14. Verify Cauchy–Schwarz inequality:

(i) $\mathbf{u} = (-4, 2, 1)$, $\mathbf{v} = (8, -4, -2)$

(ii) $\mathbf{u} = (0, -2, 2, 1)$, $\mathbf{v} = (-1, -1, 1, 1)$

[**Ans.:** (i) yes (ii) yes]

2.3 VECTOR SPACES

Let V be a non-empty set of objects on which the operations of addition and multiplication by scalars are defined. Here addition means a rule for assigning to each pair of objects $\mathbf{u}$, $\mathbf{v}$ in V a unique object $\mathbf{u} + \mathbf{v}$ and scalar multiplication means a rule for assigning to each scalar k and each object $\mathbf{u}$ in V a unique object $k\mathbf{u}$. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k_1, k_2 then V is called a vector space and the objects in V are called vectors.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V then $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
4. There is an object $\mathbf{0}$ in V, called zero vector, such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each object $\mathbf{u}$ in V, there exists an object $-\mathbf{u}$ in V called a negative of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k_1 is any scalar and $\mathbf{u}$ is an object in V, then $k_1\mathbf{u}$ is in V.
7. $k_1 (\mathbf{u} + \mathbf{v}) = k_1\mathbf{u} + k_1\mathbf{v}$
8. If k_1, k_2 are scalars and $\mathbf{u}$ is an object in V, then $(k_1 + k_2)\,\mathbf{u} = k_1\mathbf{u} + k_2\mathbf{u}$.
9. $k_1(k_2\mathbf{u}) = (k_1 k_2)\mathbf{u}$
10. $1\,\mathbf{u} = \mathbf{u}$

The operations of addition and scalar multiplication in these axioms are not always defined as standard vector operations (addition and scalar multiplication) on Euclidean space R^n.

The scalars may be real numbers or complex numbers. When the scalars are real numbers, the vector space is called *real vector space*, and when the scalars are complex numbers, the vector space is called *complex vector space*.

Some standard vector spaces are as follows:

(i) The set R^n under standard vector addition and scalar multiplication.

(ii) The set P_n of all polynomials of degree $\leq n$ together with the zero polynomial under addition and scalar multiplication of polynomials.

(iii) The set M_{mn} of all $m \times n$ matrices of real numbers under matrix addition and scalar multiplication.

(iv) The set $F[a, b]$ of all real-valued functions defined on the interval $[a, b]$ under addition and scalar multiplication of functions.

(v) The set $F[-\infty, \infty]$ of all real-valued functions defined for all real numbers under addition and scalar multiplication of functions.

Example 1: Determine whether the given set V is closed under the given operations:

(i) V is the set of all ordered triples of real numbers of the form $(0, y, z)$;

$$(0, y, z) + (0, y', z') = (0, y + y', z + z')$$
$$k(0, y, z) = (0, 0, kz)$$

(ii) V is the set of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a = d$ under matrix addition and scalar multiplication.

Solution:

(i) (a) $(0, y, z) + (0, y', z') = (0, y + y', z + z')$
Since y, y', z, z' are real numbers, $y + y', z + z'$ are also real numbers. Therefore, $(0, y + y', z + z')$ is in V.
Hence, V is closed under the addition operation.

(b) $k(0, y, z) = (0, 0, kz)$

If z is a real number then kz is also a real number. Therefore, $(0, 0, kz)$ is in V.
Hence, V is closed under multiplication operation.

(ii) (a) Let $\mathbf{u} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ where $a_1 = d_1$ and $a_2 = d_2$ be two objects in V.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

If $a_1 = d_1$, $a_2 = d_2$, then $a_1 + a_2 = d_1 + d_2$.
Therefore, $\mathbf{u} + \mathbf{v}$ is also an object in V.
Hence, V is closed under matrix addition.

(b) Let k be some scalar.

$$k\mathbf{u} = k\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}$$

If $a_1 = d_1$, then $ka_1 = kd_1$. Therefore, $k\mathbf{u}$ is also an object in V.
Hence, V is closed under scalar multiplication.

Example 2: Determine whether the set V of all pairs of real numbers (x, y) with the operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$ and $k(x, y) = (kx, ky)$ is a vector space.

