on a close curve c, except finite number of poles ~~inside~~ $z_1, z_2, z_3, \ldots z_n$ within c then

$$\oint_C f(z)\,dz = 2\pi i \times \text{sum of residue at the poles } z_1, z_2, \ldots z_n$$



Proof:- $f(z)$ is analytic inside and on c except $z_1, z_2, z_3, z_4, \ldots z_n$ then ~~by Cauchy's Inte~~ around each of the singularities draw a small circle $c_1, c_2, c_3, c_4, \ldots c_n$.

Using Cauchy's Integral theorem for multiply connected domain

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \cdots$$

Now

we know that

$$\text{Res of } f(z) \text{ at } (z = z_1) = \frac{1}{2\pi i}\oint_{C_1} f(z)\,dz$$

i.e. $$\oint_{C_1} f(z)\,dz = \text{Res}(z = z_1) \times 2\pi i$$

Similarly $$\oint_{C_2} f(z)\,dz = \text{Res}(z = z_2) \times 2\pi i$$

$\vdots$

$$\int_C f(z)\,dz = 2\pi i \times Res(z=z_1) + 2\pi i \times Res(z=z_2) + \cdots 2\pi i \times Res(z=z_n)$$

$$= 2\pi i \left(Res(z=z_1) + Res(z=z_2) + \cdots + Res(z=z_n) \right)$$

$$\left[\oint_C f(z)\,dz = 2\pi i \times \text{Sum of residues at pole} \right].$$

Residue at pole of order m

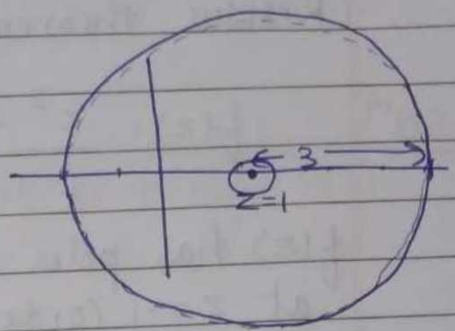
$$\text{Res}(z=a) = \frac{1}{(m-1)!} \lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]\right]$$

Evaluation of integrals using cauchy's Residue theorem

Q1 evaluate $\oint_C \frac{e^z}{(z+1)^2}\,dz$, around $|z-1|=3$

Sol^n $f(z) = \frac{e^z}{(z+1)^2}$

$z=-1$ is a pole of order 2.



$$\text{Residue at } (z=-1) = \frac{1}{1!} \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 f(z)\right]$$

$$= \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \cdot \frac{e^z}{(z+1)^2}\right]$$

$$= \lim_{z\to -1} \frac{d}{dz}(e^z) = \lim_{z\to -1} e^z = e^{-1}$$

Using Residue theorem :-

$$\oint_C f(z)\,dz = 2\pi i \times \text{Res}(z=-1)$$

$$= \frac{2\pi i}{e}$$

Q Evaluate $\oint_C \frac{2z-1}{z(z+1)(z+3)}\,dz$, $C: |z|=2$, by cauchy residue theorem

Ans $-\frac{5\pi i}{6}$

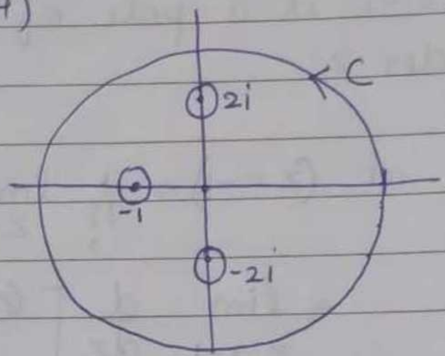
Hint: $\int_C f(z)\,dz = 2\pi i\left(\text{Res}_{z=0} f(z) + \text{Res}_{z=-1} f(z)\right)$

$$= 2\pi i\left(\frac{1}{3} - \frac{3}{4}\right)$$

Q- Evaluate $\int_C \dfrac{z^2-2z}{(z+1)^2(z^2+4)}\,dz$, $C : |z|=3$, using Residue theorem.

