(ii) Since from equation (1), any one vector can be expressed as the linear combination of the remaining two, any two of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ will form a basis for V.

Example 9: Let V be the space spanned by $\mathbf{v}_1 = \sin x$, $\mathbf{v}_2 = \cos x$, $\mathbf{v}_3 = x$. Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for V.

Solution: It is given that S spans V. To prove S linearly independent, we need to show that the Wronskian, W of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is non-zero.

$$W = \begin{vmatrix} v_1 & v_2 & v_3 \\ v_1' & v_2' & v_3' \\ v_1'' & v_2'' & v_3'' \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x & x \\ \cos x & -\sin x & 1 \\ -\sin x & -\cos x & 0 \end{vmatrix}$$

$$= \sin x(\cos x) - \cos x(\sin x) + x(-\cos^2 x - \sin^2 x)$$

$$= -x$$

This function is not zero for all values of x. This shows that S is linearly independent. Hence, S forms a basis for V.

Basis for the Subspace Span (S)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set in a vector space V then S is a basis for the subspace span (S).

2.9 FINITE DIMENSIONAL VECTOR SPACE

A vector space V is called finite dimensional if the number of vectors in its basis are finite. Otherwise, V is called infinite dimensional.

Theorem 2.11: If basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of a finite dimensional vector space V has n vectors then

(i) Every set in V having more than n vectors is linearly dependent
(ii) Every set in V having less than n vectors does not span V

Theorem 2.12: From the above theorem, we conclude that all the bases for a finite-dimensional vector space have the same number of vectors.

2.9.1 *Dimension*

The number of vectors in a basis of a non-zero finite dimensional vector space V is known as the dimension of V and is denoted by dim (V).

Note: Dimensions of some standard vector spaces can be found directly from their standard basis.

(i) dim $(R^n) = n$
(ii) dim $(P_n) = n + 1$
(iii) dim $(M_{mn}) = mn$
(iv) dim $\{\mathbf{0}\} = 0$ [$\because$ **0** is linearly dependent, vector space $\{\mathbf{0}\}$ has no basis.]

Theorem 2.13: If dim $(V) = n$ and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set in V with exactly n vectors then S is a basis for V if either S is linearly independent or S spans V.

Theorem 2.14: Let S be a non-empty set of vectors in a vector space V.

(i) If S is a linearly independent set then $S \cup \{\mathbf{v}\}$ is also linearly independent if the vector **v** does not belong to the span (S).
(ii) If **v** is a vector in S that can be expressed as a linear combination of other vectors in S then

$$\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$$

Theorem 2.15: If W is a subspace of a finite dimensional vector space V then

(i) W is finite dimensional and dim $(W) \leq$ dim (V); if dim $(W) =$ dim (V) then $W = V$.
(ii) Every basis for W is part of a basis for V.

2.10 BASIS AND DIMENSION FOR SOLUTION SPACE OF THE HOMOGENEOUS SYSTEMS

Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system of m equations in n unknowns. The basis and dimension for the solution space of this system can be found as follows:

1. Solve the homogeneous system using Gaussian elimination method. If the system has only a trivial solution then the solution space is $\{\mathbf{0}\}$, which has no basis and hence the dimension of the solution space is zero.
2. If the solution vector **x** contains arbitrary constants (parameters) $t_1, t_2, \ldots, t_p$, express **x** as a linear combination of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ with $t_1, t_2, \ldots, t_p$ as coefficients.

$$\text{i.e. } \mathbf{x} = t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + \cdots + t_p\mathbf{x}_p$$

3. The set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p\}$ form a basis for the solution space of $A\mathbf{x} = \mathbf{0}$ and hence the dimension of the solution space is p.

Note: If the row echelon form has r non-zero rows then dimension of the solution space is $p = n - r$ where n represents the number of unknowns.

Example 1: Determine the dimension and a basis for the solution space of the system

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 0 \\ -2x_1 - 2x_2 + 4x_3 &= 0 \\ -x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & 1 & -2 & 0 \\ -2 & -2 & 4 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 + 2R_1,\ R_3 + R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 + x_2 - 2x_3 = 0$$

Solving for the leading variables,

$$x_1 = -x_2 + 2x_3$$

Assigning the free variables x_2 and x_3 arbitrary values t_1 and t_2 respectively, $x_1 = -t_1 + 2t_2$, $x_2 = t_1$, $x_3 = t_2$ is the solution of the system.