Solution: Let $\mathbf{u} = (x_1, y_1)$, $\mathbf{v} = (x_2, y_2)$ and $\mathbf{w} = (x_3, y_3)$ are objects in V and k_1, k_2 are some scalars.

1. $\mathbf{u} + \mathbf{v} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + 1, y_1 + y_2 + 1)$
   Since x_1, x_2, y_1, y_2 are real numbers $x_1 + x_2 + 1$ and $y_1 + y_2 + 1$ are also real numbers.
   Therefore, $\mathbf{u} + \mathbf{v}$ is also an object in V.

2. $$\begin{aligned}\mathbf{u}+\mathbf{v}&=(x_1+x_2+1,\ y_1+y_2+1)\\&=(x_2+x_1+1,\ y_2+y_1+1)\\&=\mathbf{v}+\mathbf{u}\end{aligned}$$

   Hence, vector addition is commutative.

3. $$\begin{aligned}\mathbf{u}+(\mathbf{v}+\mathbf{w})&=(x_1,y_1)+[(x_2,y_2)+(x_3,y_3)]\\&=(x_1,y_1)+(x_2+x_3+1,\ y_2+y_3+1)\\&=[x_1+(x_2+x_3+1)+1,\ y_1+(y_2+y_3+1)+1]\\&=[(x_1+x_2+1)+x_3+1,\ (y_1+y_2+1)+y_3+1]\\&=(x_1+x_2+1,\ y_1+y_2+1)+(x_3,y_3)\\&=(\mathbf{u}+\mathbf{v})+\mathbf{w}\end{aligned}$$

   Hence, vector addition is associative.

4. Let (a, b) be an object in V such that

$$\begin{aligned}(a,b)+\mathbf{u}&=\mathbf{u}\\(a,b)+(x_1,y_1)&=(x_1,y_1)\\(a+x_1+1,\ b+y_1+1)&=(x_1,y_1)\\a+x_1+1=x_1\quad&,\quad b+y_1+1=y_1\\a=-1\quad&,\quad b=-1\end{aligned}$$

   Also, $\mathbf{u}+(a,b)=\mathbf{u}$

   Hence, $(-1,-1)$ is the zero vector in V.

5. Let (a, b) be an object in V such that

$$\begin{aligned}\mathbf{u}+(a,b)&=(-1,-1)\\(x_1,y_1)+(a,b)&=(-1,-1)\\(x_1+a+1,\ y_1+b+1)&=(-1,-1)\\x_1+a+1=-1\quad&,\quad y_1+b+1=-1\\a=-x_1-2\quad&,\quad b=-y_1-2\end{aligned}$$

   Also, $(a,b)+\mathbf{u}=(-1,-1)$

   Hence, $(-x_1-2,\ -y_1-2)$ is the negative of $\mathbf{u}$ in V.

6. $$\begin{aligned}k_1\mathbf{u}&=k_1(x_1,y_1)\\&=(k_1x_1,\ k_1y_1)\end{aligned}$$

   Since k_1x_1 and k_1y_1 are real numbers, $k_1\mathbf{u}$ is an object in V

   Hence, V is closed under scalar multiplication.

7. $$\begin{aligned}k_1(\mathbf{u}+\mathbf{v})&=k_1(x_1+x_2+1,\ y_1+y_2+1)\\&=(k_1x_1+k_1x_2+k_1,\ k_1y_1+k_1y_2+k_1)\\&\neq k_1\mathbf{u}+k_1\mathbf{v}\end{aligned}$$

V is not distributive under scalar multiplication.
Hence, V is not a vector space.

Example 3: Determine whether the set R^+ of all positive real numbers with operations

$$x + y = xy \text{ and } kx = x^k \text{ is a vector space.}$$

Solution: Let x, y and z be positive real numbers in R^+ and k_1, k_2 are some scalars

