Application to fluid flow problems:-

Consider two dimensional irrotational motion of fluid in plane (x-y)

velocity v of fluid can be expressed as

$$v = v(x)\hat{i} + v(y)\hat{j} \quad \text{——— (1)}$$

velocity in x direction

velocity in v direction

motion is irrotational therefor $\exists$ a function $\phi(x, y)$ such that

$$V = \text{grad}\,\phi$$

$$V = \nabla\phi(x, y)$$

$$V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\phi$$

$$v = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} \quad \text{——— (2)}$$

i.e comparing (1) and (2)

$$v(x) = \frac{\partial\phi}{\partial x}, \quad v(y) = \frac{\partial\phi}{\partial y}$$

x component of velocity

- this Scalar function $\phi(x, y)$ is called the velocity potential.

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Note :-

* The velocity potential $\phi(x, y)$ which gives velocity components is always satisfy Laplace eq^n i.e $\phi(x, y)$ is harmonic function
i.e $\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$

as fluid is incompressible
i.e. div $v = 0$
$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y}\right) = 0$$
$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

complex potential :-

* ϕ is a real part of complex function $f(z)$ given as

$$f(z) = \phi + i\psi$$

where ϕ is velocity potential
ψ is stream function

$f(z)$ is complex potential, which represent the flow pattern.

$\phi(x, y) = c_1$ & $\psi(x, y) = c_2$ intersects orthogonally :

* Family of curves $\phi(x, y) = c_1$ are called equipotential lines. and family of curves $\psi(x, y) = c_2$ are called stream lines. and these two curves intersects orthogonally

&
$$m_1 = \frac{\left(-\frac{\partial \phi}{\partial x}\right)}{\left(\frac{\partial \phi}{\partial y}\right)} \qquad m_2 = \frac{-\left(\frac{\partial \psi}{\partial x}\right)}{\left(\frac{\partial \psi}{\partial y}\right)}$$

[C-R eq^n
$\phi_x = \psi_y$
$\phi_y = -\psi_x$]

$\boxed{m_1 m_2 = -1}$ i.e $\phi = c_1$ & $\psi = c_2$ intersects orthogonally.

Velocity of fluid :-

$$f(z) = \phi + i\psi$$

$$f'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

CR eq^n

$$\left[\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}\right]$$

$$f'(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

$$f'(z) = v_x - i v_y$$

$$\boxed{\overline{f'(z)} = v_x + i v_y}$$

is ~~calle~~ the expression for velocity of fluid.

Speed or magnitude of resultant velocity

$$\left|\overline{f'(z)}\right| = \sqrt{v_x^2 + v_y^2}$$

(1) In two dimentional fluid flow if velocity potential is $\phi = x^4 - 6x^2y^2 + y^4$ then
(i) find the stream function φ & corresponding complex potential.

(2) write expression for velocity & hence find speed.

(3) verify that family of curves $\phi(x,y) = c_1$ & $\varphi(x,y) = c_2$ intersect orthogonally.

Soln:- $\phi = x^4 - 6x^2y^2 + y^4$

$\phi_x = 4x^3 - 12xy^2 \qquad \phi_y = 4y^3 - 12x^2y$

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy$$

$$d\varphi = -\frac{\partial \phi}{\partial y}dx + \frac{\partial \phi}{\partial x}dy$$

$$d\varphi = (-4y^3 + 12x^2y)\,dx + (4x^3 - 12xy^2)\,dy \quad \text{—— (1)}$$

∴ diff eqn (1) is exact

$$\int d\varphi = \int_{y\ \text{const}} (-4y^3 + 12y x^2)\,dx + \int_{\text{ignoring } x} (4x^3 - 12xy^2)\,dy$$

