GRADIENT, DIVERGENCE & CURL-PHYSICAL SIGNIFICANCE

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GRADIENT OF A SCALAR FIELD

GRADIENT

- The gradient of a scalar field S is a vector whose magnitude at any point is equal to the maximum rate of increase of S at that point and whose direction is along the normal to the level surface at that point.

- Grad S = $\partial S/\partial n\ \hat{\boldsymbol{n}}$

GRADIENT

- Gradient is a *differential operator* by means of which we can associate a vector field with a scalar field.

GRADIENT

- Let $S(x, y, z)$ be a scalar point function depending on the three Cartesian coordinates in space.
- Suppose $\partial S/\partial x$, $\partial S/\partial y$ and $\partial S/\partial z$ be the partial derivatives along the three perpendicular axes respectively.
- grad $S = \nabla S = \mathbf{i}\,(\partial S/\partial x) + \mathbf{j}\,(\partial S/\partial y) + \mathbf{k}\,(\partial S/\partial z)$

  where $\nabla = \mathbf{i}\,(\partial/\partial x) + \mathbf{j}\,(\partial/\partial y) + \mathbf{k}\,(\partial/\partial z)$

Gradient- Physical Significance

- **Gradient**: is a vector; hence, we can fully identify it using two pieces of information:
- *A-Direction*: it points in the direction of the biggest increase in the function.
- *B-Magnitude*: its magnitude determines the slope(the derivative) of that direction.

GRADIENT

- Gradient tells you how much something changes as you move from one point to another (such as the pressure in a stream).
- The gradient always points in the direction of the maximum rate of change in a field.

EXAMPLE

- Intensity of electric field **E**,(a vector quantity) is the gradient of potential V(a scalar quantity) with negative sign i.e.,
- $$\mathbf{E} = -\text{grad } V$$
- The negative sign indicates that the direction of field intensity is opposite to the direction of increase of potential

Divergence of a Vector Field

DIVERGENCE

- The divergence of a vector field at any point is defined as
  " the amount of flux per unit volume diverging from that point."

DIVERGENCE

- Divergence of **A** =

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A}$$

$$= (\mathbf{i}\,(\partial/\partial x) + \mathbf{j}\,(\partial/\partial y) + \mathbf{k}(\partial/\partial z)) \cdot (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)$$

$$= (\partial A_x/\partial x) + (\partial A_y/\partial y) + (\partial A_z/\partial z)$$