1. $x + y = xy$, is also a positive real number
   R^+ is closed under vector addition.
2. $x + y = xy = yx = y + x$
   Vector addition is commutative.
3. $x+(y+z) = x(y+z) = x(yz) = (xy)z = (x+y)z = (x+y)+z$
   Vector addition is associative.
4. Let a be an object in R^+ such that
   $$\begin{aligned} a + x &= x \\ ax &= x \\ a &= 1 \end{aligned}$$
   Also $x + a = x$
   Hence, $\mathbf{0} = 1$ is the zero vector in V.
5. Let a be an object in R^+ such that
   $$\begin{aligned} x + a &= 1 \\ xa &= 1 \\ a &= \frac{1}{x} \end{aligned}$$
   Also, $a + x = 1$
   Hence, $\frac{1}{x}$ is the negative of x in R^+.
6. If k_1 is real then $k_1 x = x^{k_1}$ is a positive real number for all x in R^+.
   R^+ is closed under scalar multiplication.
7. $$\begin{aligned} k_1(x+y) &= k_1(xy) = (xy)^{k_1} \\ &= x^{k_1} y^{k_1} = (k_1 x)(k_1 y) = k_1 x + k_1 y \end{aligned}$$
   Scalar multiplication is distributive with respect to vector addition in R^+.
8. $$\begin{aligned} (k_1 + k_2)x &= x^{k_1+k_2} = x^{k_1} x^{k_2} \\ &= (k_1 x)(k_2 x) = k_1 x + k_2 x \end{aligned}$$
   Scalar multiplication is distributive with respect to scalar addition in R^+.
9. $$\begin{aligned} k_1(k_2 x) &= k_1 x^{k_2} = (x^{k_2})^{k_1} \\ &= x^{k_2 k_1} = x^{k_1 k_2} = (k_1 k_2)x \end{aligned}$$
   Scalar and vector multiplications are compatible with each other.
10. $1x = x^1 = x$

All axioms are satisfied by R^+ under given operations. Hence, R^+ is a vector space under given operations.

Example 4: Why are the following sets not vector spaces under the given operations? Justify your answer.

(i) The set of all pairs of real numbers (x, y) with the operation $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k(x, y) = (2kx, 2ky)$.

(ii) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (z_1 + z_2, y_1 + y_2, x_1 + x_2)$

Solution: (i) 1 is a scalar.

$$1(x, y) = (2x, 2y) \neq (x, y)$$

Axiom 10 fails. Hence, given set is not a vector space.

(ii) $(x_1, y_1, z_1) + \{(x_2, y_2, z_2) + (x_3, y_3, z_3)\}$

$$\begin{aligned}
&= (x_1, y_1, z_1) + (z_2 + z_3, y_2 + y_3, x_2 + x_3) \\
&= \{z_1 + (x_2 + x_3), y_1 + (y_2 + y_3), x_1 + (z_2 + z_3)\} \\
&= \{(z_1 + x_2) + x_3, (y_1 + y_2) + y_3, (x_1 + z_2) + z_3\} \\
&= (x_1 + z_2, y_1 + y_2, z_1 + x_2) + (z_3, y_3, x_3) \\
&= \{(z_1, y_1, x_1) + (x_2, y_2, z_2)\} + (z_3, y_3, x_3)
\end{aligned}$$

Given set is not associative under vector addition. Axiom 3 fails. Hence, the given set is not a vector space.

Example 5: Check whether $V = R^2$ is a vector space with respect to the operations $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 - 2, y_1 + y_2 - 3)$ and $k(x, y) = (kx + 2k - 2, ky - 3k + 3)$, k is a real number.

Solution: Let $\mathbf{u} = (x_1, y_1)$, $\mathbf{v} = (x_2, y_2)$ and $\mathbf{w} = (x_3, y_3)$ are objects in R^2 and k_1, k_2 are some real scalars.

1. $\mathbf{u} + \mathbf{v} = (x_1, y_1) + (x_2, y_2)$

$$= (x_1 + x_2 - 2, y_1 + y_2 - 3) \text{ which is also in } R^2.$$

R^2 is closed under vector addition.

2. $\mathbf{u} + \mathbf{v} = (x_1 + x_2 - 2, y_1 + y_2 - 3)$

$$\begin{aligned}
&= (x_2 + x_1 - 2, y_2 + y_1 - 3) \\
&= (x_2, y_2) + (x_1, y_1) \\
&= \mathbf{v} + \mathbf{u}
\end{aligned}$$

Vector addition is commutative.