The solution vector is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t_1 + 2t_2 \\ t_1 \\ t_2 \end{bmatrix}$$

$$= t_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= t_1\mathbf{x}_1 + t_2\mathbf{x}_2$$

Hence, Basis = $\{\mathbf{x}_1, \mathbf{x}_2\} = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$

Dimension = 2

Example 2: Find the dimension and a basis for the solution space of the system

$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 0$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$\left(\frac{1}{3}\right)R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 - 5R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{8}{3} & -\frac{2}{3} & -\frac{8}{3} & 0 \end{array}\right]$$

$$\left(-\frac{3}{8}\right)R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 + \frac{1}{3}x_4 = 0$$

$$x_2 + \frac{1}{4}x_3 + x_4 = 0$$

Solving for the leading variables,

$$x_1 = -\frac{1}{3}x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4$$

$$x_2 = -\frac{1}{4}x_3 - x_4$$

Assigning the free variables x_3 and x_4 arbitrary values t_1 and t_2 respectively.

$$\begin{aligned} x_2 &= -\frac{1}{4}t_1 - t_2 \\ x_1 &= -\frac{1}{3}\left(-\frac{1}{4}t_1 - t_2\right) - \frac{1}{3}t_1 - \frac{1}{3}t_2 \\ &= -\frac{1}{4}t_1 \end{aligned}$$

Hence, $x_1 = -\frac{1}{4}t_1, x_2 = -\frac{1}{4}t_1 - t_2, x_3 = t_1, x_4 = t_2$ is the solution of the system.

The solution vector is

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{4}t_1 \\ -\frac{1}{4}t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} \\ &= t_1 \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \\ &= t_1\mathbf{x}_1 + t_2\mathbf{x}_2 \end{aligned}$$

Hence,

$$\begin{aligned} \text{Basis} &= \{\mathbf{x}_1, \mathbf{x}_2\} \\ &= \left\{ \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \\ \text{Dimension} &= 2 \end{aligned}$$

Example 3: Find the dimension and a basis for the solution space of the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 0 \\ x_1 - x_2 + 2x_3 &= 0 \end{aligned}$$

Solution: The matrix form of the system is

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & 2 & -3 & 0 \\ 2 & 5 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right]$$

Reducing the augmented matrix to row echelon form,

$$R_2 - 2R_1,\ R_3 - R_1$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & -3 & 5 & 0 \end{array}\right]$$

$$R_3 + 3R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 26 & 0 \end{array}\right]$$

$$\left(\frac{1}{26}\right) R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_2 + 7x_3 &= 0 \\ x_3 &= 0 \end{aligned}$$

Hence, $x_1 = 0, x_2 = 0, x_3 = 0$ is the solution of the system.
The solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \{\mathbf{0}\}$$

Hence, the solution space has no basis and dimension = 0.

Example 4: Determine the dimension and basis for the following subspaces of R^3 and R^4.

(i) the plane $3x - 2y + 5z = 0$
(ii) the line $x = 2t$, $y = -t$, $z = 4t$
(iii) all vectors of the form (a, b, c, d) where $d = a + b$ and $c = a - b$

Solution: (i) $3x - 2y + 5z = 0$
Solving for x,

$$x = \frac{2}{3}y - \frac{5}{3}z$$

Assigning y and z arbitrary values t_1 and t_2 respectively,

$$x = \frac{2}{3}t_1 - \frac{5}{3}t_2$$

Any vector $\mathbf{x}$ lying on the plane is

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{2}{3}t_1 - \frac{5}{3}t_2 \\ t_1 \\ t_2 \end{bmatrix}$$

$$= t_1\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$= t_1\mathbf{x}_1 + t_2\mathbf{x}_2$$

Thus, $\mathbf{x}_1$ and $\mathbf{x}_2$ span the given plane. Also, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent as they are not scalar multiples of each other.

Hence, $$\text{Basis} = \left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Dimension} = 2$$

(ii) Any vector $\mathbf{x}$ lying on the line $x = 2t$, $y = -t$, $z = 4t$ is

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ 4t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = t\mathbf{x}_1$$

Thus, $\mathbf{x}_1$ spans the given line and is also linearly independent as it is a non-zero vector.

$$\text{Hence, Basis} = \{\mathbf{x}_1\} = \left\{\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}\right\}$$

Dimension = 1

(iii) Let

$$\mathbf{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ a-b \\ a+b \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= a\mathbf{x}_1 + b\mathbf{x}_2$$

Thus, $\mathbf{x}_1$ and $\mathbf{x}_2$ span the given set of vectors. Also, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent as one is not the scalar multiple of another.

