Special Functions

- Beta and Gamma functions
- Interrelation between beta and gamma functions
- Evaluation of multiple integrals using gamma and beta functions.
- Dirichlet's integral
- Error functions complementary error functions

Improper integral

For the existence of Riemann integral (definite integral) $\int_a^b f(x)dx$, we require that the limit of integration a and b are finite and function $f(x)$ is bounded. In case

(i) limit of integration a or b or both become infinite (improper integral of first kind),

(ii) integrand $f(x)$ has singular points (discontinuity) i.e. $f(x)$ becomes infinite at some points in the interval $a \leq x \leq b$ (improper integral of second kind),

then the integral $\int_a^b f(x)dx$ is called improper integral.

Beta and Gamma functions (improper integral)

The first and second Eulerian Integrals which are also called "Beta and Gamma functions" respectively are defined as follows:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$$

$\beta(m,n)$ is read as "Beta m,n" and Γn is read as "Gamma n". Here the quantities m and n are positive numbers which may or may not be integers.

Properties of Beta function

- The function $\beta(m,n)$ is symmetrical with respect to m and n

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Put $x = 1 - y$. Then

$$\beta(m,n) = -\int_1^0 (1-y)^{m-1}y^{n-1}dy = \int_0^1 y^{n-1}(1-y)^{m-1}dy = \beta(n,m).$$

- $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta, n > 0$

Put $x = sin^2\theta$, $dx = 2sin\ \theta\ cos\ \theta\ d\theta$ in $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$.

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1} 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Which is another form of $\beta(m,n)$.

- $\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$
  (Home work)

Properties of Gamma function

The improper integral of the form $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$, $n > 0$ is called gamma function.

1. $\Gamma(1) = \int_0^\infty e^{-x} dx = 1.$

2. Reduction formula $\Gamma(n+1) = n\Gamma(n)$

Proof: $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx = [-x^n e^{-x}]_0^\infty + n \int_0^\infty e^{-x} x^{n-1} dx$

$$= n\Gamma(n) = n!.$$

3 Relation between β and Γ functions

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Proof: $\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} dt = 2\int_0^\infty e^{-x^2} x^{2m-1} dx$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1} dx \int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx dy$$

$$= 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} r^{2(m+n)-1} cos^{2m-1}\theta \ sin^{2n-1}\ \theta \ dr d\theta$$

$$= 2\int_0^\infty e^{-r^2} r^{2(m+n)-1} dr \times 2\int_0^{\pi/2} cos^{2m-1}\theta \ sin^{2n-1}\ \theta d\theta$$

$$= \Gamma(m+n)\beta(m,n).$$

4 $\Gamma(1/2) = \sqrt{\pi}$.

Proof: $\Gamma(1/2) = \int_0^\infty e^{-x} x^{-1/2}\, dx = 2\int_0^\infty e^{-y^2}\, dy$ (by putting $x = y^2$).

$$(\Gamma(1/2))^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx dy$$

$$= 4\int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta = \frac{4\pi}{2}\int_0^\infty e^{-r^2} r dr = \pi$$

5. $\int_0^{\pi/2} cos^m\theta \; sin^n \; \theta d\theta$

$$= \frac{1}{2}\beta(\frac{m+1}{2}, \frac{n+1}{2}) = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$$

Special case:

$$\left.\begin{array}{l} \displaystyle\int_0^{\pi/2} \sin^n x \, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \\ \displaystyle\int_0^{\pi/2} \cos^n x \, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \end{array}\right\}$$

(Gamma for Negative Real Numbers). For $p < 0$, $p \neq 0, -1, -2, \ldots$, we have

$$\Gamma(p) = \frac{1}{p} \cdot \Gamma(p+1) = \frac{1}{p} \cdot \frac{1}{p+1} \cdots \frac{1}{p+k} \cdot \Gamma(p+k+1),$$

where k is the least positive integer such that $p + k + 1 > 0$.

$$\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$$

Put $x^2 = \sin\theta$, *i.e.*, $x = \sin^{1/2}\theta$

so that $dx = 1/2 \sin^{-1/2}\theta \cos\theta \, d\theta$

$$= \int_0^{\pi/2} \frac{1}{2} \cdot \frac{\sin^{-1/2}\theta \, . \cos\theta \, d\theta}{\sqrt{(1-\sin^2\theta)}} = \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\int_0^{\pi/2} \sqrt{(\tan\theta)}\, d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \frac{\Gamma\left(\dfrac{\frac{1}{2}+1}{2}\right)\Gamma\left(\dfrac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\dfrac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

Multiple Integrals using Beta and Gamma Function

Example: The cylinder $x^2 + z^2 = 1$ is cut by the plane $x = 0, y = 0$ and $y = x$. Find the volume of the region in the first octant.