3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (x_1, y_1) + \{(x_2, y_2) + (x_3, y_3)\}$

$$\begin{aligned}
&= (x_1, y_1) + (x_2 + x_3 - 2, y_2 + y_3 - 3) \\
&= (x_1 + (x_2 + x_3 - 2) - 2, y_1 + (y_2 + y_3 - 3) - 3) \\
&= ((x_1 + x_2 - 2) + x_3 - 2, (y_1 + y_2 - 3) + y_3 - 3) \\
&= (x_1 + x_2 - 2, y_1 + y_2 - 3) + (x_3, y_3) \\
&= \{(x_1, y_1) + (x_2, y_2)\} + (x_3, y_3) \\
&= (\mathbf{u} + \mathbf{v}) + \mathbf{w}
\end{aligned}$$

Vector addition is commutative.

4. Let (a, b) be an object in R^2 such that

$$\begin{aligned} (a, b) + \mathbf{u} &= \mathbf{u} \\ (a, b) + (x_1, y_1) &= (x_1, y_1) \\ (a + x_1 - 2, b + y_1 - 3) &= (x_1, y_1) \\ a + x_1 - 2 = x_1 \quad &, \quad b + y_1 - 3 = y_1 \\ a = 2 \quad &, \quad b = 3 \end{aligned}$$

   Also, $\mathbf{u} + (a, b) = \mathbf{u}$
   Hence, $(2, 3)$ is the zero vector in V.

5. Let (a, b) be an object in R^2 such that

$$\begin{aligned} \mathbf{u} + (a, b) &= (2, 3) \\ (x_1, y_1) + (a, b) &= (2, 3) \\ (x_1 + a - 2, y_1 + b - 3) &= (2, 3) \\ x_1 + a - 2 = 2 \quad &, \quad y_1 + b - 3 = 3 \\ a = -x_1 + 4 \quad &, \quad b = -y_1 + 6 \end{aligned}$$

   Also, $(a, b) + \mathbf{u} = (2, 3)$
   Hence, $(-x_1 + 4, -y_1 + 6)$ is the negative of $\mathbf{u}$ in V.

6. If k_1 is a real number then $k_1(x_1, y_1) = (k_1x_1 + 2k_1 - 2, k_1y - 3k_1 + 3)$ is also in R^2. R^2 is closed under scalar multiplication.

7. $$\begin{aligned} k_1(\mathbf{u} + \mathbf{v}) &= k_1\{(x_1, y_1) + (x_2, y_2)\} \\ &= k_1(x_1 + x_2 - 2, y_1 + y_2 - 3) \\ &= \big(k_1(x_1 + x_2 - 2) + 2k_1 - 2, k_1(y_1 + y_2 - 3) - 3k_1 + 3\big) \\ &= (k_1x_1 + 2k_1 - 2 + k_1x_2 - 2k_1, k_1y_1 - 3k_1 + 3 + k_1y_2 - 3k_1) \\ &\neq k_1\mathbf{u} + k_1\mathbf{v} \end{aligned}$$

   Scalar multiplication is not distributive with respect to vector addition in R^2.
   Hence, R^2 is not a vector space.

Exercise 2.2

1. Determine whether the given set V is closed under the given operations.

   (i) The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2.

   (ii) The set of all polynomials of the form $a_0 + a_1x + a_2x^2$ where a_0, a_1, a_2 are real numbers and $a_2 = a_3 + 1$ with operations defined as

$$\begin{aligned} &(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) \\ &\quad = (a_0 + b_0) + (a_1 + b_1)x + \\ &\quad\quad (a_2 + b_2)x^2 \\ &k(a_1 + a_2x + a_3x^2) \\ &\quad = (ka_0) + (ka_1)x + (ka_2)x^2 \end{aligned}$$

   (iii) The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with the standard matrix addition and scalar multiplication.