Divergence
Physical Significance

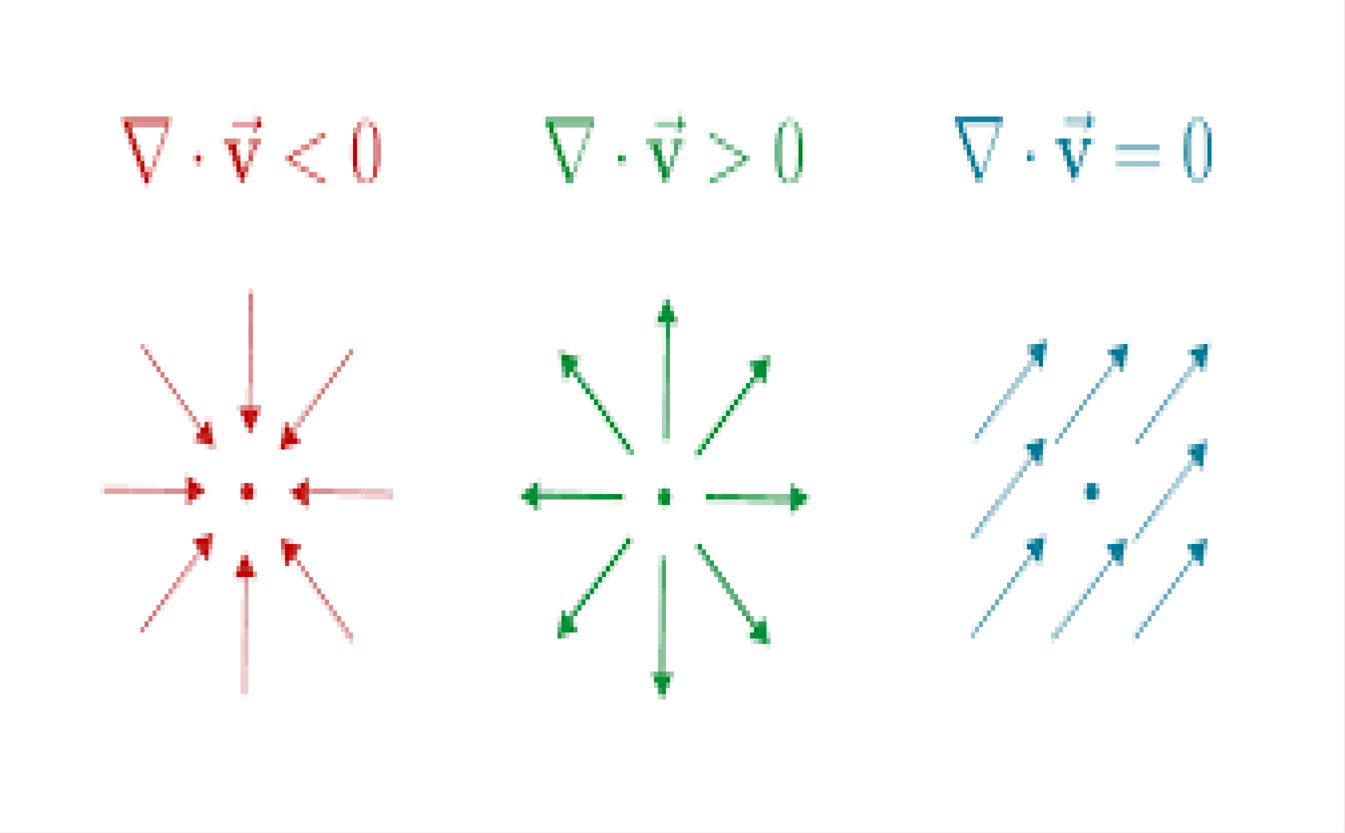
- Divergence of vector quality indicates how much the vector quality **spreads out from the certain point.**
- Imagine a fluid, with the vector field representing the velocity of the fluid at each point in space. Divergence measures the net flow of fluid out of (i.e., diverging from) a given point. If fluid is instead flowing into that point, the divergence will be negative.

DIVERGENCE

PHYSICAL SIGNIFICANCE

- A point or region with positive divergence is often referred to as a "source" (of fluid, or whatever the field is describing), while a point or region with negative divergence is a "sink".
- If div **V** = **0**, then the fluid entering and leaving is the same . i.e., the fluid is incompressible and the vector is called Solenoidal vector

DIVERGENCE



EXAMPLES

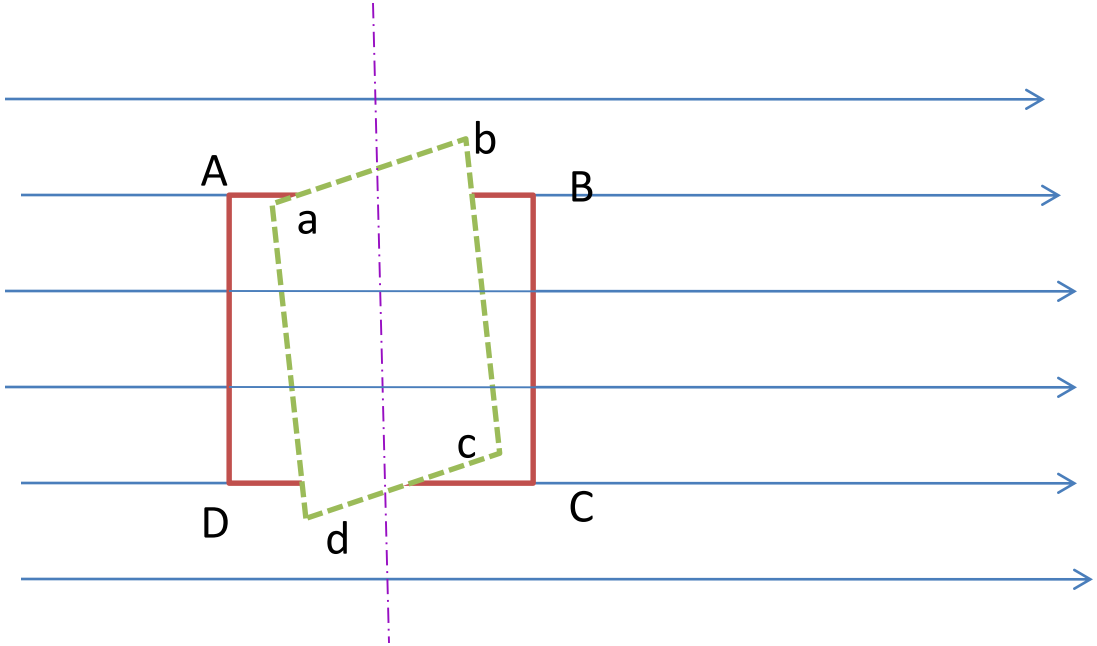
- The divergence of current density J gives the amount of charge flowing per unit volume per second from small element of closed surface around that point.
- div J =0 , indicates medium is of free charges.