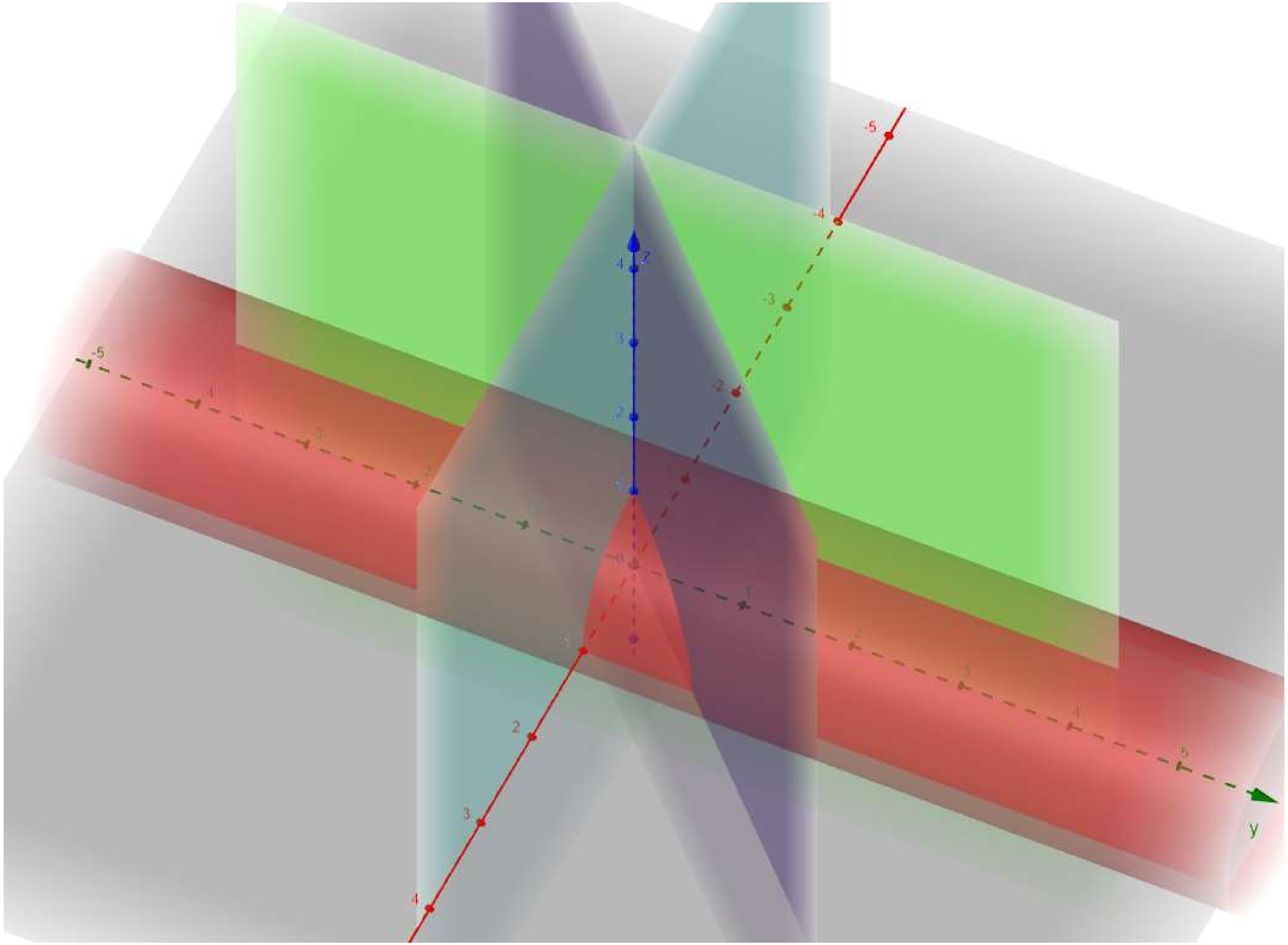
Ans: In the first octant the equation of the cylinder is given by,

$$z = \sqrt{1 - x^2}$$

The required volume is given by:

$$V = \int_{x=0}^{1} \int_{y=0}^{x} \sqrt{1 - x^2}\, dy\, dx = \int_{x=0}^{1} x\sqrt{1 - x^2} dx$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta \cos\theta \cos\theta d\theta = \frac{1}{2}\int_{\theta=0}^{\frac{\pi}{2}} 2\sin\theta \cos^2\theta d\theta = \frac{1}{2}\beta\left(1, \frac{3}{2}\right) = \frac{\frac{1}{2}\Gamma(1)\Gamma\left(\frac{3}{2}\right)}{\Gamma(\frac{5}{2})} = \frac{1}{3}$$


z
y

Dirichlet Integral

$\int\int_D x^{l-1}y^{m-1}dxdy = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}h^{l+m}$, where D is domain $x \geq 0, y \geq 0, x+y \leq h$.