[**Ans. :** (i) yes (ii) no (iii) no]

2. Determine which sets are vector spaces under the given operations:

(i) The set of all ordered triples of real numbers (x, y, z) with the operations

$$(x_1, y_1, z_1)+(x_2, y_2, z_2) = (x_2, y_1+y_2, z_2)$$
$$k(x, y, z) = (kx, ky, kz)$$

(ii) The set of all ordered triples of real numbers of the form $(0, 0, z)$ with the operations

$$(0, 0, z_1)+(0, 0, z_2) = (0, 0, z_1+z_2)$$
$$k(0, 0, z) = (0, 0, kz)$$

(iii) The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with the operations defined as

$$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}+\begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix}=\begin{bmatrix} a+c & 1 \\ 1 & b+d \end{bmatrix}$$
$$k\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}=\begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix}$$

(iv) The set of all ordered pairs of real numbers (x, y), where $x \le 0$, with the usual operations in R^2

[**Ans. :** (i) no (ii) yes (iii) yes (iv) no]

3. Show that the set V of all pairs of real numbers of the form $(1, x)$ with the operations defined as

$$(1, x_1)+(1, x_2) = (1, x_1+x_2)$$
$$k(1, x) = (1, kx)$$

is a vector space.

4. Show that the set M_{nn} of all $n \times n$ matrices with real entries is a vector space under the matrix addition and scalar multiplication.

2.4 SUBSPACES

A non-empty subset W of a vector space V is called a subspace of V if W is itself a vector space under the operations defined on V.

Note: Every vector space has at least two subspaces, itself and the subspace $\{\mathbf{0}\}$. The subspace $\{\mathbf{0}\}$ is called the zero subspace consisting only of the zero vector.

Since W is the part of a vector space V, most of the axioms are true for W as they are true for V. The following theorem shows that to prove W a subspace of a vector space V, we need to verify only the closure property with respect to the operations defined on V.

Theorem 2.2: If W is a non-empty subset of vector space V, then W is a subspace of V if and only if the following axioms hold:

Axiom 1: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W then $\mathbf{u}+\mathbf{v}$ is in W.

Axiom 2: If k is any scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

Example 1: Show that $W = \{(x, y) \mid x = 3y\}$ is a subspace of R^2. State all possible subspaces of R^2.

Solution: Let $\mathbf{u}=\{(x_1, y_1)\,|\,x_1=3y_1\}$ and $\mathbf{v}=\{(x_2, y_2)\,|\,x_2=3y_2\}$ are in W and k is any scalar.

Axiom 1:

$$\begin{aligned}\mathbf{u}+\mathbf{v}&=(x_1, y_1)+(x_2, y_2)\\&=(x_1+x_2, y_1+y_2)\end{aligned}$$

But

$$x_1=3y_1 \text{ and } x_2=3y_2$$

$$\therefore x_1+x_2=3(y_1+y_2)$$

$$\mathbf{u}+\mathbf{v}=\{(x_1+x_2, y_1+y_2)\,|\,x_1+x_2=3(y_1+y_2)\}$$

Thus, $\mathbf{u}+\mathbf{v}$ is in W.

Axiom 2:

$$\begin{aligned}k\mathbf{u}&=k(x_1, y_1)\\&=(kx_1, ky_1)\end{aligned}$$

But

$$x_1=3y_1$$

$$\therefore kx_1=3(ky_1)$$

$$k\mathbf{u}=\{(kx_1, ky_1)\,|\,kx_1=3\,(ky_1)\}$$

Thus, $k\mathbf{u}$ is in W.
Hence, W is a subspace of R^2.
All possible subspaces of R^2 are

(i) $\{\mathbf{0}\}$ (ii) R^2 (iii) Lines passing through the origin.

Example 2: Check whether the following are subspaces of R^3. Justify your answer. State all possible subspaces of R^3.

(i) $W=\{(x, 0, 0)\,|\,x\in R\}$

(ii) $W=\{(x, y, z)\,|\,x^2+y^2+z^2\le 1\}$

(iii) $W=\{(x, y, z)\,|\,y=x+z+1\}$

Solution: (i) Let $\mathbf{u}=\{(x_1, 0, 0)\,|\,x_1\in R\}$ and $\mathbf{v}=(x_2, 0, 0)\,|\,x_2\in R\}$ be in W, and k be any scalar.

Axiom 1:

$$\begin{aligned}\mathbf{u}+\mathbf{v}&=(x_1, 0, 0)+(x_2, 0, 0)\\&=(x_1+x_2, 0, 0)\end{aligned}$$

Since R is closed under addition, x_1+x_2 is in R.
Thus, $\mathbf{u}+\mathbf{v}$ is in W.