Sol^n. $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$

$f(z)$ has poles
at $z=-1$ (order 2)
at $z=2i$ (simple pole)
$z=-2i$ (simple pole)
all lies inside the circle $|z|=3$



using Residue theo.

$$\oint_C f(z)\,dz = 2\pi i\,(\text{sum of residue at poles})$$

$$= 2\pi i\,\big(\text{Res}(z=-1) + \text{Res}(z=2i) + \text{Res}(z=-2i)\big)$$

(a) Residue at $z=-1$:-

$$\text{Res}(z=-1) = \frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}\left((z+1)^2 \cdot \frac{z^2-2z}{(z+1)^2(z^2+4)}\right)$$

$$= \lim_{z\to -1}\frac{d}{dz}\left(\frac{z^2-2z}{z^2+4}\right)$$

$$\text{Res}(z=-1) = \frac{-14}{25} \quad \text{——— (1)}$$

(b) Residue at $z = 2i$

$$= \lim_{z \to 2i} (z - 2i) f(z)$$

$$= \lim_{z \to 2i} \cancel{(z - 2i)} \left(\frac{z^2 - 2z}{(z+1)^2 (z+2i) \cancel{(z-2i)}} \right)$$

$$= \frac{i - 1}{(2i+1)^2}$$

(c) Residue at $(z = -2i)$

$$= \lim_{z \to -2} (z + 2i) f(z)$$

$$= \frac{-(1+i)}{(1-2i)^2}$$

$$\int_C \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} dz = 2\pi i \,(\text{sum of residues})$$

$$= 2\pi i \left(\frac{-14}{25} + \frac{i - 1}{(2i+1)^2} - \frac{1+i}{(1-2i)^2} \right)$$

Q Evaluate $\int_C \frac{dz}{(z^2+4)^2}$, $c : |z - i| = 2$, using Residue theo

Ans $\frac{1}{32i}$

$z = 2i$, $z = -2i$ are poles of order 2.

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Q:- Evaluate $\oint_C \frac{1}{z\sin z}\,dz$, c: unit circle about origin.

Soln:- c: $|z|=1$

poles are obtained by

$$z\sin z=0$$

$$z=0,\quad \sin z=0$$
$$z=n\pi,\quad n=0,\pm1,\pm2,--$$

$\Rightarrow$ $z=0,0,\quad z=n\pi,\quad n=\pm1,\pm2,--$

$z=0$ is pole of order 2.
$z=n\pi,\quad n=\pm1,\pm2,--$ all are poles of order 1.
* only $z=0$ lies inside circle $|z|=1$

$$\underset{(z=0)}{\text{Res}}\, f(z) = \frac{1}{1!}\lim_{z\to0}\frac{d}{dz}\left(z^{2}\cdot\frac{1}{\cancel{z}\sin z}\right)$$

$$=\lim_{z\to0}\cdot\frac{d}{dz}\left(\frac{z}{\sin z}\right)$$

$$=\lim_{z\to0}\frac{z\cdot\cos z-\sin z}{(\sin z)^{2}}$$

$$=\lim_{z\to0}\frac{-z\sin z+\cancel{\cos z}-\cancel{\cos z}}{2\sin z\cos z}$$

$$=\lim_{z\to0}\frac{-z}{2\cos z}$$

$$\underset{z=0}{\text{Res}}\, f(z) = 0.$$

i.e $\int_C \frac{1}{z\sin z}\,dz = 2\pi i \times 0 = 0$

$$\boxed{\oint_C \frac{1}{z\sin z}\,dz = 0}$$

Q evaluate $\oint_C \frac{e^z}{\cos \pi z}\,dz$, $c : |z| = 1$ by Residue theorem.

Soln Poles are obtained by putting

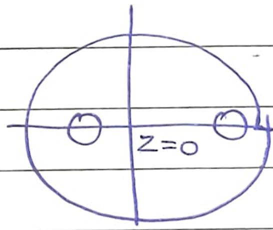
$\cos \pi z = 0$

$\cos \pi z = \cos (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2$

$\pi z = (2n+1)\frac{\pi}{2}$

$\left[z = \frac{(2n+1)}{2} \right]$, $n = 0, \pm 1, \pm 2, \ldots$

poles $z = \frac{1}{2}, \frac{-1}{2}$ lies inside the circle $|z| = 1$ and both are simple poles