Curl of a Vector Field

CURL

- The maximum line integral of the vector per unit area.
- Curl **A** = ∇ X **A**
- $(\mathbf{i}\,(\partial/\partial x) + \mathbf{j}\,(\partial/\partial y) + \mathbf{k}(\partial/\partial z))$ **X** $(\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$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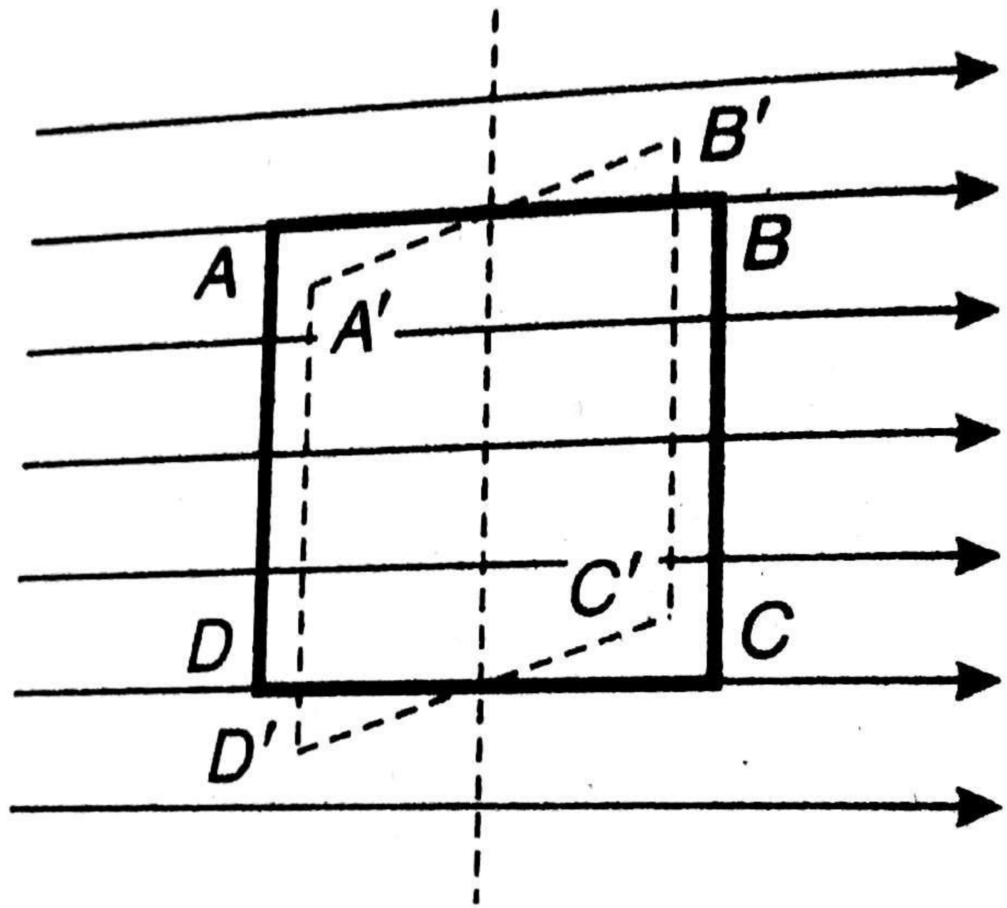
B′
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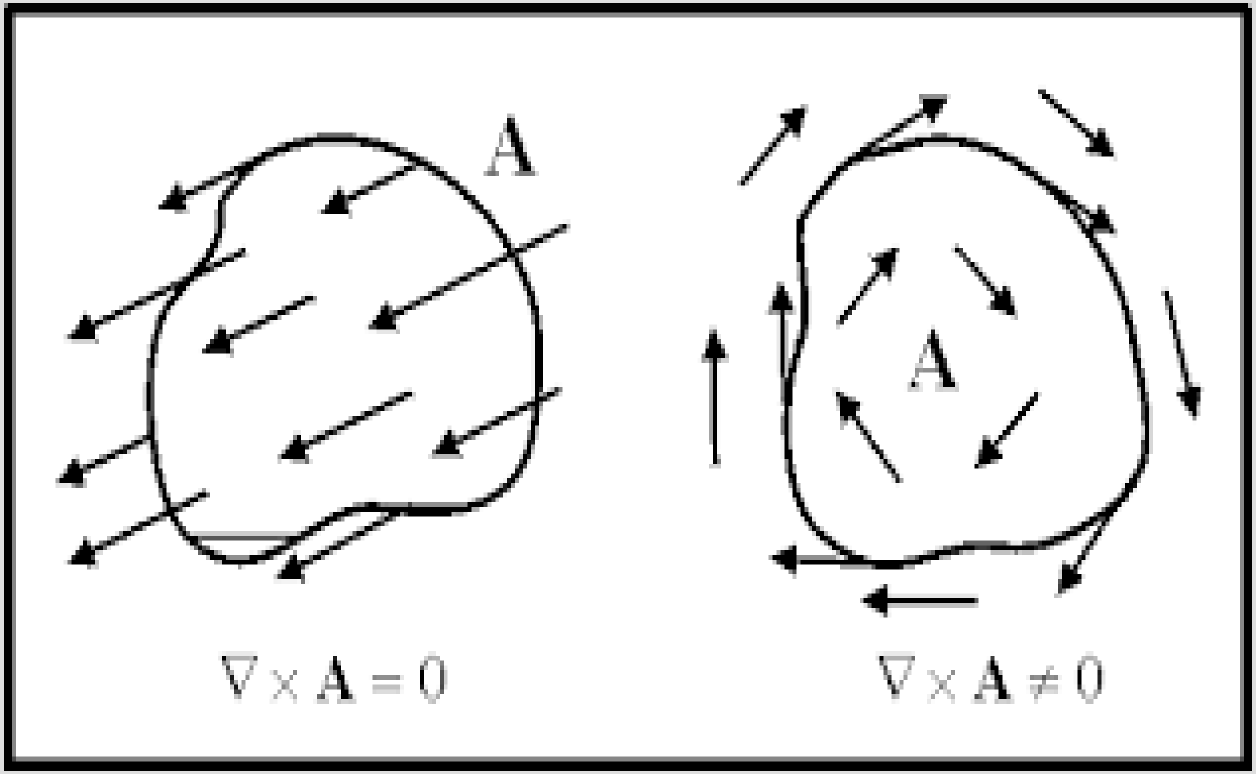
C′
D
C
D′

Curl – Physical Significance

- *Curl of vector quality indicates that how much the vector quality **curls or twist around**.*
- Example: Rotating water in a bucket has curl. You can measure curl by putting a piece of dust in the liquid and seeing if it spins around its own axis.

CURL



Examples

- Curl of an electric field results the rate of change of magnetic field in perpendicular direction.
- Curl of magnetic field results the flow of electricity.
- Curl **V** = 2 **ω**

  Where **V** is the linear velocity and **ω** is the angular velocity of a rigid body.