$$\varphi = -4y^3x + 4x^3y + 0 + c$$

$$\boxed{\varphi = 4x^3y - 4y^3x + c} \rightarrow \text{stream function.}$$

$$f(z) = \phi + i\varphi$$

$$f(z) = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3 + c)$$

$$f(z) = (x+iy)^4 + ic \qquad f(z) = z^4 + ic$$

$$\boxed{f(z) = z^4 + ic} \quad \text{complex potential}$$

(II) velocity of fluid is

$$= \overline{f'(z)}$$

$$f'(z) = 4z^3$$

$$\boxed{\overline{f'(z)} = 4\bar{z}^3} \quad \underline{\text{velocity}}$$

$$\overline{f'(z)} = 4(x-iy)^3$$

$$= 4(x^3 + iy^3 - 3ix^2y - 3xy^2)$$

$$\overline{f'(z)} = 4[(x^3-3xy^2) - i(-y^3+3x^2y)]$$

Speed is $|\overline{f'(z)}|$

$$|\overline{f'(z)}| = \sqrt{[4(x^3-3xy^2)]^2 + [4(-y^3+3x^2y)]^2}$$

$$= 2\sqrt{x^6+y^6+9x^2y^4-6x^4y^2+9x^4y^2-6x^2y^4}$$

$$= 2\sqrt{x^6+y^6+3x^4y^2+3x^2y^4}$$

$$= 2\sqrt{(x^2+y^2)^3} = 2(x^2+y^2)^{3/2} \quad \text{Speed}$$

$$\boxed{\text{Speed} = 2(x^2+y^2)^{3/2}}$$

(III) To show $\phi = c_1$ & $\psi = c_2$ are orthogonal

$$m_1 = \frac{-\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}} = \frac{-(4x^3-12xy^2)}{(4y^3-12x^2y)}$$

$$m_2 = \frac{-\left(\frac{\partial \psi}{\partial x}\right)}{\left(\frac{\partial \psi}{\partial y}\right)} = \frac{(4y^3 - 12x^2y)}{4x^3 - 12xy^2}$$

$$m_1 . m_2 = \frac{-(4x^3 - 12xy^2)}{(4y^3 - 12x^2y)} \times \frac{(4y^3 - 12x^2y)}{(4x^3 - 12xy^2)}$$

$\boxed{m_1 \; m_2 = -1}$ i.e $\phi = c_1$ & $\psi = c_2$ are orthogonal.

Q:- If $w = \phi + i\psi$ represents the complex potential for electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine the function ϕ.

Soln:- $w = f(z) = \phi + i\psi$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} \cdot dy$$

$\partial \psi$

$$\boxed{d\phi = \left(+\frac{\partial \psi}{\partial y}\right) dx - \left(\frac{\partial \psi}{\partial x}\right) dy} \quad (\text{C-R eq}^n)$$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

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$$d\phi = \left(-2y - \frac{2xy}{(x^2+y^2)^2}\right)dx + \left(-2x - \frac{y^2-x^2}{(y^2+x^2)^2}\right)dy$$

This is exact diff. eq^n

sol^n is

$$\int d\phi = \int_{y\,const}\left(-2y - \frac{2xy}{(x^2+y^2)^2}\right)dx + \int_{\substack{\text{ignoring}\\ x\text{ terms}}} 0\, dy + C$$

$$\boxed{\phi = -2yx + \frac{y}{x^2+y^2} + C} \rightarrow \text{velocity potential.}$$

:- Show that $\psi = x^2 - y^2 - 3x - 2y + 2xy$ can represent the stream function of an incompressible fluid flow. Also find corresponding velocity potential ϕ and hence complex potential $f(z) = \phi + i\psi$.

int!:- To show ψ can be represent as the stream function, it should satisfy Laplace eq^n.

i.e To show $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0.$

In electric field problems

$w = u + iv$

complex potential

u is called potential function

v is called flux function

In electric field problems

$w = u + iv$

complex potential | u is called potential function | v is called flux function

Q1 B2

Q1(B)

Q3 (

* Consider the complex potential

$w = \phi(x,y) + i\,\psi(x,y)$

(b)

(i) The curves $\phi(x,y) = a$ and $\psi(x,y) = b$ are called equipotential lines and stream lines in the field of fluid flow.

(c

2) The curves $\phi(x,y) = a$ and $\psi(x,y) = b$ are called equipotential lines and lines of force in the field of electrostatic and gravitational field.

(c

3. In heat flow problems $\phi(x,y) = a$ and $\psi(x,y) = b$ are called isothermal and heat flow lines.