Calculus
(BMAT101L)

Module 2

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Chapter 3

Continuity and Differentiability of Multi-variable Functions

3.1 Geometry of Functions of Two Variables

Definition 3.1.1 (Functions of Two Variables). Let $\mathscr{D} \subset \mathbb{R}^2$. A function $f : \mathscr{D} \to \mathbb{R}$ is called a real-valued function of *two* variables. The elements of $\mathscr{D}$ are ordered pairs (x, y) of real numbers, where x and y are called *input variables* and the real number $w = f(x, y)$, an *output variable* of the function f.

Definition 3.1.2 (Surface and Trace). Let $f(x, y)$ be a two-variable function defined on $\mathscr{D} \subset \mathbb{R}^2$. The the *graph* $\{(x, y, f(x, y)) : (x, y) \in \mathscr{D}\}$ of $f(x, y)$ defines a *surface* $z = f(x, y)$ in space. The curve of intersection of the surface $z = f(x, y)$ and the plane $z = c$ is known as a *trace* or *plane section* of the surface in the plane $z = c$. Other traces are similarly defined.

Definition 3.1.3 (Level Curves and Contours). Let $f(x, y)$ be a two-variable function defined on $\mathscr{D} \subset \mathbb{R}^2$. Let c be a constant. The curve lying in the xy-plane, given by $f(x, y) = c$ is called a *level curve* of f. Thus f takes on a constant value on its level curve. A *contour* $f(x, y) = c$ is regarded as a curve of intersection of the surface $z = f(x, y)$ and the plane $z = c$.

For $c > 0$, it lies at a height of c units from the xy-plane, and hence is a line of *constant elevation*. Whereas, for $c < 0$, it lies at a depth of c units from the xy-plane, and hence is a line of *constant depression*.

Isobars are the curves of constant pressure, *isotherms* are curves of constant temperature, and *equipotential curves* are curves of constant electrostatic potential.

3.2 Continuity of Two-variable Function $u = f(x, y)$

Definition 3.2.1 (Continuity). A function $f(x, y)$ is said to be continuous at a point $P(x_0, y_0)$ in the domain $\mathscr{D}$ of f, if the limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists and is equal to $f(x_0, y_0)$. The function f is said to be continuous on the domain $\mathscr{D}$, if it is continuous at every point of $\mathscr{D}$.

Theorem 3.2.1 (Existence of the Limit). If the f has different limits along different paths in $\mathscr{D}$ as (x, y) approaches (x_0, y_0), then the limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Example 3.2.1. Show that the limit of each of the following functions does not exist at $(0, 0)$, and hence f is discontinuous at $(0, 0)$:

(a) $f(x, y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(b) $f(x, y) = \begin{cases} \frac{x-y}{x+y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

Solution.

(a) Along the line $y = mx$, $m \neq 0$, $f(x, y) = 1/\sqrt{1 + m^2}$. Then $\lim_{(x,y)\to(0,0)} f(x, y) = \frac{1}{\sqrt{1+m^2}}$, which changes with each value of the slope m. Thus the limit of f does not exist as $(x, y) \to (0, 0)$. Hence f is discontinuous at $(0, 0)$.

(b) Along the line $y = mx$, $m \neq 0$, $\lim_{(x,y)\to(0,0)} f(x, y) = \frac{1-m}{1+m}$, which changes with each value of m. Thus the limit does not exist as $(x, y) \to (0, 0)$. Hence f is discontinuous at $(0, 0)$.

Exercise 3.2.1. Show that the following functions have no limit as $(x, y) \to (0, 0)$, and hence are discontinuous at the origin $(0, 0)$:

(a) $f(x, y) = \begin{cases} \frac{xy}{|xy|}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(b) $f(x, y) = \begin{cases} \frac{x^2-y}{x-y}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(c) $f(x, y) = \begin{cases} \frac{x^4}{x^4+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

Answers.

(a) Along the line $y = mx$ with $m \neq 0$, $\lim_{(x,y)\to(0,0)} f(x, y) = \frac{m}{|m|} = \begin{cases} -1, & \text{if } m < 0 \\ 1, & \text{if } m > 0. \end{cases}$. Thus for different paths as $(x, y) \to (0, 0)$ we get different values of the limit as $(x, y) \to (0, 0)$. Hence f is discontinuous at $(0, 0)$.

(b) Along the line $y = mx$ with $m \neq 1$, as $(x, y) \to (0, 0)$, $f(x, y)$ tends to $-m/(1 - m)$, which is different for different choices of m. Thus the limit of f does not exist as $(x, y) \to (0, 0)$. Hence f is discontinuous at $(0, 0)$.