Proof. Put $x = Xh$, $y = Yh$, $dxdy = h^2 dXdY$. Then

$$\int\int_D x^{l-1}y^{m-1}dxdy = \int\int_D (Xh)^{l-1}(Yh)^{m-1}dXdY$$

$$= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1}Y^{m-1}dXdY$$

$$= h^{l+m}\int_0^1 X^{l-1}dX\int_0^{1-X} Y^{m-1}dY$$

$$= h^{l+m}\int_0^1 X^{l-1}dX[\frac{Y^m}{m}]_0^{1-X}$$

$$= \frac{h^{l+m}}{m} \int_0^1 X^{l-1}(1-X)^m dX$$

$$= \frac{h^{l+m}}{m} \beta(l, m+1)$$

$$= \frac{h^{l+m}}{m} \frac{\Gamma(l)\Gamma(m+1)}{\Gamma(l+m+1)}$$

$$= h^{l+m} \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}$$

$\int\int\int_V x^{l-1}y^{m-1}z^{n-1}dxdydz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$, where V is region $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$.

Proof. Put $y + z \leq 1 - x = h$. then $z \leq h - y$.

$$\int\int\int_V x^{l-1}y^{m-1}z^{n-1}dxdydz = \int_0^1 x^{l-1}dx \int_0^{1-x} y^{m-1}dy \int_0^{1-x-y} z^{n-1}dz$$

$$= \int_0^1 x^{l-1}dx[\int_0^h \int_0^{h-y} y^{m-1}dyz^{n-1}dz]$$

$$= \int_0^1 x^{l-1}dx[h^{m+n}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}]$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}\int_0^1 x^{l-1}(1-x)^{m+n}dx$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}\beta(l, m+n+1)$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

Alternative

$$\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} x^{l-1} y^{m-1} z^{n-1} \, dz \, dy \, dx = \int_{x=0}^{1} \int_{y=0}^{1-x} x^{l-1} y^{m-1} \left[\frac{z^n}{n}\right]_0^{1-x-y} dy \, dx$$

$$= \frac{1}{n} \int_{x=0}^{1} \int_{y=0}^{1-x} x^{l-1} y^{m-1} (1-x-y)^n \, dy \, dx.$$

Let $y = (1-x)t$

$dy = (1-x)\,dt$

y	0	$1-x$
t	0	1

$$= \frac{1}{n} \int_{x=0}^{1} \int_{t=0}^{1} x^{l-1} (1-x)^{m-1} t^{m-1} \left((1-x) - (1-x)t\right)^n (1-x) \, dt \, dx$$

$$= \frac{1}{n} \int_{x=0}^{1} \int_{t=0}^{1} x^{l-1} (1-x)^{m+n} t^{m-1} (1-t)^n \, dx \, dt$$

$$= \frac{1}{n} \int_{x=0}^{1} x^{l-1} (1-x)^{m+n} \, dx \cdot \int_{t=0}^{1} t^{m-1} (1-t)^{n} \, dt$$

$$= \frac{1}{n} \, \beta(l, m+n+1) \cdot \beta(m, n+1)$$

$$= \frac{1}{n} \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)} \cdot \frac{\Gamma(m) \cdot \Gamma(n+1)}{\Gamma(m+n+1)}$$

$\Gamma(n+1) \rightarrow n\,\Gamma(n)$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

$$\left[\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \right.$$

Prob: Find the value of

$$I = \iiint_V x^{l-1} y^{m-1} z^{n-1} dxdydz$$

Where V is given by:

$$V = \left\{(x, y, z) \mid x, y, z \geqslant 0 \text{ and } \left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} + \left(\frac{z}{c}\right)^{\gamma} \leqslant 1\right\}$$

Ans: We consider $\left(\frac{x}{a}\right)^{\alpha} = X,\ \left(\frac{y}{b}\right)^{\beta} = Y,\ \left(\frac{z}{c}\right)^{\gamma} = Z$