Axiom 2:

$$\begin{aligned}k\mathbf{u}&=k(x_1, 0, 0)\\&=(kx_1, 0, 0)\end{aligned}$$

Since R is closed under scalar multiplication, kx_1 is in R.
Thus, $k\mathbf{u}$ is in W.
Hence, W is a subspace of R^3.

(ii) Let $\mathbf{u}=(1, 0, 0)$ and $\mathbf{v}=(0, 0, 1)$ be two vectors of the set W satisfying the condition $x^2+y^2+z^2\le 1$.

Axiom 1:
$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (1, 0, 0)+(0, 0, 1)\\ &= (1, 0, 1)\end{aligned}$$

Here $x^2 + y^2 + z^2 = 2 > 1$. Thus, $\mathbf{u} + \mathbf{v}$ is not in W.
W is not closed under addition and hence is not a subspace of R^3.

(iii) Let $\mathbf{u} = \{(x_1, y_1, z_1) \mid y_1 = x_1 + z_1 + 1\}$ and $\mathbf{v} = \{(x_2, y_2, z_2) \mid y_2 = x_2 + z_2 + 1\}$ be in W.

Axiom 1:
$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (x_1, y_1, z_1)+(x_2, y_2, z_2)\\ &= (x_1+x_2, y_1+y_2, z_1+z_2)\end{aligned}$$

But
$$\begin{aligned}y_1 &= x_1+z_1+1,\ y_2 = x_2+z_2+1\\ \therefore\ y_1+y_2 &= (x_1+z_1+1)+(x_2+z_2+1)\\ &= (x_1+x_2)+(z_1+z_2)+2\end{aligned}$$

Thus, $\mathbf{u} + \mathbf{v}$ is not in W.
W is not closed under addition and hence is not a subspace of R^3.
All possible subspaces of R^3 are (i) $\{\mathbf{0}\}$ (ii) Lines passing through the origin. (iii) Planes through the origin (iv) R^3.

Example 3: Show that the set of solution vectors of a homogenous linear system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns, is a subspace of R^n.

Solution: Let W be the set of solution vectors of $A\mathbf{x} = \mathbf{0}$.
Case I: If system has only a trivial solution ($\mathbf{x} = \mathbf{0}$) then W has at least one vector $\mathbf{0}$ and hence is a subspace of R^3.
Case II: In case of non-trivial solution, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solution vectors in W and k is any scalar.

Axiom 1:
$$\begin{aligned}A(\mathbf{x}_1+\mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2\\ &= \mathbf{0}+\mathbf{0} \qquad [\because A\mathbf{x}_1 = \mathbf{0},\ A\mathbf{x}_2 = \mathbf{0}]\\ &= \mathbf{0}\end{aligned}$$

Thus, $\mathbf{x}_1 + \mathbf{x}_2$ is also a solution vector in W.

Axiom 2:
$$\begin{aligned}A(k\mathbf{x}_1) &= k(A\mathbf{x}_1) \qquad [\because k \text{ is a scalar}]\\ &= \mathbf{0}\end{aligned}$$

Thus, $k\mathbf{x}_1$ is also a solution vector in W.
Hence, W is a subspace of R^n.

Example 4: Show that the following sets are the subspaces of the respective real vector space V under the standard operations:

(i) $W = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_0 = 0\}$, $\qquad V = P_3$

(ii) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a+b+c+d=0 \right\}$, $V = M_{22}$

(iii) $W = \{ A_{nn} \mid AB = BA \quad \text{for a fixed } B_{nn} \}$, $V = M_{nn}$

(iv) $W = \{ f \mid f(x) = a_1 + a_2 \sin x, \quad \text{where } a_1 \text{ and } a_2 \text{ are real numbers} \}$,
$V = F(-\infty, \infty)$

Solution: (i) Let $\mathbf{p}_1 = a_0 + a_1x + a_2x^2 + a_3x^3$ and $\mathbf{p}_2 = b_0 + b_1x + b_2x^2 + b_3x^3$ be in W such that $a_0 = 0$, $b_0 = 0$ and k is any scalar.

Axiom 1: $$\begin{aligned} \mathbf{p}_1 + \mathbf{p}_2 &= (a_0 + a_1x + a_2x^2 + a_3x^3) + (b_0 + b_1x + b_2x^2 + b_3x^3) \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 \end{aligned}$$

But $$a_0 = 0, b_0 = 0$$
$$\therefore a_0 + b_0 = 0$$

Thus, $\mathbf{p}_1 + \mathbf{p}_2$ is in W.