(c) Along the parabola $y = x^2$, $f(x, y) \to 1/2$, while along the x-axis, $f(x, y)$ tends to 1, as $(x, y) \to (0, 0)$. Thus for two different paths $f(x, y)$ approaches to different numbers. Thus the limit does not exist as $(x, y) \to (0, 0)$. Hence f is discontinuous at $(0, 0)$.

Example 3.2.2. By an appropriate substitution, show that the limit of

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

varies from -1 to 1 along the line $y = mx$ as $(x, y) \to (0, 0)$.

Solution. Along the line $y = mx$, $m \neq 1$, $l = \lim_{(x,y)\to(0,0)} f(x, y) = 2m/(1 + m^2)$. Then substitute $m = \tan\theta$ in this, we get $l = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$, which varies with the angle θ of inclination. Since $\sin 2\theta$ lies in the interval $[-1, 1]$, l varies between -1 and 1 for each real θ.

Example 3.2.3. Using the polar coordinates

$$x = r\cos\theta, \ r = r\sin\theta, \tag{3.2.1}$$

show that

$$f(x, y) = \begin{cases} \tan^{-1}\left(\frac{|x|+|y|}{x^2+y^2}\right), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

is not continuous at the origin.

Solution. Using (3.2.1), $f(x, y) = \tan^{-1}\left\{\frac{r(|\cos\theta|+|\sin\theta|)}{r^2}\right\} \to \tan^{-1}\infty = \pi/2$ as $r \to 0$. Thus the limit is $l = \frac{\pi}{2}\cdot$ Since $l \neq f(0, 0)$, f is not continuous at $(0, 0)$.

Example 3.2.4. Use the polar coordinates (3.2.1) to show that

$$f(x,y) = \begin{cases} \frac{2x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

is not continuous at the origin.

Solution. Using (3.2.1), we see that

$$f(x,y) = \frac{2r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + \sin^2\theta}, \quad r \neq 0.$$

Along the line $\theta =$ constant, as $r \to 0$, the double limit is 0. While along the parabolic path $y = x^2$, we have $r\cos\theta = r^2\sin 2\theta$ so that

$$f(x,y) = \frac{2r^2\cos^{\theta}\cdot r^2\cos^{\theta}}{r^4\cos^4\theta + r^4\cos^4\theta} = 1.$$

As, for two different paths of approach, $l = \lim_{(x,y)\to(0,0)} f(x,y)$ is different, l does not exist, and hence f is not continuous at $(0,0)$.

Exercise 3.2.2. Using (3.2.1), examine the continuity of $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ at the origin:

Solution. In view of (3.2.1), $f(x,y) = \cos^2\theta - \sin^2\theta = \cos 2\theta \in [-1,1]$. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist and hence f is not continuous at $(0,0)$.

Exercise 3.2.3 (self-check)**.** Given that $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$, justify whether it is possible to define f at $(0,0)$ so that it will be continuous at the origin.

3.3 The Total Derivative

Let $u = f(x,y)$ be defined on a domain $\mathscr{D} \subset \mathbb{R}^2$ and $(x_0,y_0) \in \mathscr{D}$. The first order partial derivatives of f with respect to x and y are given by

$$\frac{\partial f}{\partial x}(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}, \quad \frac{\partial f}{\partial y}(x_0,y_0) = \lim_{k\to 0}\frac{f(x_0,y_0+k) - f(x_0,y_0)}{k},$$

Second order partial derivatives of f are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Higher order partial derivatives are similarly defined. The rules of finding partial derivatives of sum, difference, product and quotient are similar to those of ordinary derivatives.

Theorem 3.3.1 (Two Intermediate Variables and One Independent Variable)**.** Let $u = f(x,y)$, where $x = g_1(t)$, $y = g_2(t)$. Then u is a composite function of t through the intermediate variables x and y. The derivative of w with respect to t, given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \tag{3.3.1}$$

is called the *total derivative* of f with respect to t.

Example 3.3.1. Let $\theta = xy - 2$ be the temperature at the point (x, y) on the circle

$$x = 2\sqrt{2}\cos t, y = 2\sqrt{2}\sin t,\ 0 \le t \le 2\pi.$$

Find the points where the total derivative $\frac{d\theta}{dt}$ becomes zero.