Now, $\frac{x}{a} = X^{\frac{1}{\alpha}}$ Similarly, $y = bY^{\frac{1}{\beta}},\ z = cZ^{\frac{1}{\gamma}}$

$$\Rightarrow dy = \frac{b}{\beta} Y^{\frac{1}{\beta}-1} dY$$

$$dx = \frac{a}{\alpha} X^{\frac{1}{\alpha}-1} dX$$

$$dz = \frac{c}{\gamma} Z^{\frac{1}{\gamma}-1} dZ$$

So, due to this substitution, the new region is:

$$V' = \{(X, Y, Z):\ X, Y, Z \geqslant 0 \text{ and } X + Y + Z \leqslant 1\}$$

Hence,

$$I = a^{l-1} b^{m-1} c^{n-1} \cdot \frac{abc}{\alpha\beta\gamma} \iiint_{V'} X^{\frac{l-1}{\alpha}} Y^{\frac{m-1}{\beta}} Z^{\frac{n-1}{\gamma}} X^{\frac{1}{\alpha}-1} Y^{\frac{1}{\beta}-1} Z^{\frac{1}{\gamma}-1} dX dY dZ$$

$$= \frac{a^l b^m c^n}{\alpha\beta\gamma} \iiint_{V'} X^{\frac{l-1}{\alpha}+\frac{1}{\alpha}-1} Y^{\frac{m-1}{\beta}+\frac{1}{\beta}-1} Z^{\frac{n-1}{\gamma}+\frac{1}{\gamma}-1} dX dY dZ$$

$$= \frac{a^l b^m c^n}{\alpha\beta\gamma} \iiint_{V'} X^{\frac{l}{\alpha}-1} Y^{\frac{m}{\beta}-1} Z^{\frac{n}{\gamma}-1} dX dY dZ$$

$$= \frac{a^l b^m c^n}{\alpha\beta\gamma} \frac{\Gamma\left(\frac{l}{\alpha}\right)\Gamma\left(\frac{m}{\beta}\right)\Gamma\left(\frac{n}{\gamma}\right)}{\Gamma\left(\frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}+1\right)}$$

Exercise: Find the volume of the sphere $x^2 + y^2 + z^2 = 1$ in the first Octant.

The error function (useful in probability theory)

– Error function: $$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt\,.$$

– Standard model or Gaussian cumulative distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2} + \frac{1}{2}\mathrm{erf}(x/\sqrt{2}) \quad (say\ P(-\infty, x))$$

$$\Phi(x) - \frac{1}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt = \frac{1}{2}\mathrm{erf}(x/\sqrt{2}).$$

Proof: Put $t = \sqrt{2}u$ and proceed, you might reach a step of

$P(-\infty, x) = P(-\infty, 0) + P(0, x)$, **where** $$P(0,x) = \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-u^2} du$$

Here you can prove that $P(0,x) = \frac{1}{2} erf(\frac{x}{\sqrt{2}})$. **This can be done by using the definition of error function in (1).**

Now you need to find $P(-\infty,0)=\frac{I}{\sqrt{\pi}}$ **where** $I=\int_{-\infty}^{0} e^{-u^2}du$ **. To find this integral you have to put *u=x* first, then *u= y* and multiply the two resulting integrals. Make the change of variables to polar coordinate you get**

$$I^2=\int_{-\infty}^{0} e^{-r^2} rdr \int_{0}^{\pi/2} d\theta$$

From this latter integral you get

$$I=\frac{\sqrt{\pi}}{2} \text{ and } \Rightarrow P(-\infty,0)=\frac{1}{2}.$$

$$\boxed{\therefore P(-\infty,x)=\frac{1}{2}+\frac{1}{2}erf\left(\frac{x}{\sqrt{2}}\right)}$$

Q. E. D.

– Complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2/2} dt = 1 - \mathrm{erf}(x/\sqrt{2}),$$

Proof: Put $t^2 = u$ and use the definition (1) of error function and the definition of $\Gamma(1/2)$.

- $\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) = \frac{\sqrt{2}}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2/2} dt\,.$

– in terms of the standard normal cumulative distribution function

$$\mathrm{erf}(x) = 2\Phi\left(x\sqrt{2}\right) - 1.$$

- Several useful facts

$$\mathrm{erf}(-x) = -\mathrm{erf}(x)$$

$$\mathrm{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{2}{\sqrt{\pi}}\frac{1}{2}\sqrt{\pi} = 1.$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\int_0^x\left(1 - t^2 + \frac{t^4}{2!} - \cdots\right)dt = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \cdots\right). \quad (|x| << 1)$$