$\mathrm{Res}(z = \frac{1}{2}) = \lim_{z \to \frac{1}{2}} (z - \frac{1}{2}) \cdot \frac{e^z}{\cos \pi z}$ $\left(\frac{0}{0}\right)$

$= \lim_{z \to \frac{1}{2}} \frac{(z - \frac{1}{2}) e^z + e^z}{-\pi \sin \pi z}$ $\left(\frac{0}{0}\right)$

$= \frac{e^{\frac{1}{2}}}{-\pi \sin \frac{\pi}{2}} = \frac{e^{\frac{1}{2}}}{-\pi}$

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$$\underset{z=-\frac{1}{2}}{\text{Res}}\, f(z) = \lim_{z\to -\frac{1}{2}} \left(z+\tfrac{1}{2}\right) \frac{e^z}{\cos \pi z} \quad \left(\tfrac{0}{0}\right) \text{form}$$

$$= \lim_{z\to -\frac{1}{2}} \frac{e^z + \left(z+\frac{1}{2}\right)e^z}{-\pi \sin \pi z}$$

$$= \frac{e^{-\frac{1}{2}}}{\pi}$$

$$\oint_C \frac{e^z}{\cos \pi z}\, dz = 2\pi i \,(\text{sum of residues at poles})$$

$$= 2\cancel{\pi} i \left(\frac{e^{1/2} - e^{-1/2}}{\cancel{\pi}} \right)$$

$$= 2\times 2 i \left(\frac{e^{1/2} - e^{-1/2}}{2} \right)$$

$$\boxed{\oint_C \frac{e^z}{\cos \pi z}\, dz = 4 i \sinh \tfrac{1}{2}}$$

Q:- Evaluate $\oint_C \frac{1}{\sinh z}\, dz$, $c : |z| = 4$

poles are

$\sinh z = 0$

$\sinh z = -i \sin iz = 0 \Rightarrow \sin iz = 0$

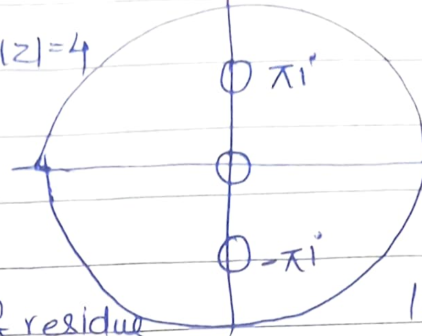
$\cancel{-i \sin iz = \sin n\pi}$

$\sin iz = \sin n\pi$

$iz = n\pi$

$z = -n\pi i, \quad n = 0, \pm 1, \pm 2, \ldots$

$z = 0, \pi i, -\pi i$ lies inside $|z|=4$



$|z|=4$

i.e

$$\oint_C \frac{1}{\sinh z}\, dz = 2\pi i \times (\text{Sum of residue at poles})$$

(a) $$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} z \cdot \frac{1}{\sinh z} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{z\to 0} \frac{1}{\cosh z} = 1$$

$$\left[\text{or } \operatorname*{Res}_{z=a} f(z) = \frac{\phi(a)}{\psi'(a)}\right]$$

(b) $$\operatorname*{Res}_{z=\pi i} f(z) = \frac{1}{\cosh(\pi i)} = \frac{1}{\cos i(\pi i)} = \frac{1}{\cos(-\pi)} = -1$$

(c) $$\operatorname*{Res}_{z=-\pi i} f(z) = \frac{1}{\cosh(-\pi i)} = -1$$

$$\oint \frac{1}{\sinh z}\, dz = 2\pi i(1-1-1) = -2\pi i$$

$$\boxed{\oint \frac{1}{\sinh z}\, dz = -2\pi i}$$

Q $\oint_C \frac{\sin z}{z^6}\, dz$, $C: |z|=2$ by residue theo.

Ans $\frac{\pi i}{60}$

Q $\oint_C \frac{z^2 e^{zt}}{z^2+1}\, dz$, $C: |z|=2$ Ans $-2\pi i \sin t$

evaluate

Q $\oint_C \frac{1}{e^{2z} - e^{z}} \, dz$ around $|z| = 1025\pi$ oriented counterclockwise.