NATURE OF DIVERGENCE AND CURL OF FLUX VECTOR

S. No.	Nature of divergence & curl	Fluid nature
1	div A = 0	Incompressible
2	div A ≠ 0	Compressible
3	curl A = 0	Irrotational
4	curl A ≠ 0	Rotational
5	div A = 0 & curl = 0	Irrotational flow of incompressible fluid
6	div A ≠ 0 & curl = 0	Irrotatioal flow of compressible fluid
7	div A = 0 & curl A≠0	Rotatioal flow of incompressible fluid
8	div A ≠ 0 & curl A≠0	Rotational flow of compressible fluid

If $\phi(x, y, z) = 3x^2y - y^3z^2$, find the value of grad ϕ at point $(1, -2, -1)$.

[**Ans.** $-\mathbf{i}\,12 - \mathbf{j}\,9 - \mathbf{k}\,16$].

[**Hint.** $\text{grad}\,\phi = \nabla\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$

$$= \mathbf{i}\,6xy + \mathbf{j}(3x^2 - 3y^2z^2) - \mathbf{k}\,2y^3z$$

Now substitute $x = 1$, $y = -2$ and $z = -1$].

EXAMPLE *If* $\phi(x, y, z) = 3x^2 y - y^2 z^2$, *find* $\vec{\nabla}\phi$ *at* $(1, -2, 1)$.

Solution. $$\vec{\nabla}\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi$$

$$\therefore \quad \vec{\nabla}\phi = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(3x^2 y - y^2 z^2)$$

or $$\vec{\nabla}\phi = \mathbf{i}\frac{\partial}{\partial x}(3x^2 y - y^2 z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2 y - y^2 z^2) + \mathbf{k}\frac{\partial}{\partial z}(3x^2 y - y^2 z^2)$$

$$= \mathbf{i}\,(6x\,y) + \mathbf{j}\,(3x^2 - 2yz^2) - \mathbf{k}\,(2y^2 z)$$

At $(1, -2, 1)$

$$(\vec{\nabla}\phi)_{(1,-2,1)} = \mathbf{i}\,[6(1)(-2)] + \mathbf{j}\,[3(1)^2 - 2(-2)(1)^2] - \mathbf{k}\,[2(-2)^2(1)]$$

$$= -12\,\mathbf{i} + 7\,\mathbf{j} - 8\,\mathbf{k}$$

Evaluate div **F**, *where*

$$\mathbf{F} = 2x^3 z\,\mathbf{i} - x y^2 z\,\mathbf{j} + 3 y^2 x\,\mathbf{k}$$

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \cdot (2x^3 z\,\mathbf{i} - x y^2 z\,\mathbf{j} + 3 y^2 x\,\mathbf{k})$$

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(2x^3 z) - \frac{\partial}{\partial y}(x y^2 z) + \frac{\partial}{\partial z}(3 y^2 x)$$

$$= 6x^2 z - 2xyz + 0$$

$$= 6x^2 z - 2xyz.$$

EXAMPLE *Find the value of constant c for which the vector* $\mathbf{A} = \mathbf{i}(x + 3y) + \mathbf{j}(y - 2z) + \mathbf{k}(x + cz)$ *is solenoidal.*

Solution. The vector **A** is solenoidal if the divergence is zero. We know that

$$\text{div } \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + cz)$$

$$= 1 + 1 + c = 2 + c$$

$$2 + c = 0 \quad \text{or} \quad c = -\mathbf{2}.$$

PROBLEM *Find curl* **A**, *where* $\mathbf{A} = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. *Is the motion rotational or irrotational?*

Solution. Given $\mathbf{A} = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. We know that

$$\text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 0 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2y)\right] + \mathbf{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(2x)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(2x)\right]$$

$$= \mathbf{i}\,(0) + \mathbf{j}\,(0) + \mathbf{k}\,(0)$$

$$\therefore \text{curl } \mathbf{A} = 0$$

Therefore, vector **A** is irrotational.

Thank You