Module 6.3: Properties of Fourier Transforms

1. Linear Property:

If $F(P)$ and $G(P)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$F\{a f(x) + b g(x)\} = a F(P) + b G(P),$$

where a and b are constants.

Note: (i) $F_s\{a f(x) + b g(x)\} = a F_s(P) + b G_s(P)$

(ii) $F_c\{a f(x) + b g(x)\} = a F_c(P) + b G_c(P)$

2. Change of scale property:

If $F(P)$ is the complex Fourier transform of $f(x)$, then $F\{f(ax)\} = \frac{1}{a} F\left(\frac{P}{a}\right) (a>0)$.

Proof: We have

$$F(P) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx.$$

So,

$$F\{f(ax)\} = \int_{-\infty}^{\infty} e^{ipx} f(ax)\,dx$$

$$= \int_{-\infty}^{\infty} e^{ip\cancel{t}/a} \cdot f(t) \frac{dt}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} e^{i\left(\frac{p}{a}\right)t} f(t)\,dt$$

Taking $ax = t$, then we get $a\,dx = dt$

or, $dx = \frac{1}{a} dt$

And $x \to -\infty$, $t \to -\infty$

$x \to \infty$, $t \to \infty$

$$\therefore F\{f(ax)\} = \frac{1}{a} F\left(\frac{p}{a}\right).$$

Note: (i) $F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{p}{a}\right)$

and (ii) $F_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{p}{a}\right)$

3. Shifting Property:

If $F\{f(x)\} = F(p)$, then $F\{f(x-a)\} = e^{ipa} F(p)$.

[Note: (ii) [illegible]]

Proof: We have

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x) e^{ipx} dx$$

So, $$F\{f(x-a)\} = \int_{-\infty}^{\infty} f(x-a) e^{ipx} dx$$

$$= \int_{-\infty}^{\infty} f(t) e^{ip(t+a)} dt$$

$$= e^{ipa} \int_{-\infty}^{\infty} e^{ip\cancel{t}} f(\cancel{t})\, dt$$

Taking $x - a = t$, then

$dx = d\cancel{t}$

and $x \to -\infty$, $t \to -\infty$

$x \to \infty$, $t \to \infty$

$\therefore F\{f(x-a)\} = e^{ipa} . F(p)$

4. Modulation Theorem:

If $F\{f(x)\} = F(p)$, then

$$F\{f(x)\cos ax\} = \frac{1}{2}\left[F(p+a) + F(p-a)\right]$$

Proof: we have $F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{ipx} f(x)\,dx$

So, $F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{ipx} f(x)\cos ax\,dx$

$$= \int_{-\infty}^{\infty} e^{ipx} f(x) . \left(\frac{e^{iax} + e^{-iax}}{2}\right) dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left(f(x) . e^{i(p+a)x} + f(x) . e^{i(p-a)x}\right) dx$$

$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x)\, e^{i(p+a)x}\,dx + \int_{-\infty}^{\infty} f(x)\, e^{i(p-a)x}\,dx\right)$$

$$= \frac{1}{2}\left[F(p+a) + F(p-a)\right].$$

Note: (i) $F_s\{f(x)\cos ax\} = \frac{1}{2}\{F_s(p+a) + F_s(p-a)\}$

(ii) $F_s\{f(x)\sin ax\} = \frac{-1}{2}\{F_c(p+a) - F_c(p-a)\}$

(iii) $F_c\{f(x)\cos ax\} = \frac{1}{2}\{F_c(a+p) + F_c(a-p)\}$

(iv) $F_c\{f(x).\sin ax\} = \frac{1}{2}\{F_s(p+a) - F_s(p-a)\}$

5. $F\{x^n f(x)\} = (-i)^n \frac{d^n}{dp^n}[F(p)]$ $\left(\boxed{F\{f(x)\} = F(p)}\right)$

6. $F\left\{\frac{d^n}{dx^n} f(x)\right\} = (-ip)^n F(p)$

(Assuming
$f(x) \to 0$ as $x \to \pm\infty$
$f'(x) \to 0$ as $x \to \pm\infty$
$\vdots$
$f^{(n-1)}(x) \to 0$ as $x \to \pm\infty$)

Proof: we have

$$F\{f'(x)\} = \int_{-\infty}^{\infty} e^{ipx}\left(\frac{d}{dx} f(x)\right) dx$$

$$= \left[e^{ipx}.\, f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (ip) e^{ipx} f(x)\, dx$$

$$= -ip \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

$$= (-ip)\, F(p)$$

Similarly, we can prove

$$F\{f''(x)\} = (-ip)^2 F(p)$$

$\vdots$

$$F\left\{\frac{d^n}{dx^n} f(x)\right\} = (-ip)^n . F(p)$$

7. (i) $F_s\{f'(x)\} = -p\,F_c(p)$

and (ii) $F_c\{f'(x)\} = -f(0) + p\,F_s(p)$.

Relation Between Fourier and Laplace transforms

Fourier transform of $x(t)$ is defined as

$$F\{x(t)\} = F(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and the Laplace transform of $x(t)$ is defined

as $$L\{x(t)\} = F(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt,$$

where s is a complex variable and is given by $s = \sigma + j\omega$.

Therefore, $$L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left(x(t) \cdot e^{-\sigma t}\right) e^{-j\omega t}\, dt$$

$$\Rightarrow \boxed{L\{x(t)\} = F\{x(t)\, e^{-\sigma t}\}}$$