Solution. We see that

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{\partial\theta}{\partial x}\frac{dx}{dt} + \frac{\partial\theta}{\partial y}\frac{dy}{dt} = y(-2\sqrt{2}\sin t) + x(2\sqrt{2}\cos t)\\ &= (2\sqrt{2}\sin t)(-2\sqrt{2}\sin t) + (2\sqrt{2}\cos t)(2\sqrt{2}\cos t)\\ &= 8(\cos^2 t - \sin^2 t) = 8\cos 2t.\end{aligned}$$

Therefore, $\frac{d\theta}{dt} = 0$ only if $2t = \pi/2, 3\pi/2$, that is when $t = \pi/4,\ 3\pi/4$.

Example 3.3.2 (Changing Voltage in a Circuit)**.** The voltage V in a circuit that satisfies Ohm's law

$$V = IR \text{ or } I = V/R \tag{3.3.2}$$

drops slowly as the battery discharges. At the same time, the resistance R is increasing as the resistor heats up. The change in current at any instance is described by the total derivative of I with respect to the time t:

$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt} = \frac{1}{R}\frac{dV}{dt} - \frac{V}{R^2}\frac{dR}{dt} = \frac{1}{R}\left(\frac{dV}{dt} - I\frac{dR}{dt}\right).$$

Exercise 3.3.1 (Self-check)**.** Using the formula (3.3.1), find $\frac{du}{dt}$ at a given value of t, and verify your result by the direct differentiation:

(a) $u = \sin\left(\frac{x}{y}\right)$, where $x = e^t,\ y = t^2$

(b) $u = \sin^{-1}(x - y)$, where $x = 3t,\ y = 4t^3$

(c) $u = x^2 + y^2$, where $x = \cos t + \sin t,\ y = \cos t - \sin t$

(d) $u = \frac{x+y}{z}$, where $x = \cos^2 t,\ y = \sin^2 t,\ z = 1/t$ at $t = 3$

(e) $u = xy + yz + zx$, where $x = e^t,\ y = e^{-t},\ z = 1/t$

3.4 Jacobians

Definition 3.4.1 (Second Order Jacobian)**.** Consider the transformation

$$u = f(x, y),\ v = g(x, y). \tag{3.4.1}$$

The Jacobian of u and v with respect to x and y is given by

$$J\left(\frac{u, v}{x, y}\right) \equiv \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}.$$

Example 3.4.1.

(a) Let $u = ax + by,\ v = cx + dy$ Then

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix} = \begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc.$$

(b) Let $u = x^2 - y^2,\ v = 2xy$. Then

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4\left(x^2 + y^2\right)$$

Exercise 3.4.1. Find $J\left(\frac{u,v}{x,y}\right)$, where

(a) $u = x \sin y,\ v = y \sin x$

(b) $u = x^2 - 2y,\ v = x + y$

Answers.

(a) $\sin x \sin y - xy \cos x \cos y$

(b) $2\,(x + y)$

Theorem 3.4.1 (Inverse Property of Jacobians)**.** If $J\left(\frac{u,v}{x,y}\right) \neq 0$, then the Jacobian transformation (3.4.1) is invertible, and

$$J\left(\frac{u,v}{x,y}\right) \cdot J\left(\frac{x,y}{u,v}\right) = 1 \text{ or } J\left(\frac{x,y}{u,v}\right) = 1/J\left(\frac{u,v}{x,y}\right). \tag{3.4.2}$$

Example 3.4.2. Let $u = x(1 - y),\ v = xy$. Then

$$J\left(\frac{x,y}{u,v}\right) = 1/J\left(\frac{u,v}{x,y}\right) = 1/\begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = \frac{1}{x} = \frac{1}{u+v}.$$

Example 3.4.3. Verify that $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$, where $x = r\cos\theta, y = r\sin\theta$.

Solution. Note that

$$J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\sin\theta \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r.$$

The given transformation implies that $r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right)\cdot$ Hence

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}\cdot$$

This verifies the property, given in (3.4.1).

Theorem 3.4.2 (Chain Rule of Jacobians)**.** Consider the transformations:

$$u = f(r,s),\ v = g(r,s),$$

where

$$r = p(x,y),\ s = q(x,y).$$

Then

$$J\left(\frac{u,v}{x,y}\right) = J\left(\frac{u,v}{r,s}\right) J\left(\frac{r,s}{x,y}\right). \tag{3.4.3}$$

Example 3.4.4. Consider the elliptical polar transformation: $x = ar\cos\theta$, $y = br\sin\theta$. Let $u = r\cos\theta$, $v = r\sin\theta$. Then $x = au$, $y = bv$. Therefore,

$$J\left(\frac{x, y}{r, \theta}\right) = J\left(\frac{x, y}{u, v}\right) J\left(\frac{u, v}{r, \theta}\right) = (ab)(r) = abr.$$