Axiom 2: $$k\,\mathbf{p}_1 = ka_0 + ka_1x + ka_2x^2 + ka_3x^3$$

But $$a_0 = 0$$
$$\therefore\ ka_0 = 0$$

Thus, $k\mathbf{p}_1$ is in W.
Hence, W is a subspace of P_3.

(ii) Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ be in M_{22} such that $a_1 + b_1 + c_1 + d_1 = 0$, $a_2 + b_2 + c_2 + d_2 = 0$ and k is any scalar.

Axiom 1: $$\begin{aligned} A_1 + A_2 &= \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \end{aligned}$$

But $a_1 + b_1 + c_1 + d_1 = 0$, $a_2 + b_2 + c_2 + d_2 = 0$
$$\begin{aligned} &\therefore (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2) + (d_1 + d_2) \\ &= (a_1 + b_1 + c_1 + d_1) + (a_2 + b_2 + c_2 + d_2) = 0 \end{aligned}$$

Thus, $A_1 + A_2$ is in W.

Axiom 2: $$kA_1 = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & kd_1 \end{bmatrix}$$

But $a_1 + b_1 + c_1 + d_1 = 0$
$$\therefore ka_1 + kb_1 + kc_1 + kd_1 = k(a_1 + b_1 + c_1 + d_1) = 0$$

Thus, kA_1 is in W.
Hence, W is a subspace of M_{22}.

(iii) Let A_1 and A_2 be in W such that $A_1B = BA_1$, $A_2B = BA_2$ and k be any real scalar.

Axiom 1: $$\begin{aligned}(A_1 + A_2)B &= A_1B + A_2B\\ &= BA_1 + BA_2\\ &= B(A_1 + A_2)\end{aligned}$$

Thus, $A_1 + A_2$ is in W.

Axiom 2: $$\begin{aligned}(kA_1)B &= k(A_1B)\\ &= k(BA_1)\\ &= B(kA_1) \qquad [\because k \text{ is a scalar}]\end{aligned}$$

Thus, kA_1 is in W.
Hence, W is a subspace of M_{nn}.

(iv) Let $f_1(x) = a_1 + a_2 \sin x$ and $f_2(x) = b_1 + b_2 \sin x$ be in W where a_1, a_2, b_1, b_2 are real numbers and k be any scalar.

Axiom 1: $$\begin{aligned}f_1(x) + f_2(x) &= (a_1 + a_2 \sin x) + (b_1 + b_2 \sin x)\\ &= (a_1 + b_1) + (a_2 + b_2)\sin x\end{aligned}$$

Since a_1, b_1, a_2, b_2 are real numbers, $(a_1 + b_1)$ and $(a_2 + b_2)$ are also real numbers.
Thus, $f_1(x) + f_2(x)$ is in W.

Axiom 2: $$\begin{aligned}kf_1(x) &= k(a_1 + a_2 \sin x)\\ &= ka_1 + ka_2 \sin x\end{aligned}$$

Since k is a real scalar, ka_1 and ka_2 are real numbers.
Thus, $kf_1(x)$ is in W.
Hence, W is a subspace of $F(-\infty, \infty)$.

Example 5: State only one axiom that fails to hold for each of the following sets W to be subspaces of the respective real vector space V under the standard operations:

(i) $W = \{(x, y) \mid x^2 = y^2\}$, $V = R^2$

(ii) $W = \{(x, y) \mid xy \geq 0\}$, $V = R^2$

(iii) $W = \{A_{n\times n} \mid A\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}\}$, $V = M_{nn}$

(iv) $W = \{f \mid f(x) \leq 0, \forall x\}$, $V = F(-\infty, \infty)$

(v) $W = \{a_0 + a_1x + a_2x^2 + a_3x^3, \forall x$ where a_0, a_1, a_2 and a_3 are integers$\}$, $V = P_3$

Solution: (i) Let $\mathbf{u} = (-1, 1)$ and $\mathbf{v} = (2, 2)$ be two vectors of the set W such that $x^2 = y^2$.