Hence,

$$\text{Basis} = \{\mathbf{x}_1, \mathbf{x}_2\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

Dimension = 2

Example 5: Find a basis and dimension of

$$W = \{(a_1, a_2, a_3, a_4) \in R^4 \mid a_1 + a_2 = 0,\ a_2 + a_3 = 0,\ a_3 + a_4 = 0\}$$

Solution:

$$a_1 + a_2 = 0 \;\Rightarrow\; a_2 = -a_1$$
$$a_2 + a_3 = 0 \;\Rightarrow\; a_3 = -a_2 = a_1$$
$$a_3 + a_4 = 0 \;\Rightarrow\; a_4 = -a_3 = -a_1$$

Any vector $\mathbf{x}$ in W is

$$\mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_1 \\ a_1 \\ -a_1 \end{bmatrix}$$

$$= a_1\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= a_1\mathbf{x}_1$$

Thus, $\mathbf{x}_1$ spans W and is also linearly independent as it is a non-zero vector.

$$\text{Hence, Basis} = \{\mathbf{x}_1\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Dimension $= 1$

Example 6: Find the dimension and a basis for the following subspaces of P_2 and P_3.

(i) all polynomials of the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3$, where $a_0 = 0$

(ii) all polynomials of the form $ax^3 + bx^2 + cx + d$, where $b = 3a - 5d$ and $c = d + 4a$

Solution: (i) Let $\mathbf{p}$ be any polynomial in the given subspace of P_2.

$$\begin{aligned} \mathbf{p} &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ &= a_1 x + a_2 x^2 + a_3 x^3 \quad [\because a_0 = 0] \end{aligned}$$

Thus, the vectors x, x^2 and x^3 span the given subspace of P_2. Also, x, x^2 and x^3 are linearly independent which can be verified as follows:

Let $k_1 x + k_2 x^2 + k_3 x^3 = 0$

Equating corresponding coefficients,

$k_1 = k_2 = k_3 = 0.$

Thus, x, x^2 and x^3 are linearly independent.

Hence, Basis $= \{x, x^2, x^3\}$

Dimension $= 3$

(ii) Let $\mathbf{p}$ be any polynomial in the given subspace of P_3.

$$\begin{aligned} \mathbf{p} &= ax^3 + bx^2 + cx + d \\ &= ax^3 + (3a - 5d)x^2 + (d + 4a)x + d \\ &= a(x^3 + 3x^2 + 4x) + d(-5x^2 + x + 1) \\ &= a\mathbf{p}_1 + d\mathbf{p}_2 \end{aligned}$$

Thus, $\mathbf{p}_1$ and $\mathbf{p}_2$ span the given subspace of P_3. Also, $\mathbf{p}_1$ and $\mathbf{p}_2$ are linearly independent as one is not the scalar multiple of another.

Hence, Basis $= \{\mathbf{p}_1, \mathbf{p}_2\}$

$= \left\{(x^3 + 3x^2 + 4x), (-5x^2 + x + 1)\right\}$

Dimension $= 2$

2.11 REDUCTION AND EXTENSION TO BASIS

Theorem 2.16: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of non-zero vectors in a vector space V.

(i) If S spans V then S can be reduced to a basis for V by removing some vectors from S and dim $(V) < n$.
(ii) If S is linearly independent then S can be extended to a basis for V by adding some vectors into S and dim $(V) > n$.

2.11.1 *Reduction to Basis*

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of non-zero vectors in a real vector space V.
If V = span S and dim $(V) < n$ then S can be reduced to a basis for V as follows:

1. Consider, $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = \mathbf{0}$ ...(2.7)
2. Construct the augmented matrix of the homogeneous system obtained from Eq. (2.7). Reduce the homogeneous system to row echelon form.
3. The vectors corresponding to the columns containing the leading 1's form a basis for V.

Note: By changing the order of vectors in S, other possible bases can be found.

2.11.2 *Extension to Basis*

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a linearly independent set of vectors in a real vector space V. If dim $(V) = n > m$ then S can be extended to a basis for V as follows:

1. Form the set $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n.
2. Follow all the steps (1 to 3) of 2.11.1.

Note: By changing the order of standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ in S', other possible bases can be found.

Example 1: Reduce $S = \{(1, 0, 0), (0, 1, -1), (0, 4, -3), (0, 2, 0)\}$ to obtain a basis for W = span S

Solution: Consider,

$$k_1(1, 0, 0) + k_2(0, 1, -1) + k_3(0, 4, -3) + k_4(0, 2, 0) = (0, 0, 0)$$
$$(k_1, k_2 + 4k_3 + 2k_4, -k_2 - 3k_3) = (0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned} k_1 \qquad\qquad\qquad &= 0 \\ k_2 + 4k_3 + 2k_4 &= 0 \\ -k_2 - 3k_3 \qquad &= 0 \end{aligned}$$