Definition 3.4.2 (Third Order Jacobian)**.** Consider the transformation

$$u = f(x, y, z),\ v = g(x, y, z)\ w = h(x, y, z). \tag{3.4.4}$$

The Jacobian of u, v and w with respect to x, y and z is given by

$$J\left(\frac{u, v, w}{x, y, z}\right) \equiv \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Exercise 3.4.2 (Self-check)**.** Find $J\left(\frac{u,v,w}{x,y,z}\right)$, where:

(a) $u = \frac{2x-y}{2},\ v = \frac{y}{2}, w = \frac{z}{3}$

(b) $u = \frac{yz}{x},\ v = \frac{zx}{y}, w = \frac{xy}{z}$

(c) $u = x^2 - 2y,\ v = x + y + z,\ w = x - 2y + 3z.$

Remark 3.4.1. The inverse property given in Theorem 3.4.1 and the chain rule given in Theorem 3.4.2 can be extended to third and higher order Jacobians also.

Exercise 3.4.3 (Self-check)**.** Find $J\left(\frac{x,y,z}{u,v,w}\right)$, where:

(a) $u = x + y + z,\ uv = y + z,\ uvw = z$

(b) $u = \frac{yz}{x},\ v = \frac{zx}{y},\ w = \frac{xy}{z}.$

Example 3.4.5. Find $\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$, where the *cylindrical polar coordinates* r, θ and z are described by the transformation $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Also, verify that $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} \cdot \frac{\partial(r,\theta,z)}{\partial(x,y,z)} = 1$.

Solution. First we see that

$$J\left(\frac{x, y, z}{r, \theta, z}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\sin\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\left(\cos^2\theta + \sin^2\theta\right) = r,$$

by expanding the determinant using third row. Now, the inverse of the transformation is

$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\left(\frac{y}{x}\right),\ z = z,$$

provided $r \neq 0$. Therefore,

$$J\left(\frac{r, \theta, z}{x, y, z}\right) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{r}.$$

Hence $J\left(\frac{x,y,z}{r,\theta,z}\right) J\left(\frac{r,\theta,z}{x,y,z}\right) = 1$.

Exercise 3.4.4. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}$, where the *spherical polar coordinates* ρ, φ and θ are described by $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$. Verify that $\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} \cdot \frac{\partial(\rho,\varphi,\theta)}{\partial(x,y,z)} = 1$.

Answer. $\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \rho^2 \sin\varphi$ and $\frac{\partial(\rho,\varphi,\theta)}{\partial(x,y,z)} = 1/\rho^2 \sin\varphi$. Therefore, $\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} \cdot \frac{\partial(\rho,\varphi,\theta)}{\partial(x,y,z)} = 1$.

Example 3.4.6. Consider the spheroidal polar transformation: $x = a\rho \sin\varphi \cos\theta$, $y = b\rho \sin\varphi \sin\theta$, $z = c\rho \cos\varphi$. Let $u = \rho \sin\varphi \cos\theta$, $v = \rho \sin\varphi \sin\theta$, $w = \rho \cos\varphi$. Then $x = au$, $y = bv$, $z = cw$. Therefore,

$$J\left(\frac{x,y,z}{\rho,\varphi,\theta}\right) = J\left(\frac{x,y,z}{u,v,w}\right) J\left(\frac{u,v,w}{\rho,\varphi,\theta}\right) = abc\rho^2 \sin\varphi.$$

Theorem 3.4.3 (Jacobians and Functional Dependence)**.** Consider the transformation (3.4.1). Then u and v are functionally dependent of each other if and only if

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0. \tag{3.4.5}$$

Example 3.4.7. Let $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ and $v = \sin^{-1} x + \sin^{-1} y$. Verify Theorem 3.4.3 and hence find the functional relationship between u and v.

Solution. First we have

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$
$$= 1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} - 1 + \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} = 0.$$

Thus by Theorem 3.4.3, u and v are functionally related. Indeed, we observe from trigonometry that $u = \sin v$.

Exercise 3.4.5 (self-check)**.** Let $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$. Verify that u and v are functionally dependent and hence find the functional relationship between u and v.

Similarly, we have

Theorem 3.4.4 (Jacobians and Functional Dependence)**.** Consider the transformation (3.4.4). Then u, v and w are functionally dependent if and only if

$$J(\frac{u,v,w}{x,y,z}) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0. \tag{3.4.6}$$

Exercise 3.4.6 (self-check)**.** Let $u = 3x + 2y - z$, $v = x - 2y + z$, $w = x^2 + 2xy - zx$. Verify that u, v and w are functionally dependent and hence find the functional relationship among them.

Text and Reference Books

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