Axiom 1: $$\begin{aligned}\mathbf{u} + \mathbf{v} &= (-1, 1) + (2, 2)\\ &= (1, 3)\end{aligned}$$

Here $1^2 \neq 3^2$. Thus, $\mathbf{u}+\mathbf{v}$ is not in W.
W is not closed under addition and hence is not a subspace of R^2.

(ii) Let $\mathbf{u} = (-2, -3)$ and $\mathbf{v} = (3, 1)$ be two vectors of the set W such that $xy \geq 0$.

Axiom 1: $$\mathbf{u}+\mathbf{v} = (-2,-3)+(3,1) = (1,-2)$$

Here $1(-2) = -2 < 0$. Thus, $\mathbf{u}+\mathbf{v}$ is not in W.
W is not closed under addition and hence is not a subspace of R^2.

(iii) From the definition of W, it is clear that W is the set of all non-singular matrices of order n so that $A\mathbf{x} = \mathbf{0}$ has only trivial solution ($\mathbf{x} = \mathbf{0}$)

$$\text{Let } A_1 = \begin{bmatrix} 3 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}_{n\times n} \quad \text{and} \quad A_2 = \begin{bmatrix} 2 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix}_{n\times n}$$

are two matrices in W such that $|A_1| \neq 0$ and $|A_2| \neq 0$.

Axiom 1: $$A_1 + A_2 = \begin{bmatrix} 5 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n}$$

$|A_1 + A_2| = 0$. Thus, $A_1 + A_2$ is not in W.

W is not closed under addition and hence is not a subspace of M_{nn}.

(iv) W is the set of all negative functions of x. Let $f(x)$ is in W such that $f(x) \leq 0$

Axiom 2: If $k = -2$, then

$$k\,f(x) = -2f(x) > 0 \quad [\because f(x) \leq 0]$$

Thus, $kf(x)$ is not in W.
W is not closed under scalar multiplication and hence is not a subspace of $F(-\infty, \infty)$.

(v) Let $\mathbf{u} = a_0 + a_1x + a_2x^2 + a_3x^3$ be in W, where a_0, a_1, a_2, a_3, are integers.

Axiom 2: If $k = \frac{1}{2}$, then

$$\frac{1}{2}\mathbf{u} = \frac{a_0}{2} + \frac{a_1}{2}x + \frac{a_2}{2}x^2 + \frac{a_3}{2}x^3$$

Since $\frac{a_0}{2}, \frac{a_1}{2}, \frac{a_2}{2}, \frac{a_3}{2}$ are not necessarily to be integers, $\frac{1}{2}\mathbf{u}$ is not in W.

W is not closed under scalar multiplication and hence is not a subspace of P_3.

2.5 LINEAR COMBINATION

A vector **v** is called a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ if it can be expressed as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $k_1, k_2, \dots, k_r$ are scalars.

Note: If $r = 1$, then $\mathbf{v} = k_1\mathbf{v}_1$. This shows that a vector **v** is a linear combination of a single vector $\mathbf{v}_1$ if it is a scalar multiple of $\mathbf{v}_1$.

Vector Expressed as a Linear Combination of Given Vectors

The method to check if a vector **v** is a linear combination of the given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ is as follows:

1. Express **v** as linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$
$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r \tag{2.4}$$
2. If the system of equations in (1) is consistent then **v** is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$. If it is inconsistent, then v is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$.

Note: To express **v** as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$, solve the system of equations in (2.4) directly to determine scalars $k_1, k_2, \dots, k_r$.

Example 1: Which of the following are linear combinations of $\mathbf{v}_1 = (0, -2, 2)$ and $\mathbf{v}_2 = (1, 3, -1)$?

(i) (3, 1, 5) (ii) (0, 4, 5)

Solution: Let $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$

(i) $$\begin{aligned}(3, 1, 5) &= k_1(0, -2, 2) + k_2(1, 3, -1)\\ &= (k_2, -2k_1 + 3k_2, 2k_1 - k_2)\end{aligned}$$

Equating corresponding components,

$$\begin{aligned} k_2 &= 3\\ -2k_1 + 3k_2 &= 1\\ 2k_1 - k_2 &= 5\end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{rr|r} 0 & 1 & 3\\ -2 & 3 & 1\\ 2 & -1 & 5\end{array}\right]$$